Announcements

- Exam 2 on Friday
- Exam 2 Review Talk on Course Webpage
- Please Complete Exam Instructions Assignment

Last Time

- Matchings
  - Subset of edges using distinct vertices
- Hall's Theorem
  - Matching in bipartite graph using all of X unless set S with $|S| > |N(S)|$
- Every regular bipartite graph has a perfect matching
  - $\Sigma d(x) = |E| = \Sigma d(y)$

Today

- Edge Coloring of Bipartite Graphs
- Konig’s Theorem
- Introduction to Flows and Networks

Application: Edge Coloring Bipartite Graphs

Recall, that given a graph G, the edge coloring number was either Δ(G) or Δ(G)+1.

Theorem (V 6.1.1): Every finite bipartite graph G can be edge colored with Δ(G) colors.

Proof Idea

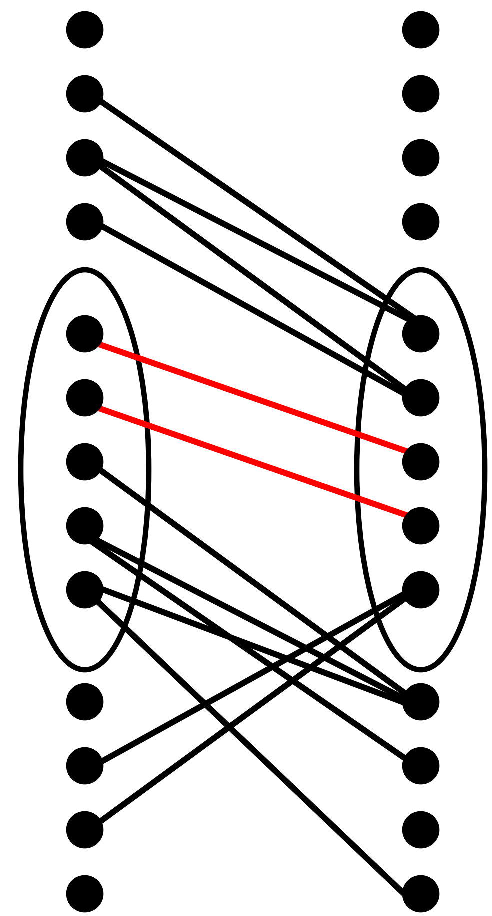
- Find a matching M which includes all of the vertices of G of maximum degree.
- Color all edges in M one color, inductively color G-M.
  - Since M includes an edge from each vertex of max degree $\Delta(G-M)=\Delta(G)-1$, can be colored with $\Delta(G)-1$ colors.

Matching M

- Need to find matching including all degree $k = \Delta(G)$ vertices.
- If all on one side: Consider G', graph with only the max degree vertices and their neighbors.
  - Similar argument to regular case
  - Matching unless set S with $|S| > |N(S)|$
  - $k|S| = \Sigma d(s) = \Sigma d(t) \leq k|N(S)|$
  - Use Hall's Theorem

In General

- Consider all degree k vertices
- Find matching M_1 among max degree vertices.
- Remaining degree k vertices don't connect.
- Match degree k vertices on each side.



Summary

- Match all vertices of top degree
  - Match as many among each other
  - Match rest (Hall’s Theorem & Counting argument)
- Recurse to find next color
- Bipartite graphs have less than the guaranteed edge coloring number.

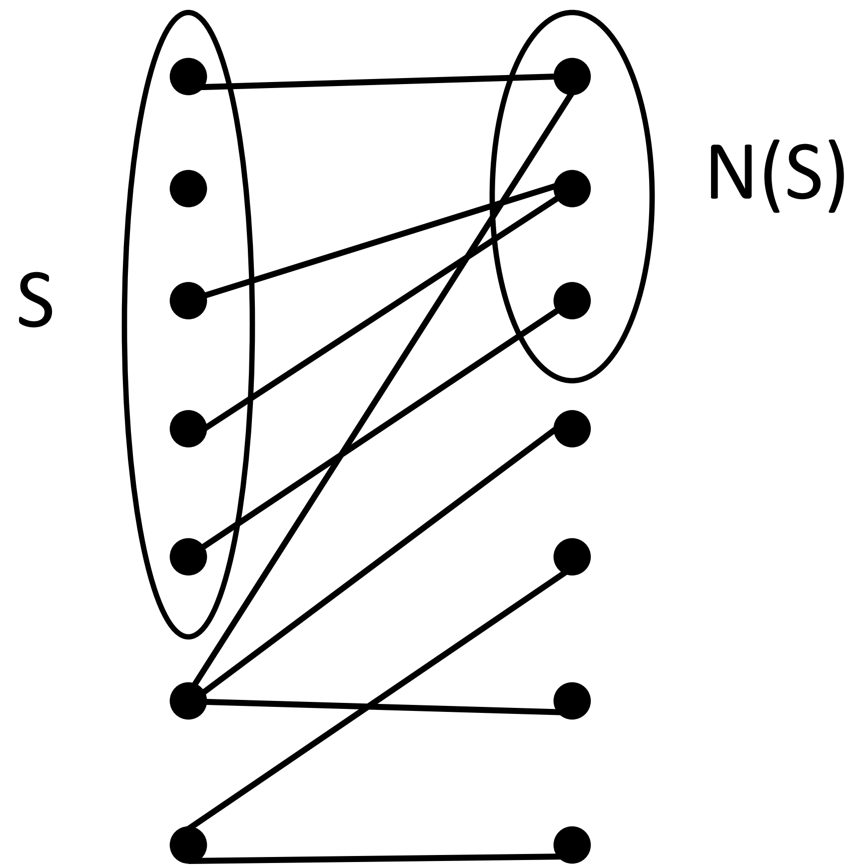
Maximum Matching

Hall's Theorem tells us when we can match *all* of X. If we cannot, what is the most that we can manage?

- Cannot get everything if some S with $|S| > |N(S)|$.
- If instead $|S| - |N(S)| \geq k$, then have to leave at least k out.

Another View

- Have set S bigger than N(S).
- Every edge hits N(S) ∪ (X-S).
  - This is a *vertex cover*.
- Matching has size at most
  | N(S) ∪ (X-S)|
  = |X| - (|S| - |N(S)|).



Summary

<u>**Definition:**</u> A *vertex cover* is a set C of vertices so that every edge is incident on some vertex of C.

<u>**Lemma:**</u> The size of the maximum matching is at most the size of the minimum vertex cover.

<u>**Proof:**</u> Each edge of M uses different vertex of C.

Is this tight?

Konig’s Theorem

Theorem (1.53): For any finite, bipartite graph G the size of the maximum matching equals the size of the minimum vertex cover.

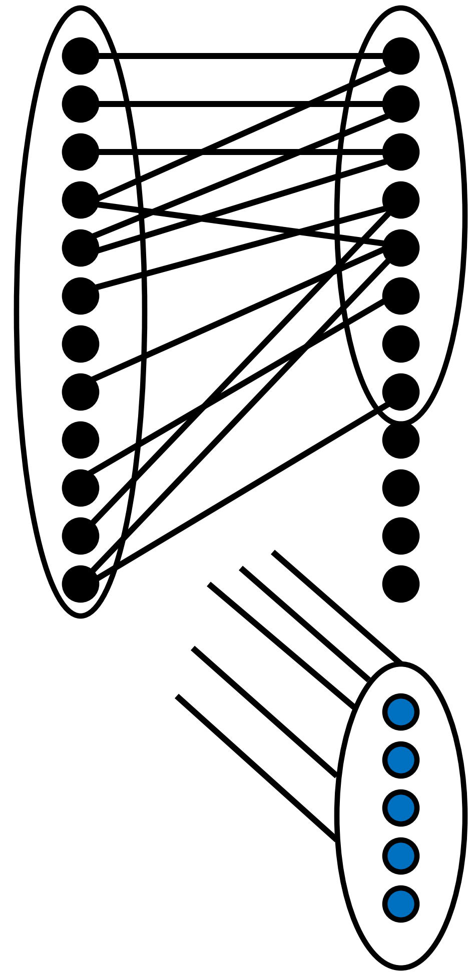
Note: This is a generalization of Hall’s Theorem.

Proof Idea

- Already know, maximum matching at most as large as minimum cover. Need other direction.
- Use Hall's Theorem.
- Matching of size $|X|-k$ iff full matching after adding k more vertices.
- Happens unless there is a set S with $|N(S)| < |S|-k$.

Proof

- Maximum matching of size |X| - k.
- Add k-1 vertices on Y connect to all X.
  - Still no full matching
- Hall's Theorem
  |S| ≥ |N(S)| + k.
- N(S) ∪ (X-S), vertex cover of size |X|-k.



Network Flows

- Introduction
- Mathematical Formulation
- Maxflow-Mincut
- Applications

Evacuation Planning

You are an emergency planner for a city. You need to plan evacuation routes to be used in the case of a disaster. You need to know two things:

1) Which routes to use
2) How quickly you can get people out of the city to the designated safe area

Note: You cannot just use all the roads out equally, some routes might get bottlenecked unless you plan carefully.