Announcements

- Exam 2 on Friday
- Please complete exam instructions assignment before then

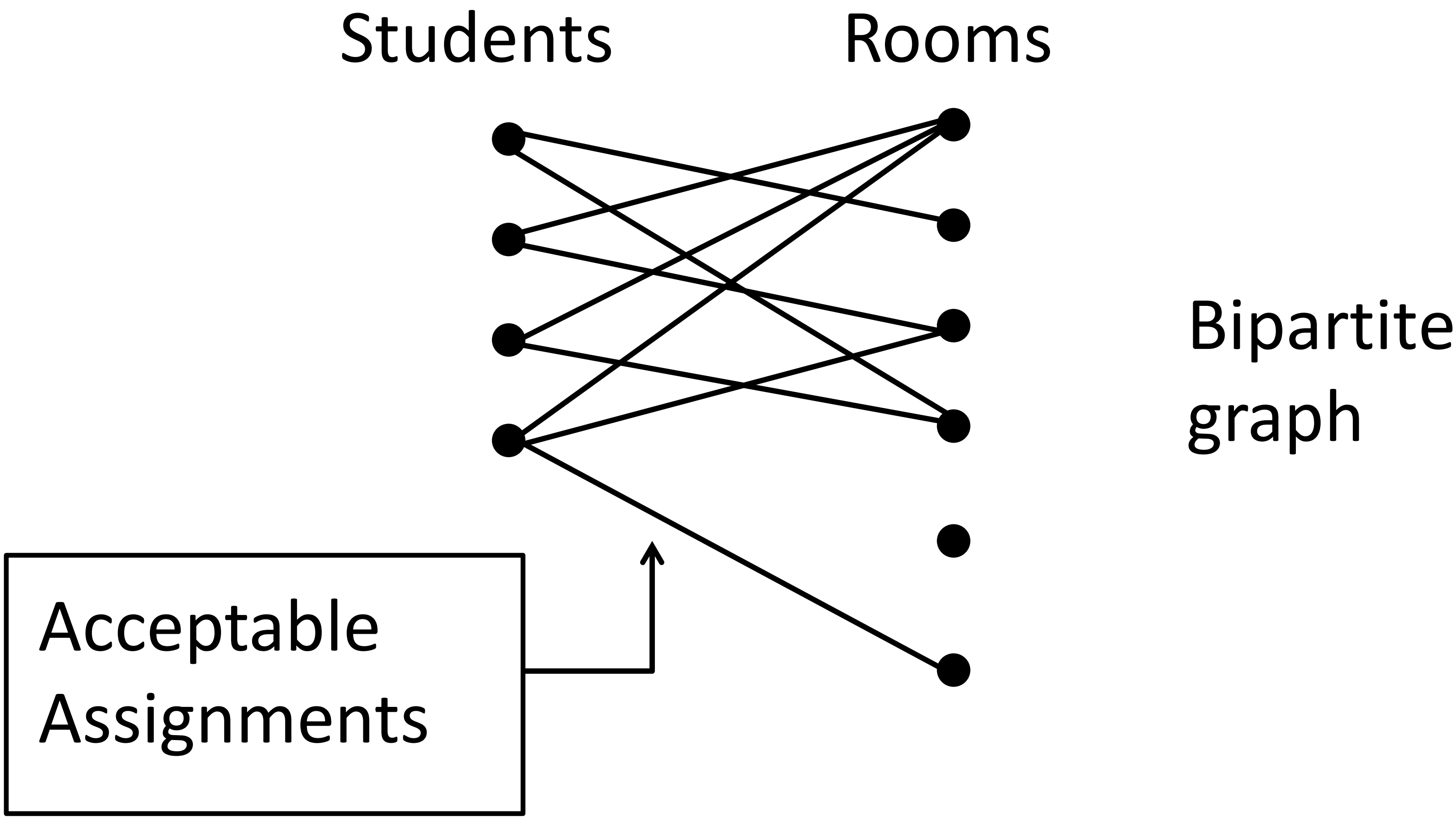
Matchings and Flows (Ch 1.7)

- Bipartite Matching
  - Hall's Theorem
  - Konig's Theorem
- Flows
  - Maxflow-Mincut & Applications
- Perfect Matchings in General
  - Tutte's Theorem

Room Assignments

You are running a dorm and trying to assign students to rooms. Beforehand you had each student fill out a survey of rooms they would be willing to be in. You want to know whether or not it is possible to find an assignment of each student to a room so that every student is assigned a room they find acceptable.

Mathematical Step



Matchings

Definition: A *matching* in a graph G is a set of edges of G no two of which share an endpoint. The *size* of a matching is the number of edges. A matching is *maximum* if its size is as large as possible.

Question: Maximum Matching

What is the size of a maximum matching in the graph below?

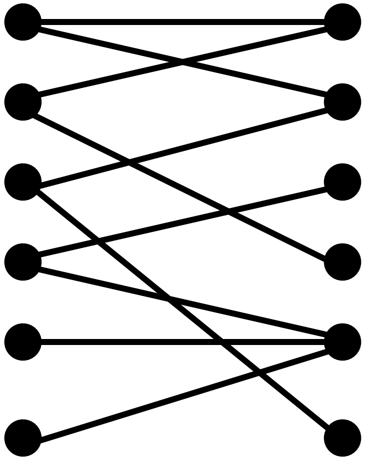
A) 3

B) 4

C) 5

D) 6

E) 7



When is a Full Matching Possible?

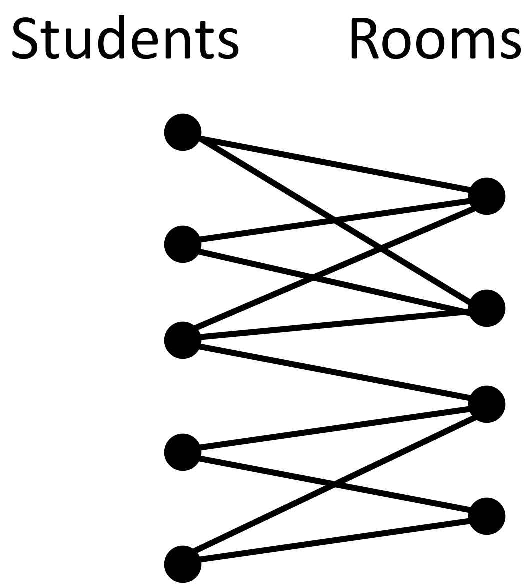
When can we match every student to a room?

Not always.

In the next few slides we'll go over some cases where it is impossible.

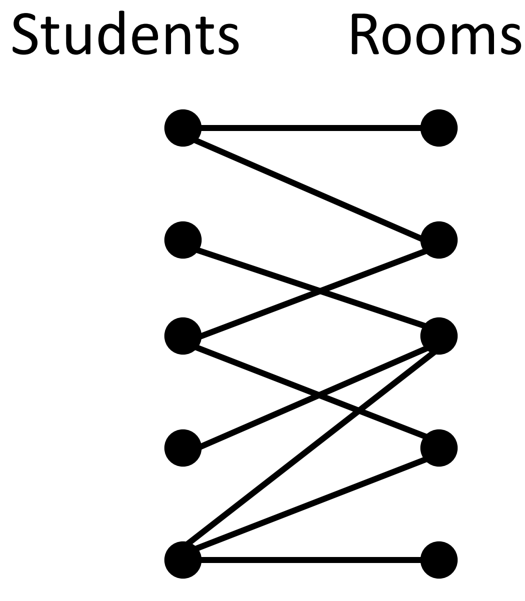
Example 1

5 students and 4 rooms. Cannot fit them all.



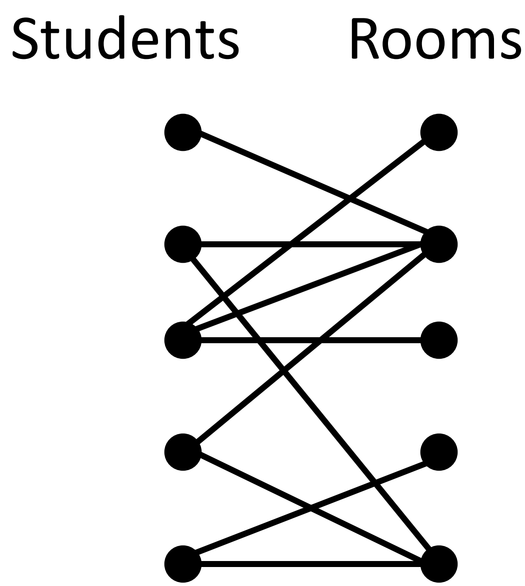
Example 2

Two students demand the same room.



Example 3

Three students all demand one of the same two rooms.



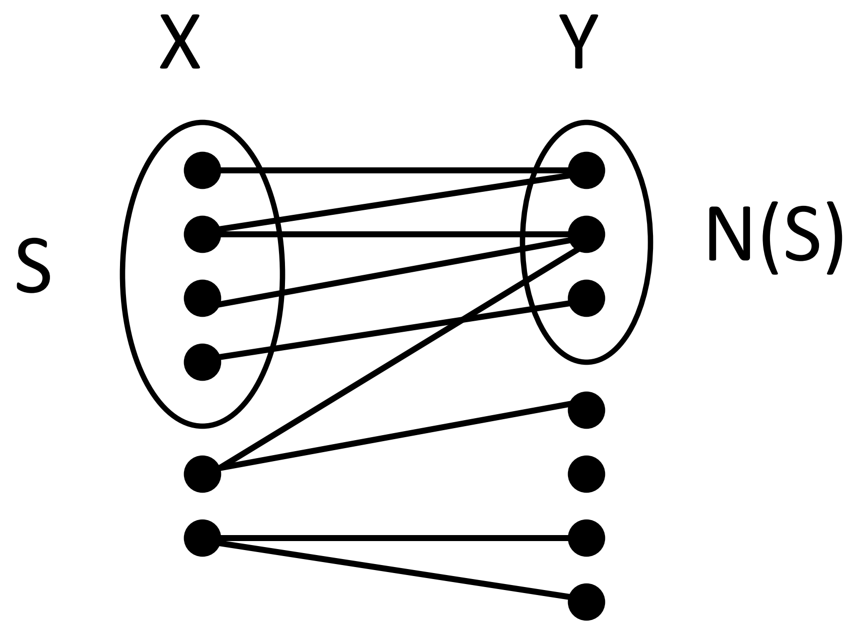
The Marriage Lemma

Theorem (1.51): Let G be a bipartite graph with parts X and Y. There is a matching of G using all the vertices in X if and only if for every subset $S \subseteq X$, $|N(S)| \geq |S|$, where N(S) is the set of neighbors of elements of S.

Easy Direction

Suppose there is an S with |N(S)| < |S|.

Each element of S can only match to an element of N(S), and there are not enough for each to get a different one.

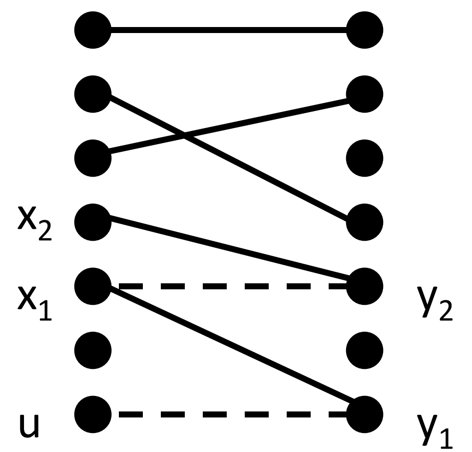


Hard Direction

- Need to show that if there is no way to match all elements of X then there must be an S with $|N(S)| < |S|$.
- Assume that there is no such matching.
- Find a maximum matching.
- Try to extend this matching, in the process find S.

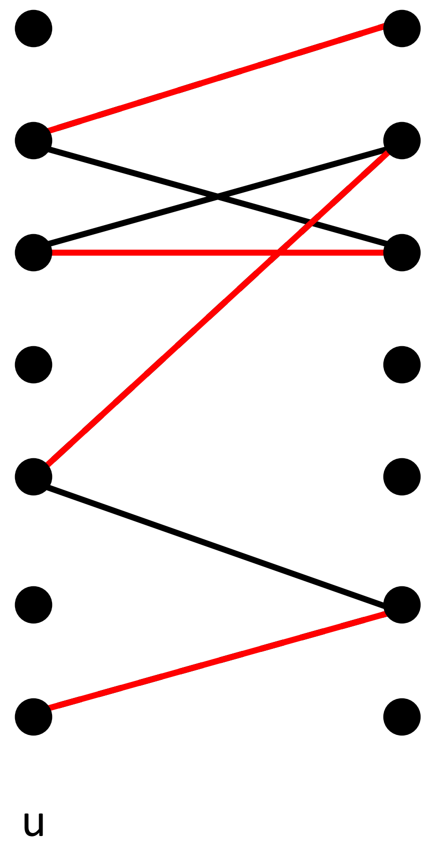
Extra Vertex

- Have maximum matching
- Take unmatched u
- Adjacent to some y_1
- Can add $\{u,y_1\}$ unless y_1 already matched to some x_1.
- Try to rematch x_1 to some other neighbor y_2.
- Works unless y_2 matched to x_2.
- Repeat this process.



Augmenting Paths

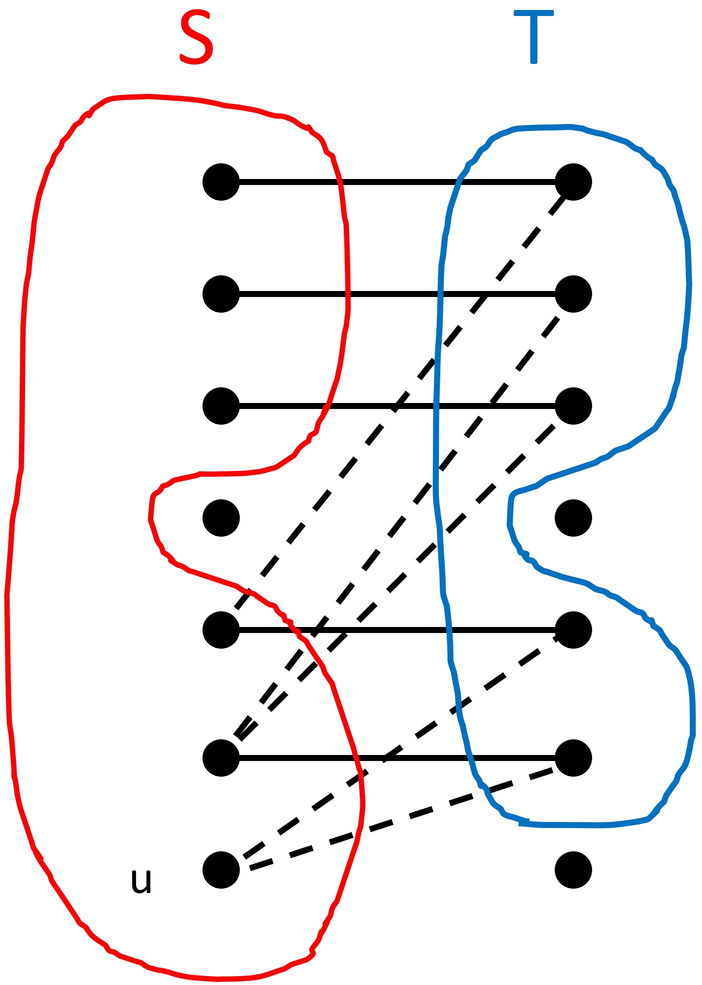
- Consider paths starting at u that take any edge of the graph from X to Y, and take matching edges back.
- If you can reach an unmatched vertex, can increase matching.



Set S

If you cannot find one ending on unmatched vertex:

- Consider all vertices you can reach with such a path.
- Let S be the set of X-vertices you can reach.
- T set of Y-vertices.



Properties

- Every neighbor of an element of S is in T
  - $T \supseteq N(S)$
- For each vertex in T, the matched vertex is in S.
  - $|S| \geq |T|$
- But additionally $u \in S$.
  - $|S| > |T| \geq |N(S)|$

Proof Summary

- Consider Maximum Matching
- If not all of X matched, try to find augmenting path (which cannot reach new vertex in Y)
- Consider the set of all vertices reached
- Gives an S that works

Question: Algorithm

Does the proof of Hall's Theorem give an algorithm for finding a matching of X or a set S?

A) Yes
B) No

Application: Regular Bipartite Graphs

Proposition (1.57): Any regular, bipartite graph has a perfect matching (i.e. a matching that uses all of the vertices).

Bipartite Graph Degrees

Note that every edge of a bipartite graph attaches to each side of the graph.

Lemma: Let G be a bipartite graph with parts X and Y, then

$$\Sigma\, d(x) = |E| = \Sigma\, d(y)$$

Proof: Direct all edges from X to Y and apply the digraph handshake lemma.

Corollary: For regular bipartite G, $|X| = |Y|$.

Proof

- Enough to find matching using all of X.
- Hall's Theorem ⇒ can find unless $S \subseteq X$ with $|S| > |N(S)|$.
- Consider induced subgraph H on $S \cup N(S)$.
  - Sums of degrees

    $$k|S| = \Sigma d(s) = \Sigma d(t) \leq k|N(S)| < k|S|$$
- Contradiction! Therefore, there must be a matching.