Announcements

- HW 5 Due Sunday

Previously

- Vertex Colorings
  - Assign color to vertices of a graph so that adjacent vertices have different colors.
  - Chromatic number, fewest number of colors needed.
  - At least clique number.

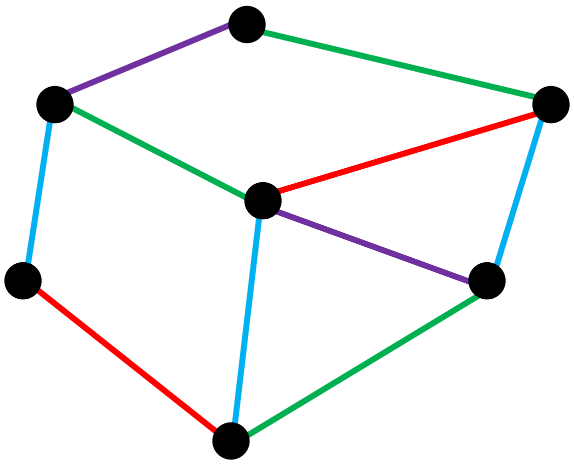
Today

Edge colorings

- Definitions
- Examples
- Vizing’s Theorem

Edge Colorings

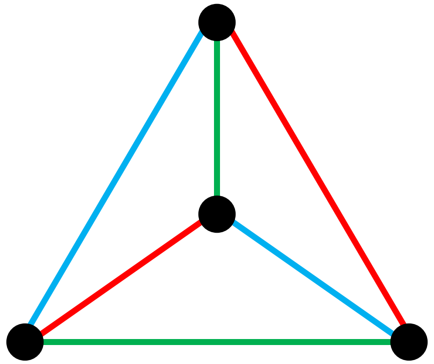
Definition: An *edge coloring* of a graph is an assignment of a color to each edge so that no two edges incident on the same vertex are the same color.



Question: Coloring a K_4

What is the minimum number of colors needed in an edge-coloring of a K_4?

A) 1
B) 2
C) 3
D) 4
E) 5



How Many Colors are Needed

Lemma: Any edge coloring of a graph G requires at least Δ(G) colors.

Proof: The Δ(G) edges at a maximum-degree vertex all need to be different colors.

Lemma: There always is a coloring with at most 2 Δ(G)-1 colors.

Proof: Use greedy coloring.

Vizing's Theorem

Theorem (V. 6.2.1): Any finite graph G has an edge coloring with at most Δ(G)+1 colors.

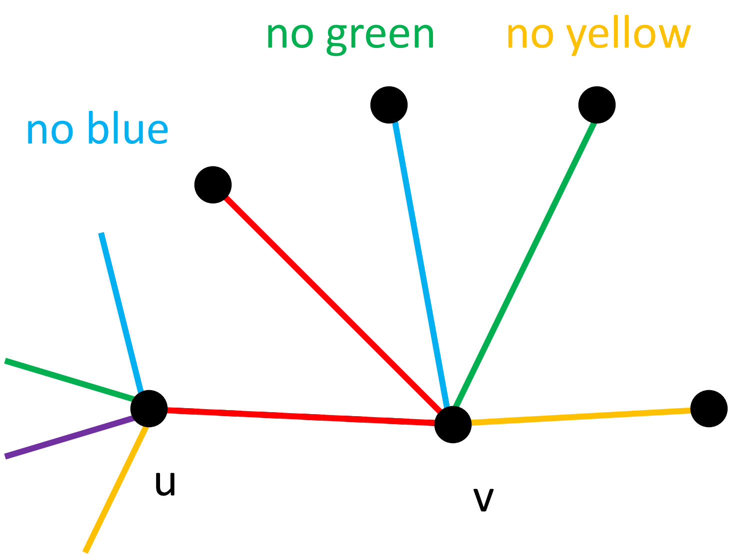
- The minimum number of colors is either Δ(G) or Δ(G)+1.
- Both are possible. C_n requires 2 colors (Δ(G)) when n is even and 3 colors (Δ(G)+1) when n is odd.

Proof Idea

- Proof by induction on |E|
- Color G-e, show how to insert last edge
- This might require some recoloring of its neighbors

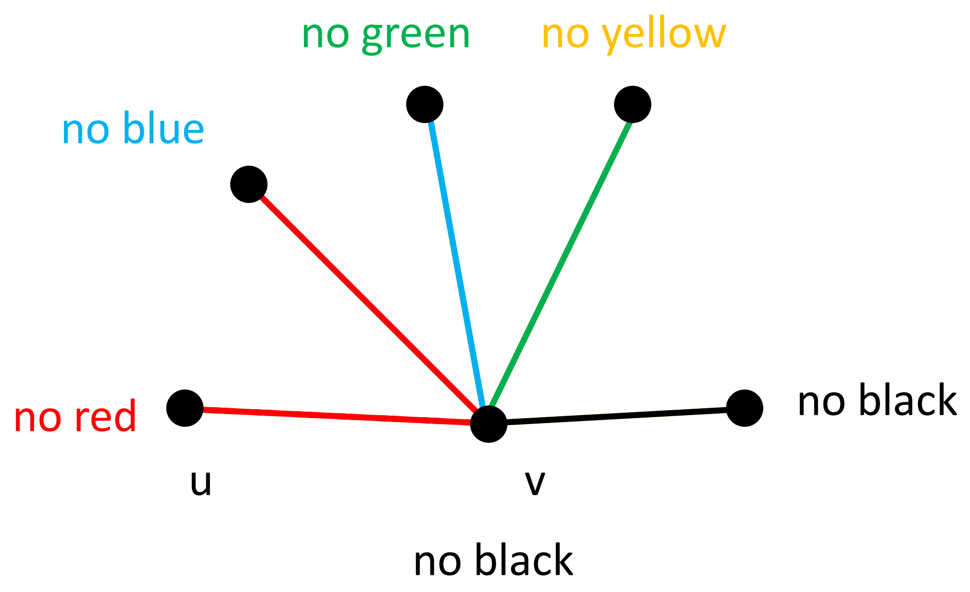
Proof

- Color G-e, e = {u,v}
- Some color (red) missing at u.
  - If also missing at v, color e
- Otherwise $e_1 = \{u_1,v\}$ already red
- u_1 missing some other color (blue)
- Try to recolor e_1 to blue
  - Get chain of recolorings



Case 1

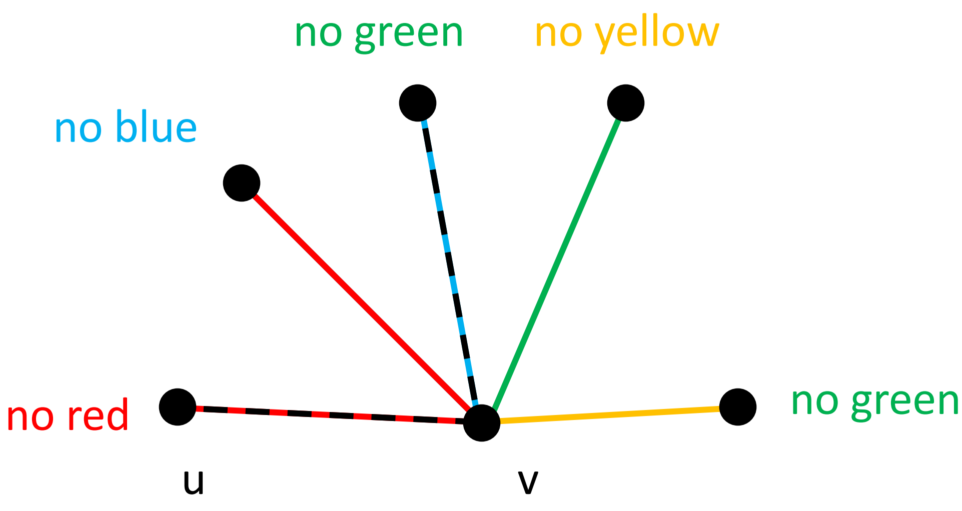
If the chain ends eventually, you can recolor all of the affected edges, inserting the new one.



Case 2

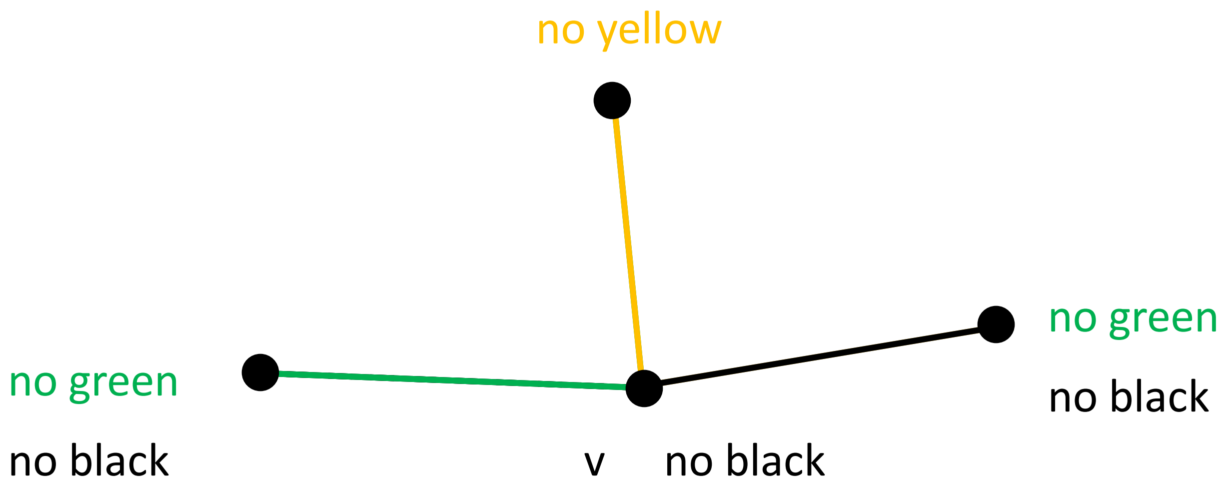
Otherwise, the chain must eventually loop back.

Recolor everything up to the loop.



Case 2 Continued

- Have cycle of missing colors about v
- v lacks some color (black)
- If something in the cycle lacks black can recolor
- Try to recolor to avoid this

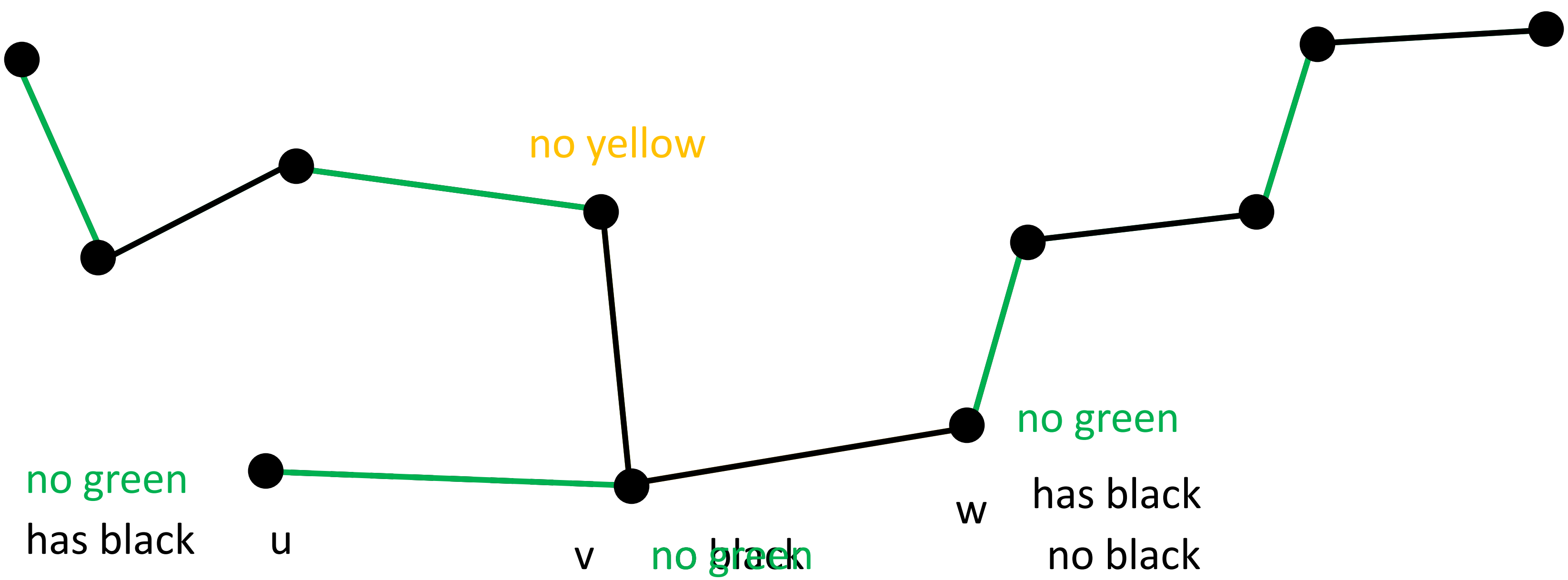


Recoloring

- Want to recolor green and black edges in order to get around this difficulty
- Consider subgraph H of green and black edges
- Degrees in H of at most 2, components are paths and cycles
- Can switch green and black in any component

Recoloring the Cycle

- u, v, w all have degree 1 in H
- One must be in own component
- Recolor that component & add edge



Proof Summary

- Color G-e
- Try to add color to e, get chain of color replacements
- If chain ends, recolor, otherwise get a cycle
- Look at 2-color graph, recolor some component of it to remove cycle & color

Question: Algorithm

Does the proof of Vizing's Theorem give an algorithm to produce a ($\Delta(G)+1$) edge-coloring of G?

A) Yes

B) No