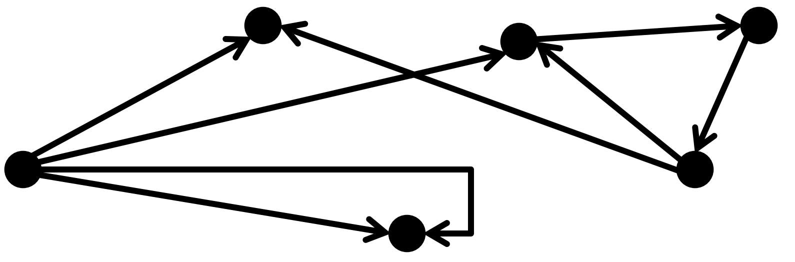
Last Time

- Graph definition
  - Vertices connected by edges
- Types of graphs:
  - Multigraph (multiple edges between same vertices)
  - Pseudograph (allows self loops)
  - Simple graph (neither of the above)
  - Hypergraph (edges between more than two vertices)
  - Directed graph (edges have orientation)

Question: Graph Generalizations

Which graph types are exhibited by the graph below?



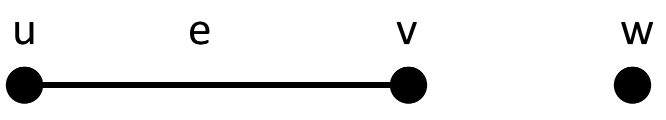
A) Multigraph
B) Pseudograph
C) Simple Graph
D) Hypergraph
E) Directed Graph

Today

- Graph Terminology
- Handshake Lemma
- Walks, paths and Cycles

Graph Terminology I

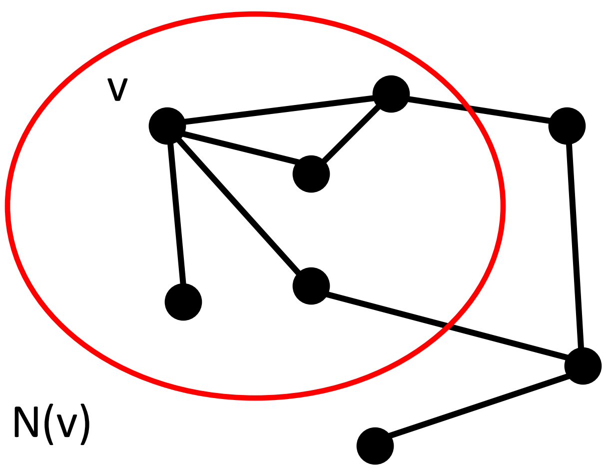
- Two vertices u and v are *adjacent* if there is an edge connecting them.
- A vertex v is *incident* on an edge e (or is an *endpoint* of e) if v is one of the vertices e connects.



u is adjacent to v
v is non-adjacent to w
u is incident on e
w is not incident on e

Graph Terminology II

- The *neighborhood* of a vertex v (denoted N(v)) is the set of vertices adjacent to v along with v.
- The *degree* of v (denoted d(v)) is the number of vertices adjacent to v.

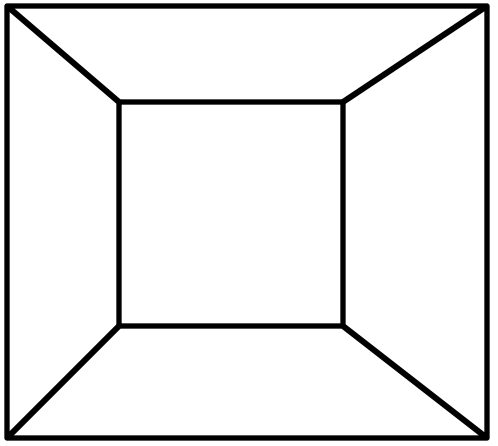


d(v) = 4

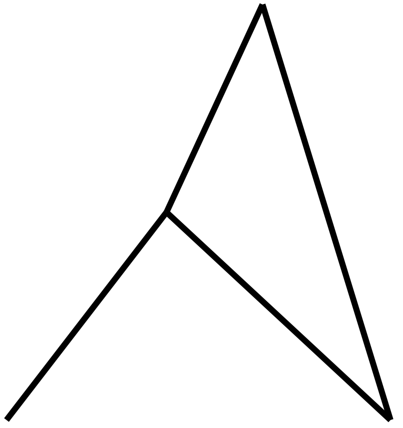
Graph Terminology III

- A graph is *d-regular* if all vertices have degree d. It is *regular* if it is d-regular for some d.

This graph is 3-regular

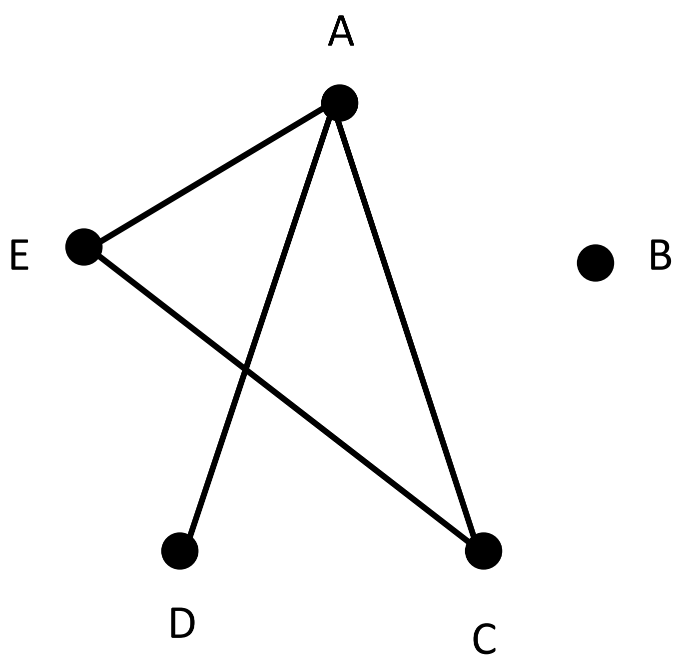


This graph is not regular



Question: Degrees

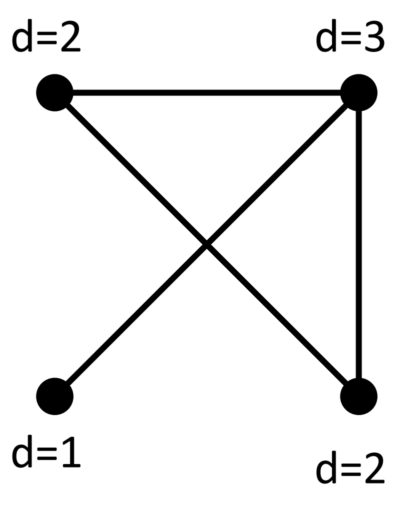
Which vertex in this graph has the smallest degree?



The Handshake Lemma

(Theorem 1.1) For any graph $G = (V, E)$,

$$\sum_{v \in V} d(v) = 2|E|.$$

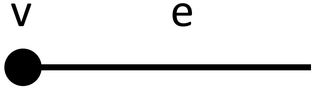


$$2 + 3 + 1 + 2 = 8 = 2 \cdot 4$$

Proof I

Strategy: Counting things in two different ways.

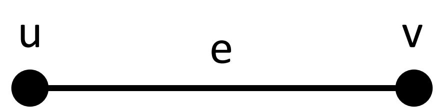
Show both sides are equal to the number of pairs of (v,e) where v is a vertex incident on an edge e.



Proof II

Right Hand Side:

Each edges e = (u,v) has two incident vertices, u and v.

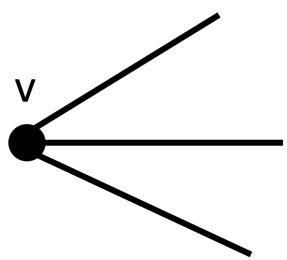


Total number of pairs is 2|E|.

Proof III

Left Hand Side:

Each vertex v is incident on d(v) edges.



Total number of pairs is $\sum_{v \in V} d(v)$.

Proof IV

Equating the two sides we find:

$$\sum_{v \in V} d(v) = \#\{\text{Incidence pairs}\} = 2|E|.$$

QED.

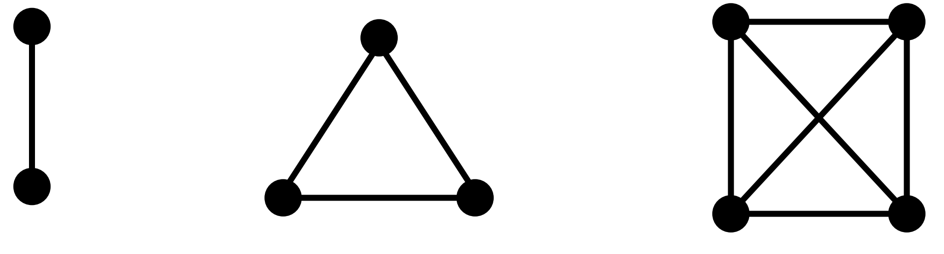
Question

How many edges does a 3-regular graph with 5 vertices have?

A) 3
B) 6
C) 7.5
D) 10
E) There is no such graph

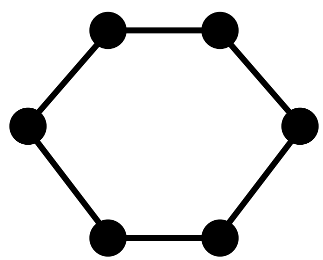
Examples of Graphs I

A *complete graph* on n vertices (denoted K_n) is a graph with n vertices and an edge between every pair of them

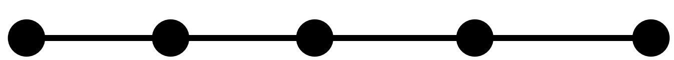


Examples of Graphs II

A *cycle* on n vertices (denoted C_n) is a graph with n vertices connected in a loop.

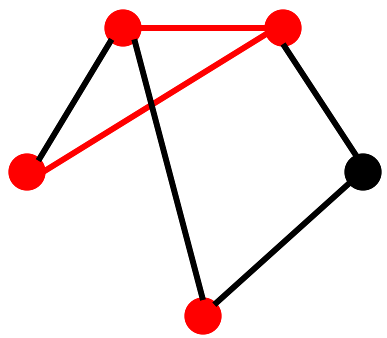


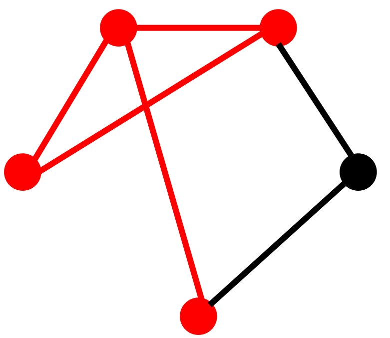
A *path* on n vertices (denoted P_n) is a graph with n vertices connected in a chain.



Examples of Graphs III

A graph H is a *subgraph* of G if V(H) ⊂ V(G) and E(H) ⊂ E(G).





A subgraph H is an *induced subgraph* if it contains *all* the edges of G connecting two vertices in V(H).

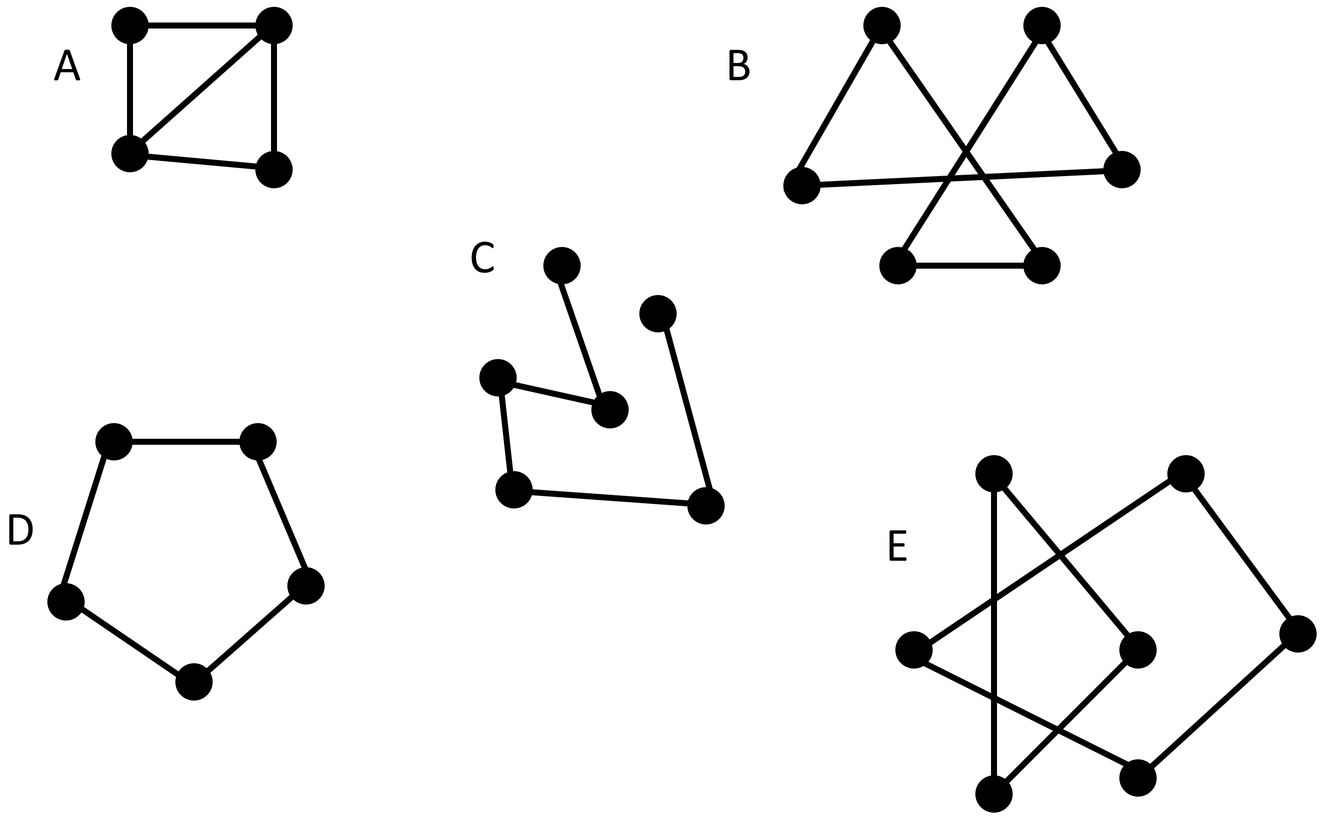
Examples of Graphs IV

A *bipartite graph* is a graph whose vertices can be split into two parts where all edges connect one part to the other.

A *complete bipartite graph* (denoted $K_{n,m}$) has an edge connecting every element of one part (of size n) to every element of the other (of size m).

Question: Cycle Identification

Which of the graphs below are cycles?



Question: Edge Counts

Which of these graphs has the greatest number of edges?

A) C_{10} (10 edges)
B) P_{12} (11 edges)
C) K_5 (10 edges)
D) $K_{3,4}$ (12 edges)