Announcements

- HW 5 Due Sunday
- Please let me know if you need to take exam 2 not-during class hours

Last Time

- Chromatic number = minimum number of colors needed to color G.
- Brook's Theorem: $\chi(G) \leq \Delta(G)$ unless G is an odd cycle or complete graph.
  - Not regular: greedy coloring ending at v.
  - Not 3-connected: Break into parts, color parts inductively, recolor to make them match.
  - 3-connected, find v with non-adjacent neighbors u,w. Greedily color so that u,w assigned same color first, v assigned color last.

Question: Algorithm

Does the proof of Brook's Theorem give an algorithm to produce a $\Delta(G)$-coloring of G?

A) Yes
B) No

Today

Coloring Planar Graphs

- Setup
- Five Color Theorem
- Four Color Theorem

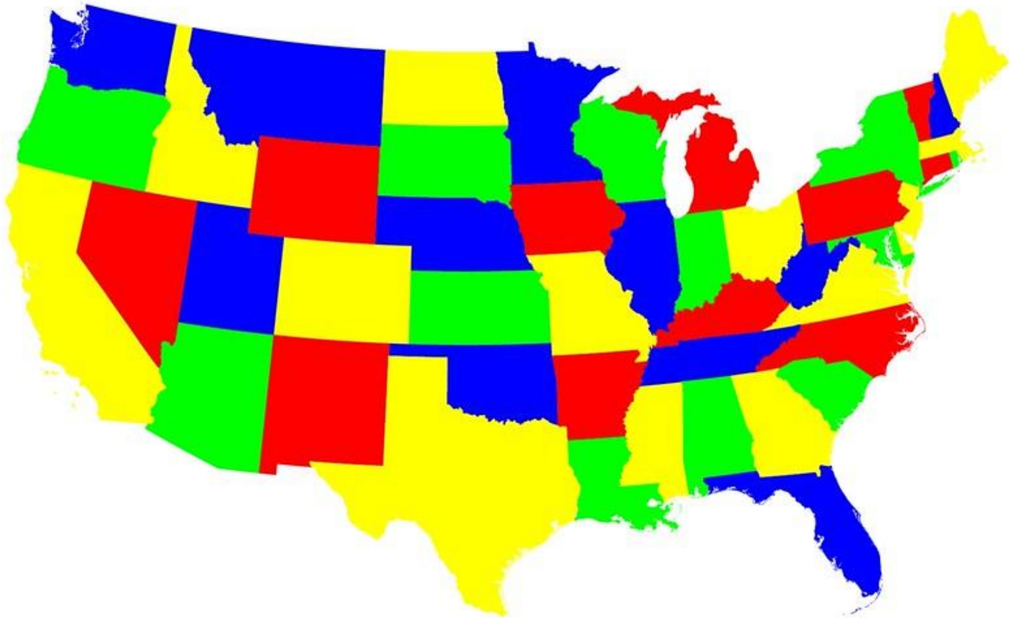
Coloring Planar Graphs

A particularly interesting and practical problem is how many colors are needed to color a planar graph.

In other words, given a map with connected regions, how many colors do you need to ensure that neighboring regions are different colors?

Map Coloring

Can color US map with 4 colors.

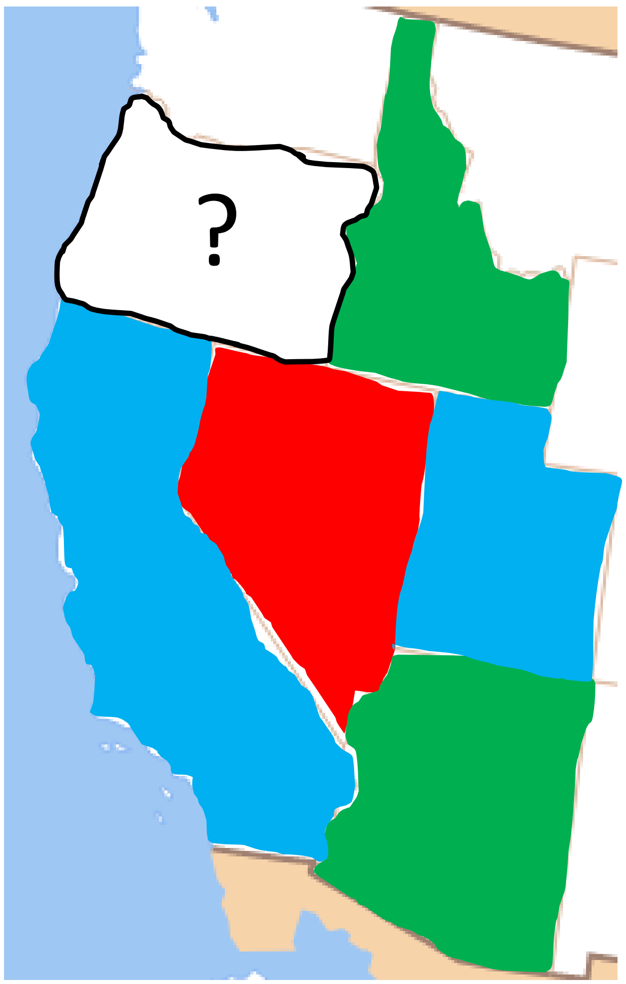


Map Coloring II

Can’t color even these western states with fewer than 4 colors.

Chromatic number of the US is 4.

What about other maps?



Warmup

Theorem: Every planar graph is 6-colorable.

Idea: We know that G has a vertex of degree at most 5. Use a greedy coloring.

Proof: By induction on number of vertices.

- If $|V| \leq 6$, can color trivially.
- If we can color all smaller graphs
  - v a vertex with $d(v) < 6$
  - Color G-v inductively
  - Give v color not used by neighbors

The 5-Color Theorem

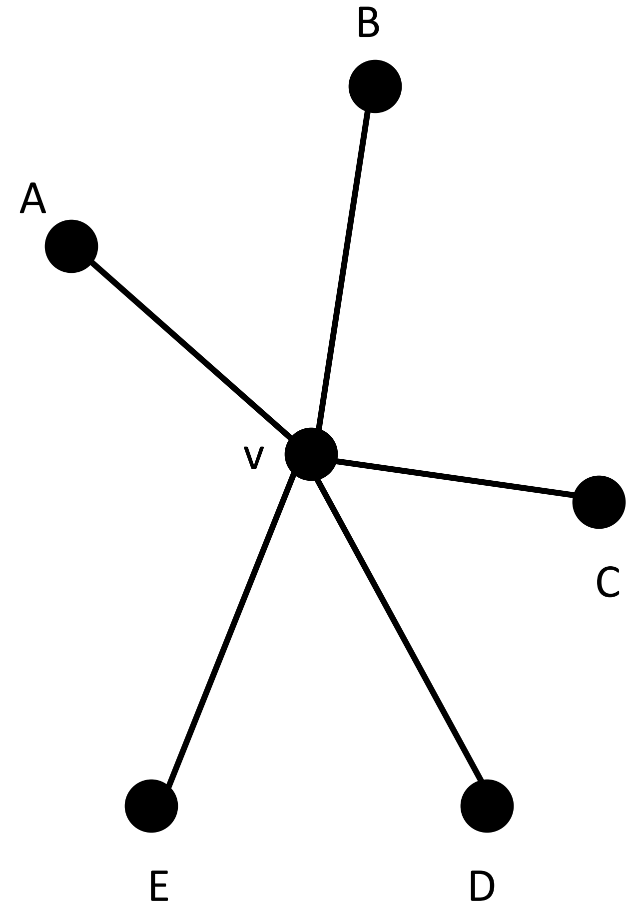
<u>Theorem 1.47 (Kemp):</u> Every planar graph is 5-colorable.

<u>Idea:</u> Induction as before.

(There will be some complications)

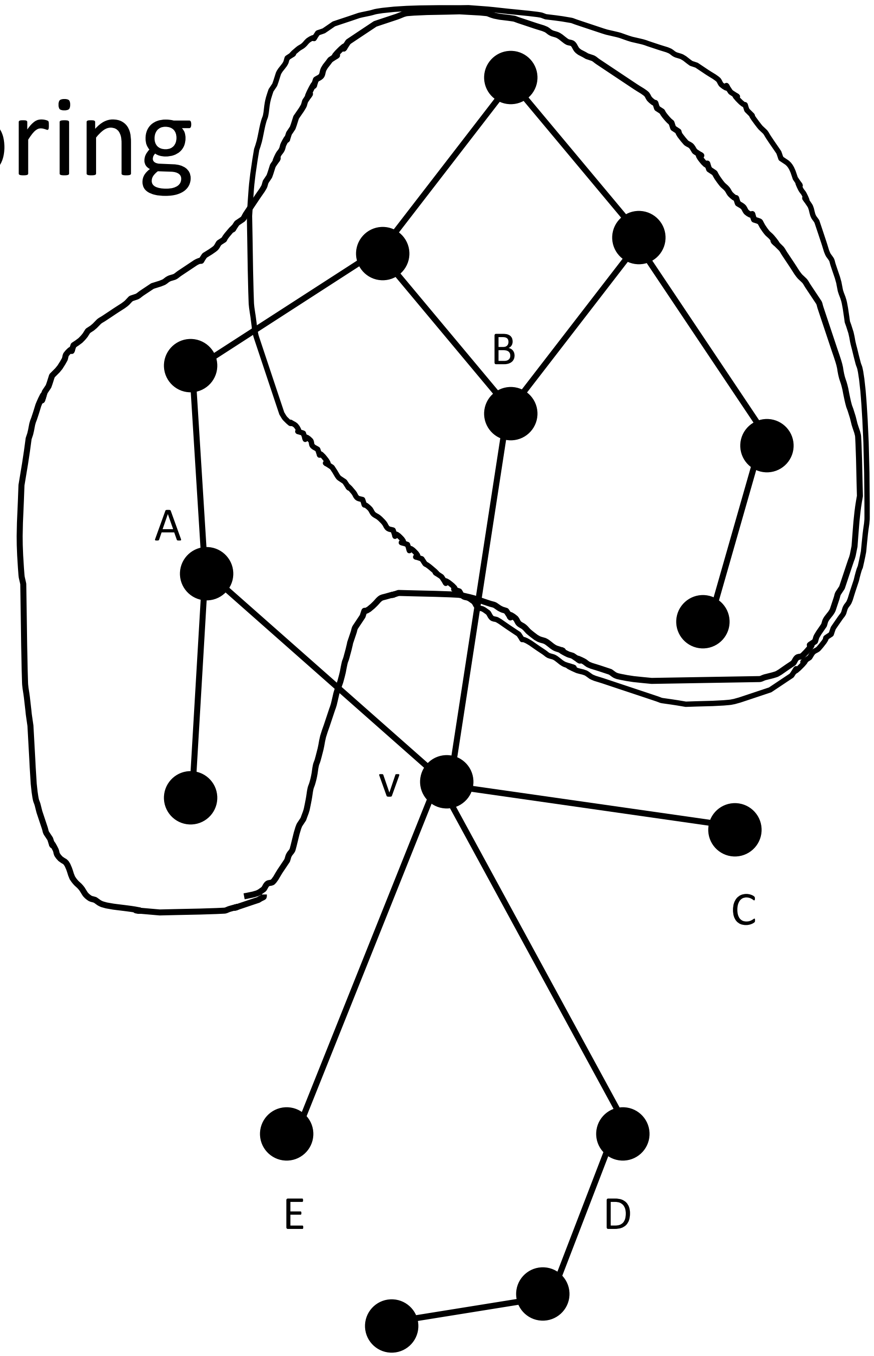
Proof

- Induct on |V|
  - Base case easy
- Take v with $d(v) \le 5$
- Color G-v
  - OK *unless* v's neighbors use all 5 colors
  - Try to recolor them



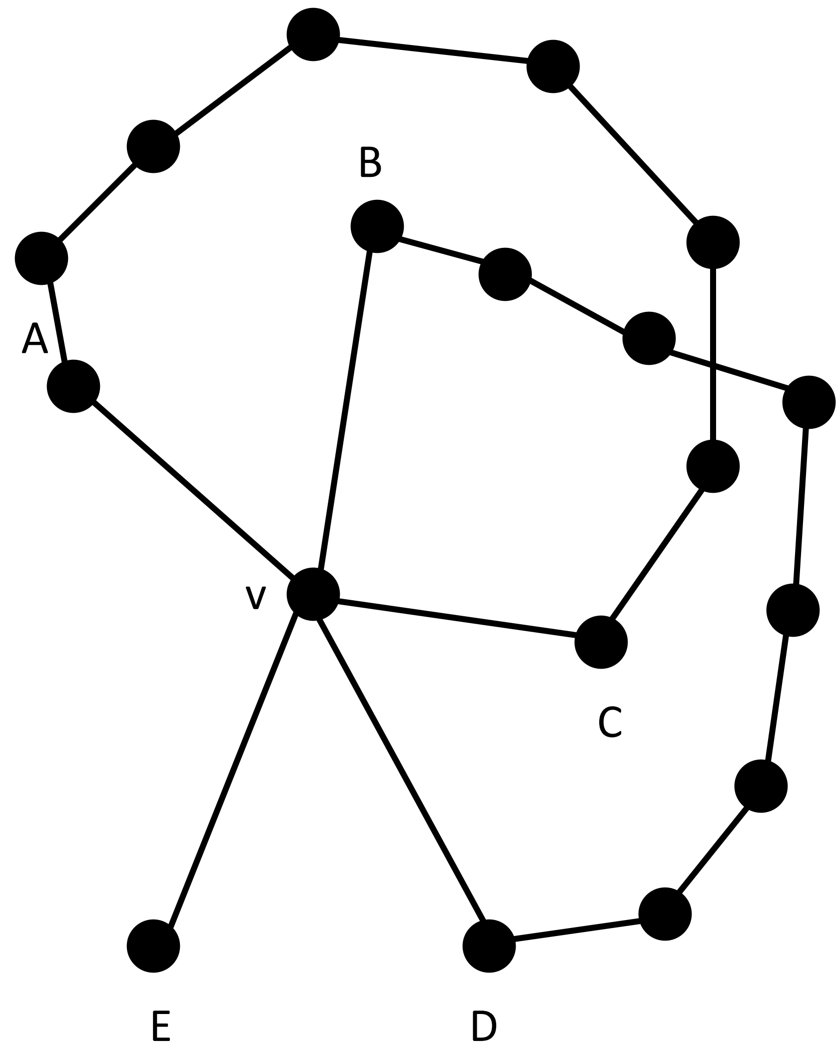
Recoloring

- Can swap two colors
  - Doesn't help
- Consider subgraph of red & blue vertices
  - Can recolor single component
  - Works unless A & B in same component



Kemp Chains

- Can recolor unless red-green chain from A to C
- Can recolor unless blue-yellow chain from B to D
- Cannot have both!
- Always a way to recolor and add v



Proof Summary

- Pick v of degree at most 5
- Color G-v
- If v’s neighbors don’t include all 5 colors, complete coloring
- Otherwise, pick neighbor and recolor, then recolor it’s neighbors, etc. trying to free up a color for v
- There’s always a way to do this

The Four Color Map Theorem

Theorem 1.46: Every planar graph is 4-colorable.

Notes:

- Optimal
- Proof along the same lines as above- add one vertex by recoloring some nearby ones
- Too many cases to check by hand. All known proofs are computer assisted.