Announcements

- Exam 1 grades released (median score 70)
- HW 4 Due Sunday
- HW 5 Online

Previously

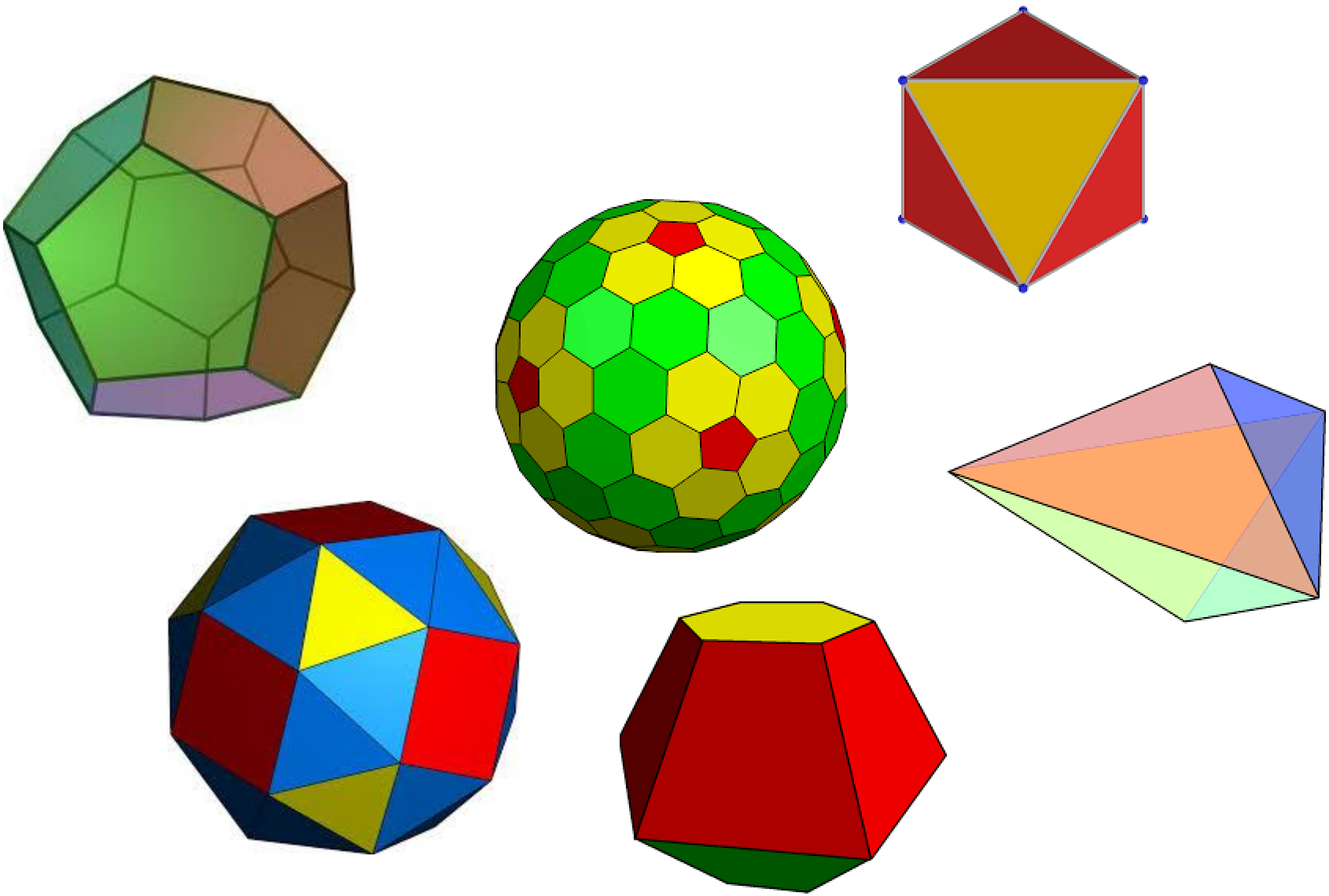
- Planar Graphs
- Euler's Formula:
  $v - e + f = 2$
- Polyhedra

Polyhedra

Definition: A *polyhedron* is a 3 dimensional figure bounded by finitely many flat *faces*. Two faces meet at an *edge* and edges meet at *vertices*.

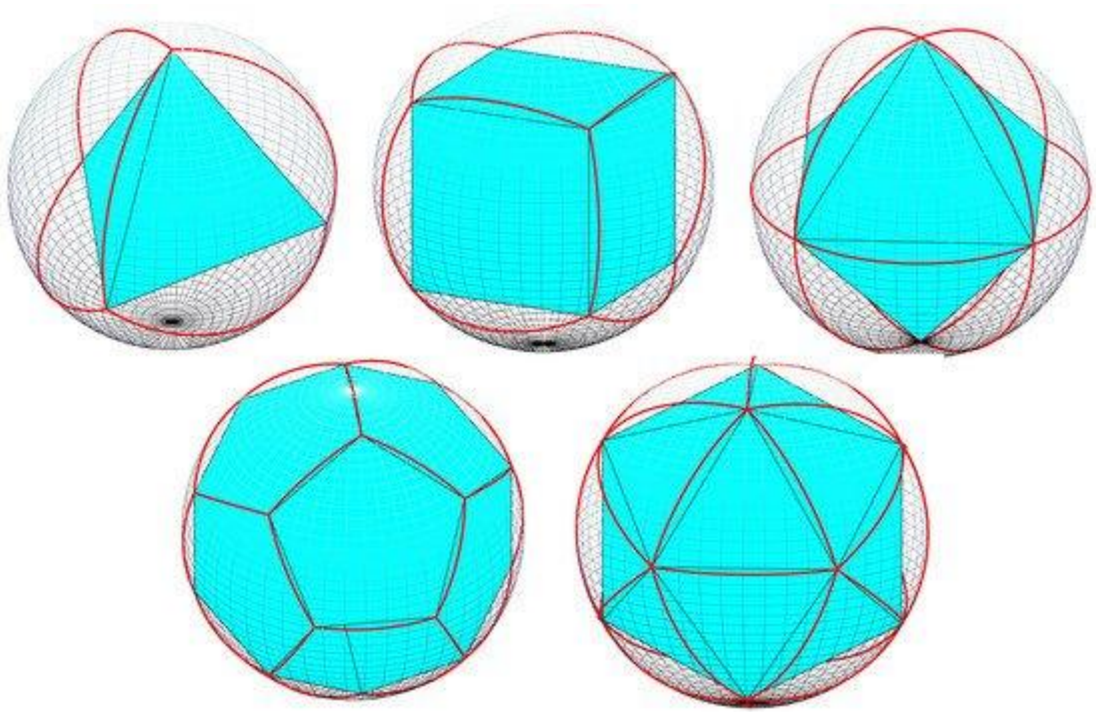
A polyhedron is *convex* if for any two points in the polyhedron the line segment connecting them is also contained in the polyhedron.

Examples



Polyhedral Graphs

Given a convex polyhedron, can turn it into a planar graph by projecting vertices/edges onto a sphere (which can then be flattened onto a plane).



Today

- Regular Polyhedra
- Classification of planar graphs

Euler’s Formula

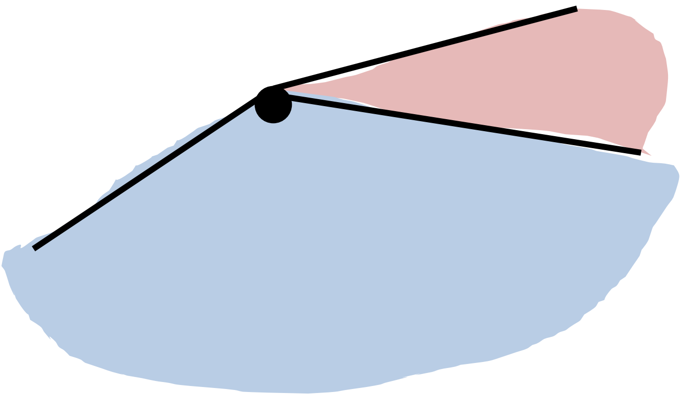
Euler’s Formula applies directly:

For any polyhedron:

$$\#Faces - \#Edges + \#Vertices = 2$$

Degrees

Note that in any polyhedron, each vertex has degree at least 3.



Handshake Lemma implies

$$2e = \Sigma\, d(v) \geq 3v$$

$$e \geq 3v/2$$

Edges and Vertices

Remember we also saw that if each face of a planar graph had at least k edges then

$$e \leq (v-2)/(1-2/k)$$

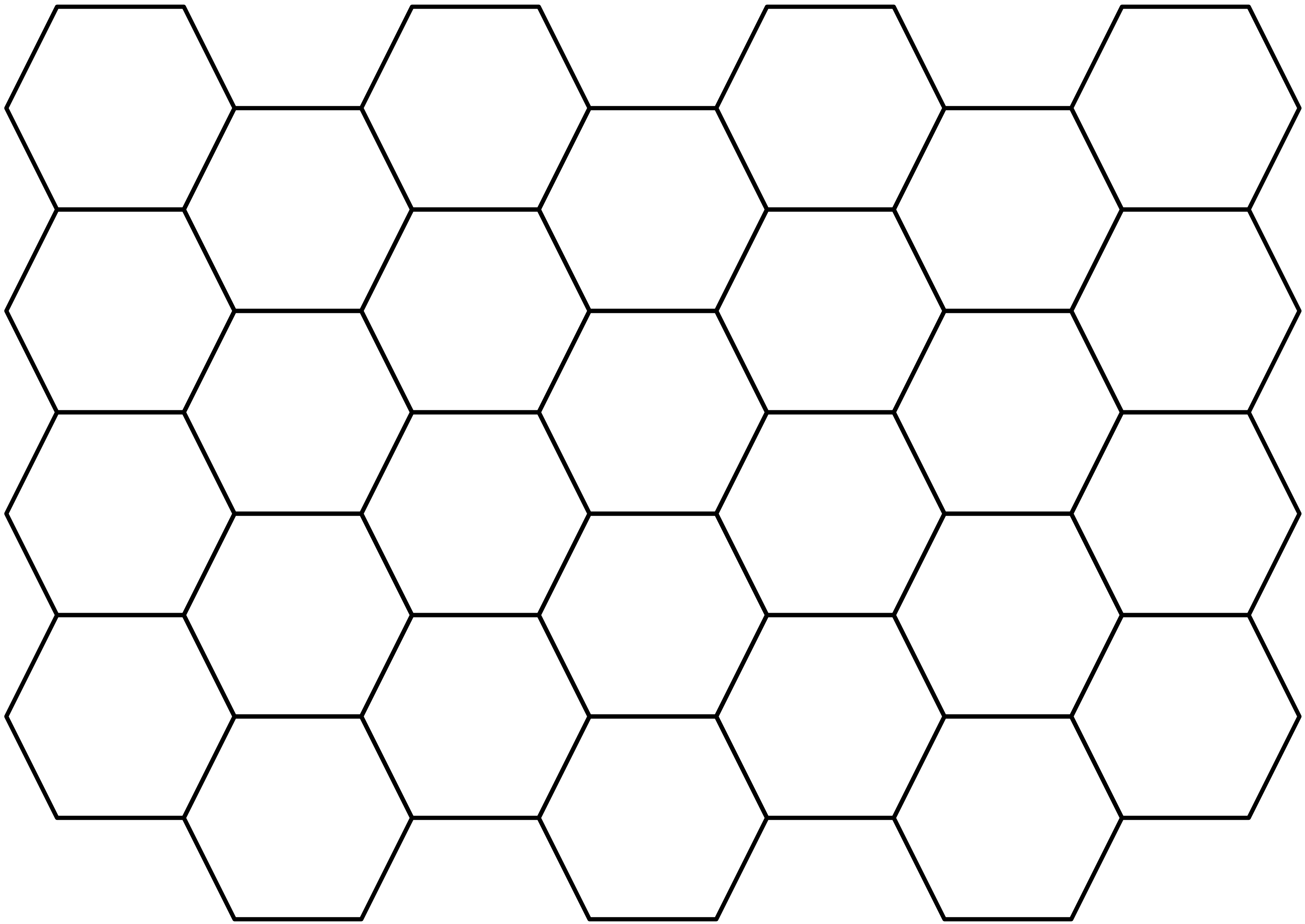
If k = 6, we have that

$$3v/2 \leq e \leq (v-2)/(2/3) = 3(v-2)/2.$$

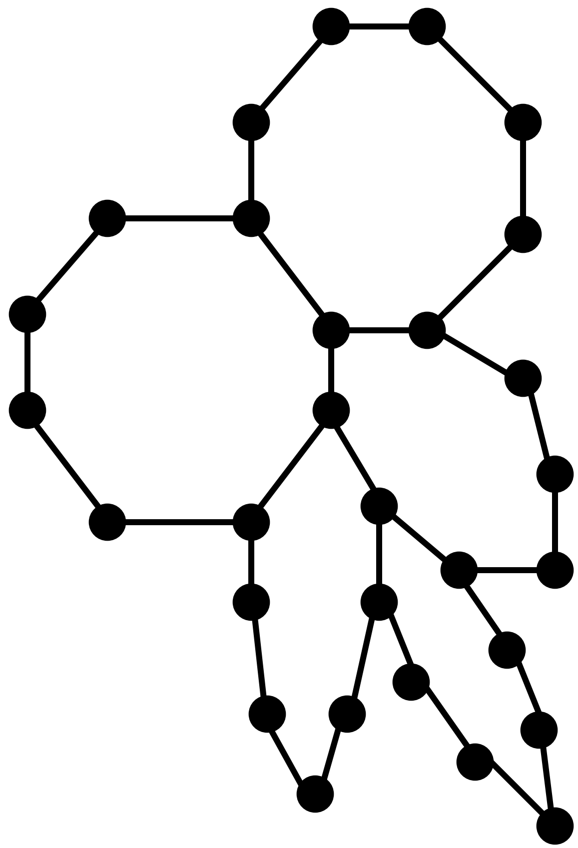
Contradiction!

Corollary: Every polyhedron has a face with at most 5 sides.

Discussion



Worse with More Sides



Regular Polyhedra

A regular polyhedron is a highly symmetric polyhedron (like a cube). In particular, it has the following properties:

- All edges are the same length.
- All faces are regular polygons with the same number – s of sides.
- The same number of faces, d, meet at each vertex.

Counting

Note that each vertex has degree-d. By the Handshake Lemma:

$$2e = \Sigma\, d(v) = dv.$$

By the dual handshake lemma:

$$2e = \Sigma\, s(f) = sf.$$

By Euler's Formula:

$$2 = v - e + f = e(2/d+2/s-1).$$

Counting Continued

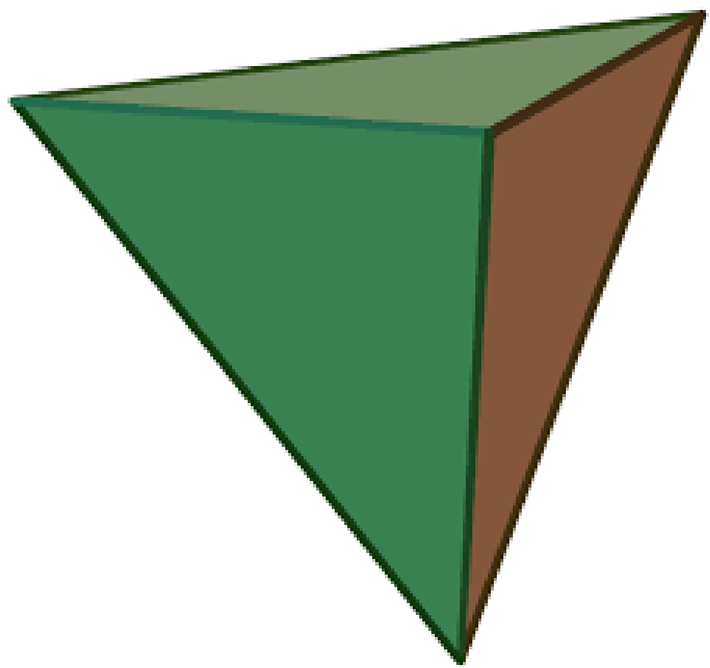
Therefore, we must have:

- $5 \geq d, s \geq 3$
- $2/d + 2/s > 1$
- $e = 2/(2/d+2/s-1)$
- $v = 2e/d$
- $f = 2e/s$

Case 1: d = s = 3

v = 4, e = 6, f = 4

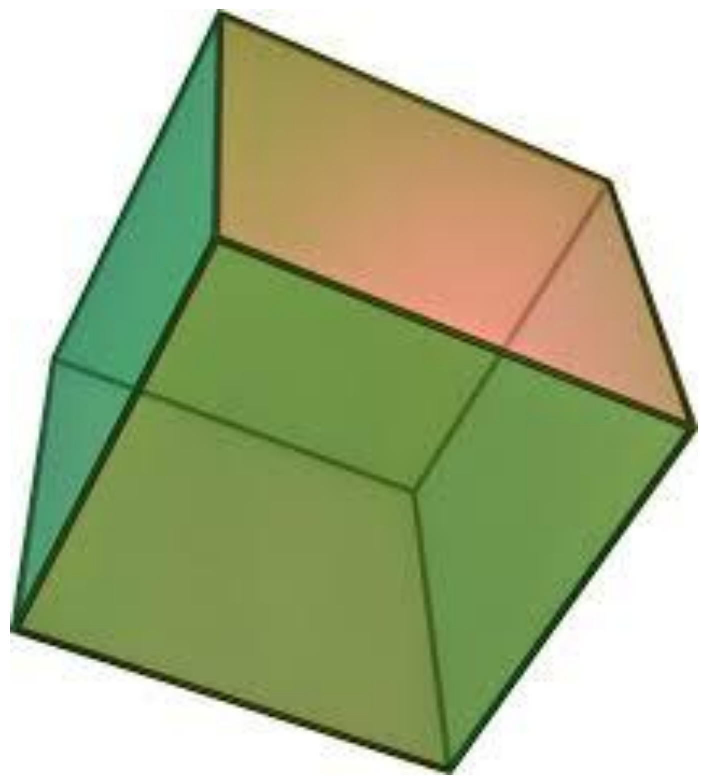
Regular tetrahedron



Case 2: $d = 3$, $s = 4$

$v = 8$, $e = 12$, $f = 6$

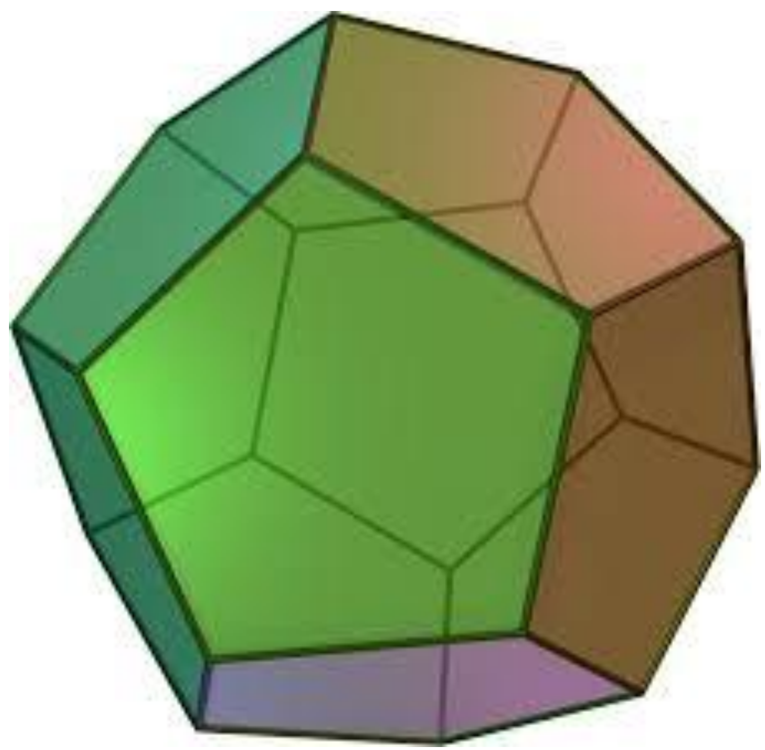
Cube



Case 3: d = 3, s = 5

v = 20, e = 30, f = 12

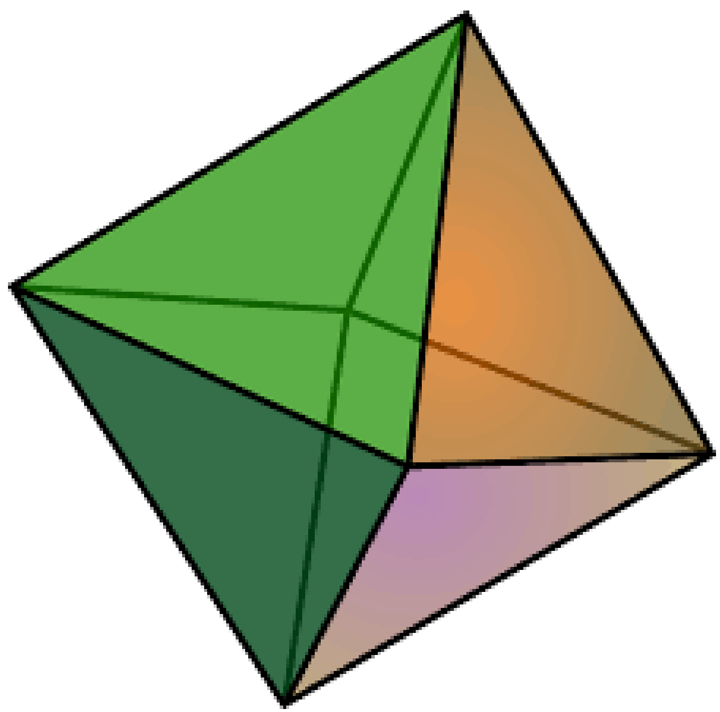
Regular dodecahedron



Case 4: d = 4, s = 3

v = 6, e = 12, f = 8

Regular Octahedron



$d = 4, s \geq 4$

Doesn't work!

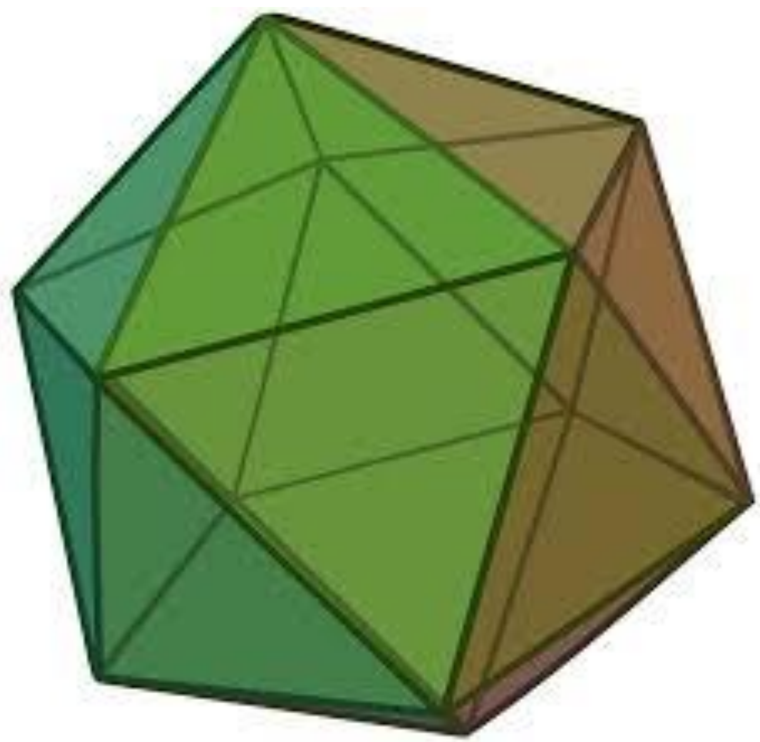
$2/d + 2/s \leq 2/4 + 2/4 = 1$.

4 squares at a vertex tessellate.

Case 5: $d = 5$, $s = 3$

$v = 12$, $e = 30$, $f = 20$

Regular icosahedron

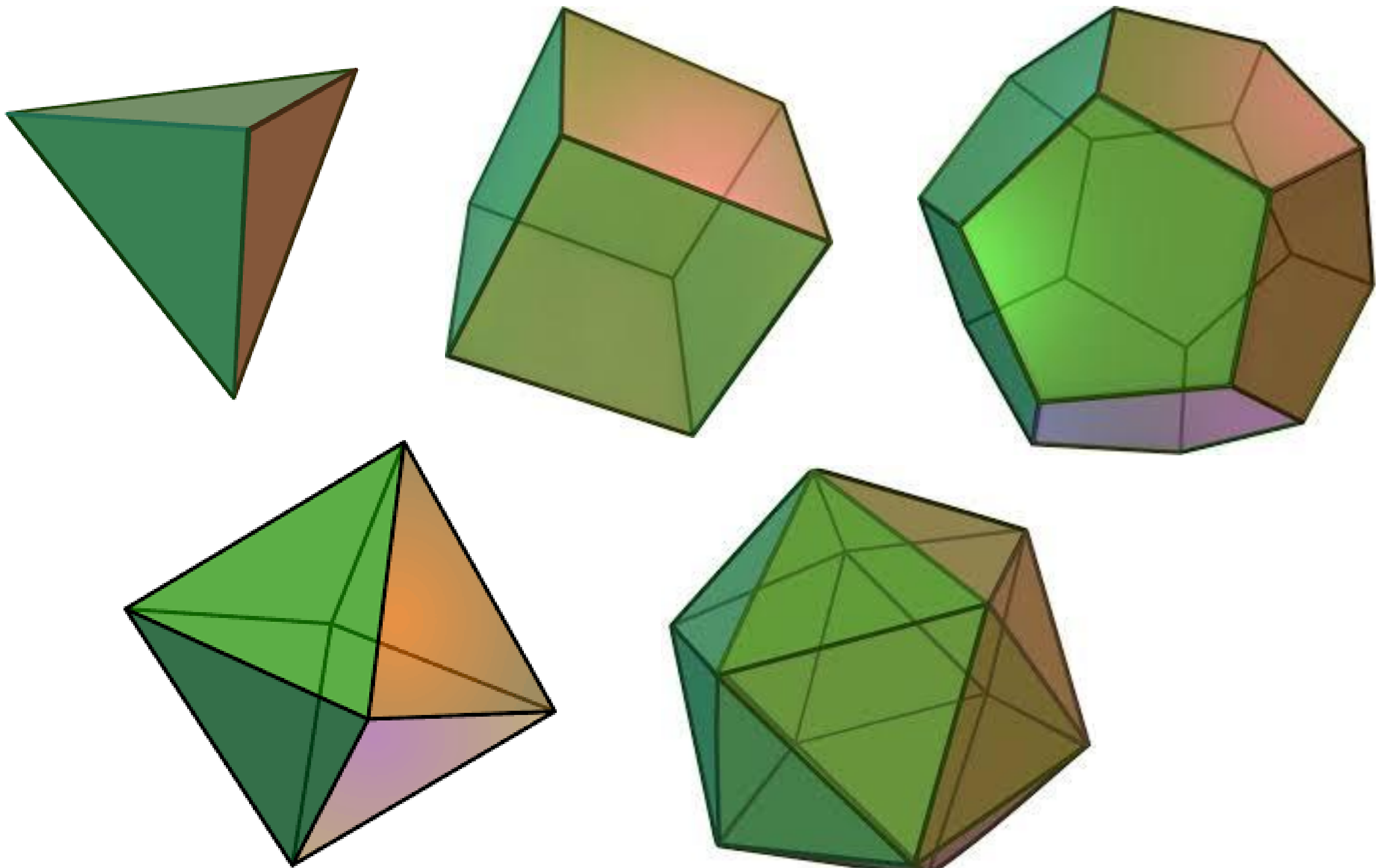


$d = 5, s \geq 4$

Doesn't work!

$2/d + 2/s \leq 2/5 + 2/4 = 9/10 < 1.$

Only 5 Platonic Solids



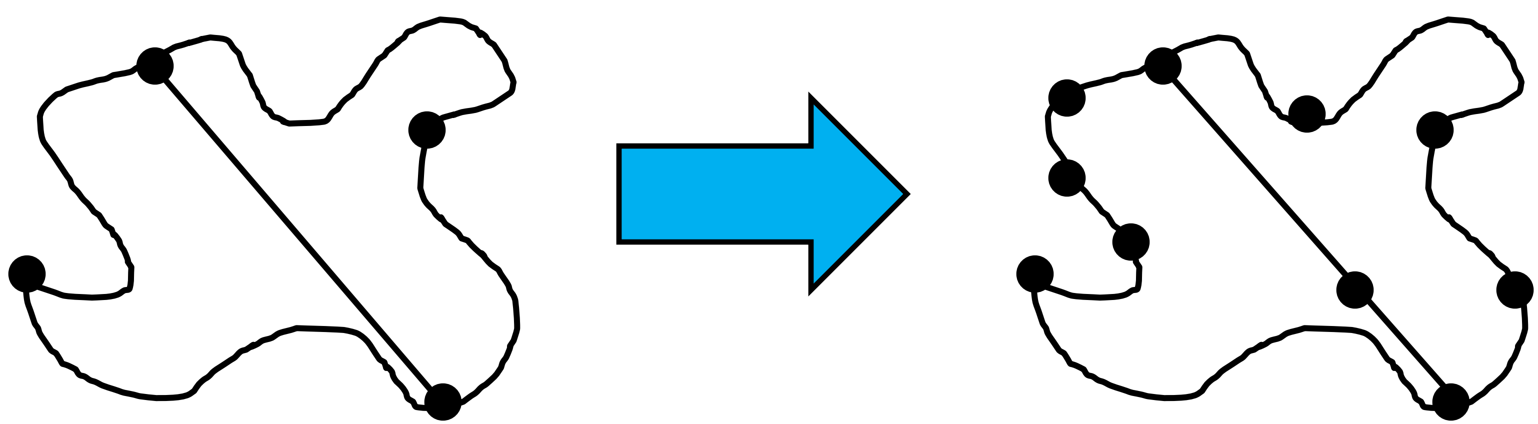
Which Graphs are Planar?

We know that K_5 and $K_{3,3}$ are *not* planar, and that polyhedral graphs and trees are, but can we come up with a complete characterization of which graphs are and are not planar?

Subdivisions

Here's one thing that doesn't much affect planarity:

Definition: A *subdivision* of a graph G is obtained by placing vertices in the middle of some of its edges.



Subdivisions II

<u>**Lemma:**</u> If G’ is a subdivision of G, then G’ is planar if and only if G is.

<u>**Proof:**</u>

- Given a plane embedding of G add vertices in the middle of edges to get embedding of G’.
- Given embedding of G’, remove vertices and join edges, to get embedding of G.

Non-Planar Graphs

Graphs that are non-planar:

- K_5 and $K_{3,3}$.
- Any subdivision of K_5 and $K_{3,3}$.
- Any graph that contains a subdivision of K_5 and $K_{3,3}$ as a subgraph.
- Nothing else!

Kuratowski's Theorem

<u>**Theorem (V. 7.0.1):**</u> A finite graph G is planar if and only if it has no subdivision of a K_5 or $K_{3,3}$ as a subgraph.

Coloring Problems (Ch 1.6)

- Introduction and Definitions
- Basic Results
- Brooke’s Theorem
- Colorings of Planar Graphs
- Edge Colorings

Bipartite Graphs

Recall, one way to think about bipartite graphs was to color the vertices black and white so that no edge connected two vertices of the same color.

What if we allow more colors?

Colorings

Definition: A (vertex) *coloring* of a graph G is an assignment of a color to each vertex of G so that no two adjacent vertices have the same color. This is an *n-coloring* if only n different colors are used.

Definition: The *Chromatic Number*, $\chi(G)$, of a graph G is the smallest number n so that G has an n-coloring.

Basic Facts

- A graph has $\chi(G) = 1$ if and only if G has no edges.
- A graph has $\chi(G) \leq 2$ if and only if G is bipartite.
- Determining $\chi(G)$ for more complicated graphs is difficult. For 2-colorings once you color a vertex, there is only one possible choice for its neighbors. For 3-colorings, you have 2.