Announcements

- Homework 4 Due on Sunday

Last Time

Basic Facts about Planar Graphs

- $e \leq 3v-6$
- Every planar graph has a vertex of degree at most 5
- K_5 and $K_{3,3}$ not planar
- Every planar graph can be triangulated (i.e. you can add edges until all faces are triangles)

Today

- Fary’s Theorem
- Polyhedra

Fary’s Theorem

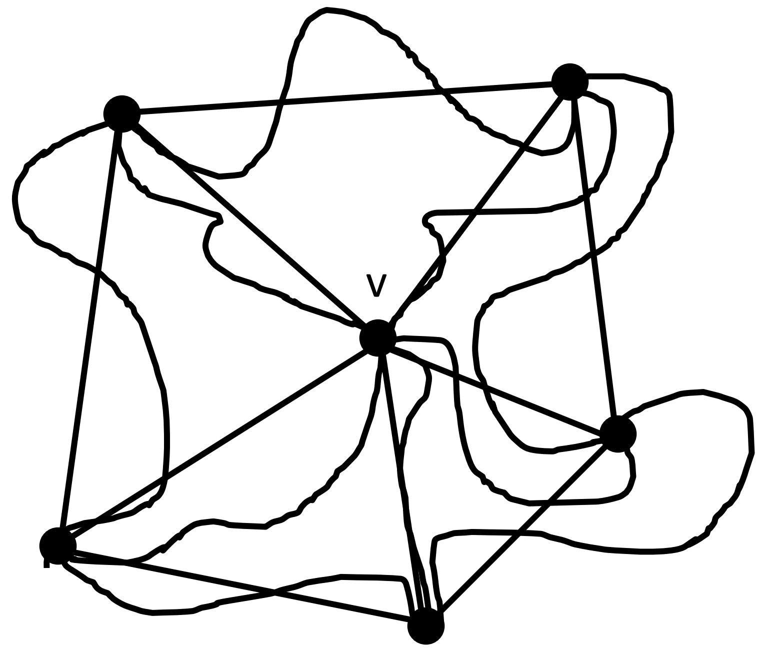
Theorem (V. 7.4.2): Any finite (simple) planar graph G has a plane embedding where all of the edges are straight line segments.

Proof Strategy

- Induct on v.
  - If v ≤ 3, easy to draw.
- Assume G is connected (ow/ draw each component separately)
- Find a vertex v of low degree.
- Draw G-v with straight lines.
- Re-insert v into drawing.

Idea

- G triangulated and $d(v) \leq 5$
- Draw G-v w/ lines.
- Find a place for v with straight lines to neighbors.



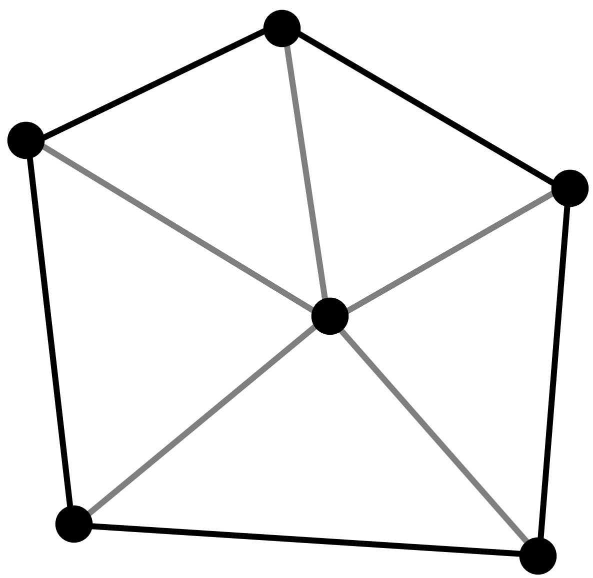
Lemma

Lemma: Given any polygon P in the plane with at most 5 sides, there is a point v inside of P with straight line paths to each of P's vertices.

Idea: Consider cases based on the locations of the non-convex (i.e. > 180 degree) angles.

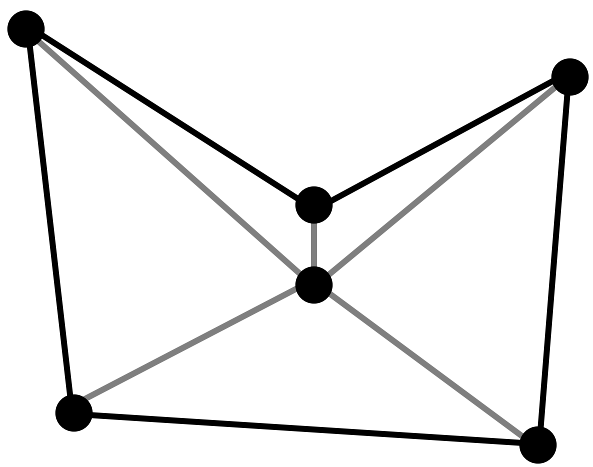
Case 1: No non-convex angles

Any point on the interior works.



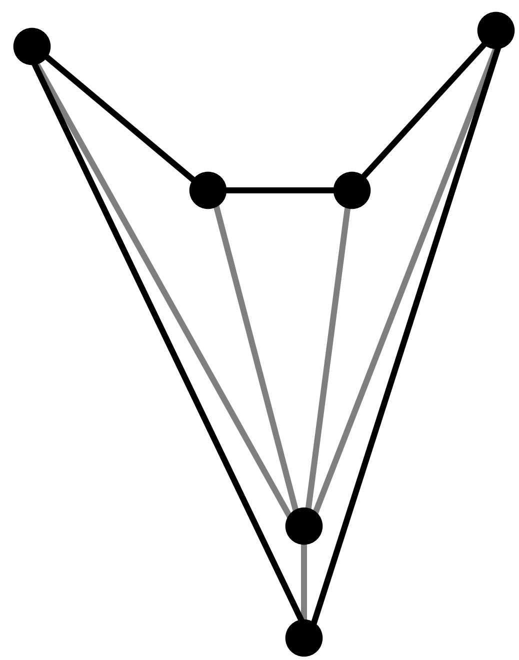
Case 2: One non-convex angle

Place v just inside the non-convex angle.



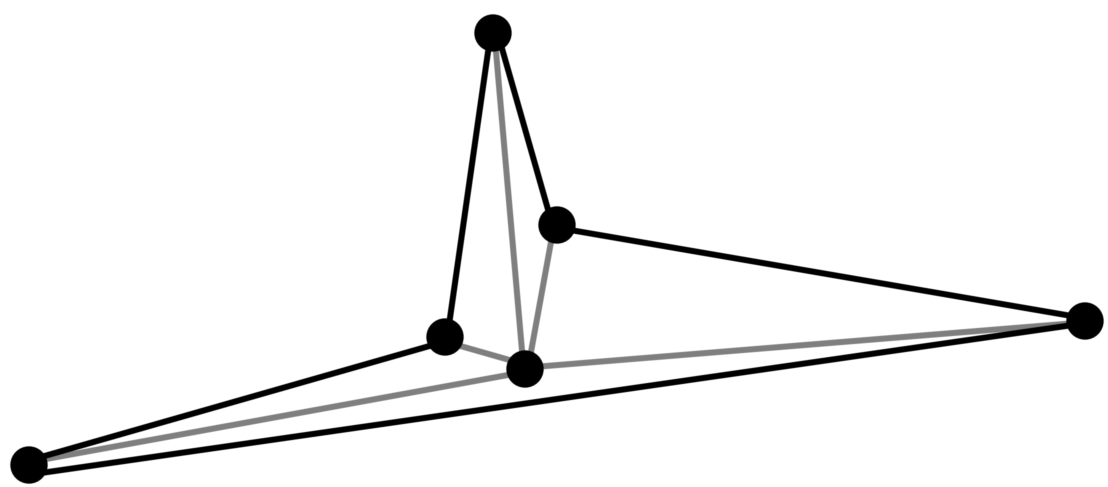
Case 3: Two adjacent non-convex angles

Place v near opposite vertex.



Case 4: Two non-adjacent non-convex angles

Place v near the far wall where it can see the hidden vertex.

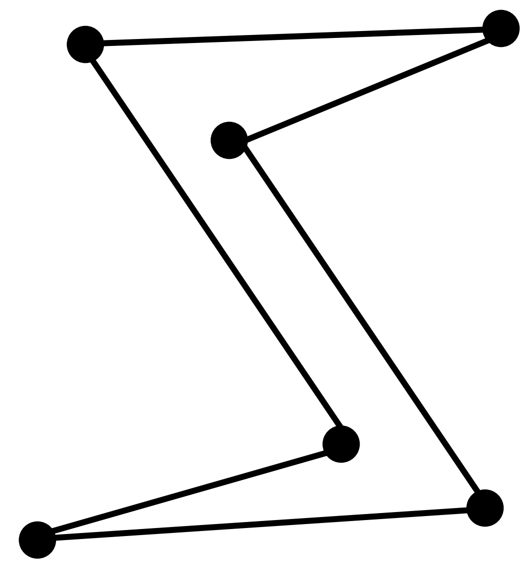


Case 5: More than two non-convex angles

Not possible! Sum of angles is 180(n-2) degrees for an n-gon.

Question: Hexagons?

Is this lemma true for hexagons?



A) Yes

B) No

Proof of Fary's Theorem

- G connected, planar graph.
- If $v \leq 3$ can draw.
- Triangulate G.

Claim: Any such G will have an *interior* vertex of degree at most 5.

Proof of Claim

Handshake Lemma:

$$\Sigma\, d(v) = 2e = 6v - 12$$

Rearrange:

$$\Sigma\, (6 - d(v)) = 12$$

The three boundary vertices each contribute at most 3. So *some* other vertex must have 6-d(v) positive.

Low Degree v

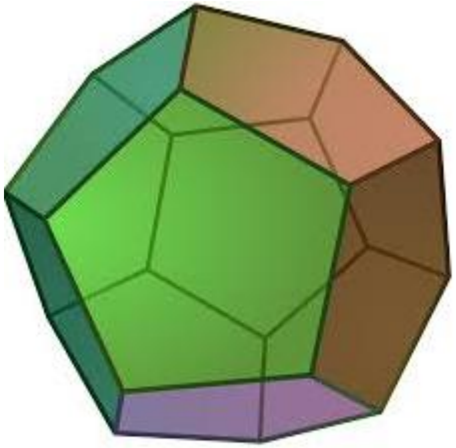
- By Inductive Hypothesis, can draw G-v with straight lines.
- Neighborhood of v a polygon with at most 5 sides.
- By Lemma, can insert v somewhere where it has straight line paths to its neighbors.
- Using this location we can draw G with straight line edges.

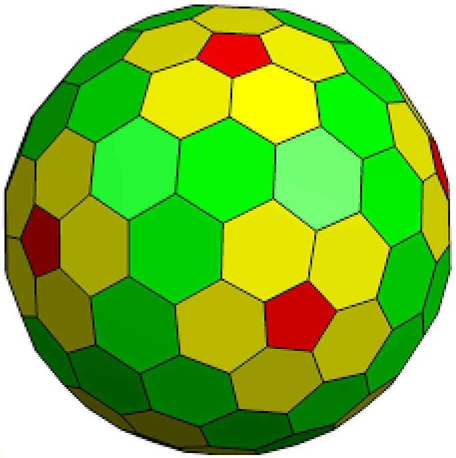
Polyhedra

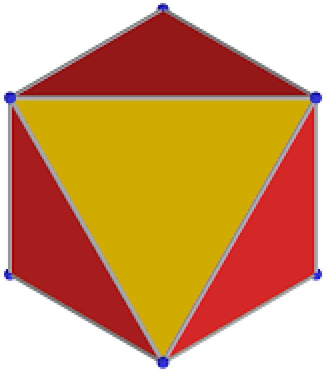
<u>Definition:</u> A *polyhedron* is a 3 dimensional figure bounded by finitely many flat *faces*. Two faces meet at an *edge* and edges meet at *vertices*.

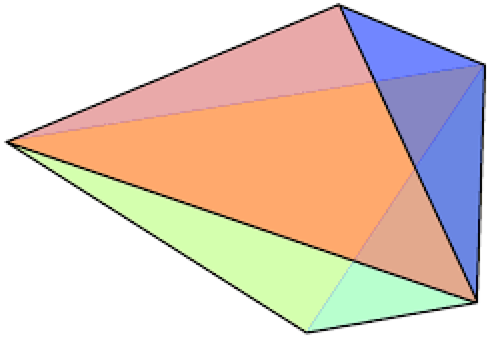
A polyhedron is *convex* if for any two points in the polyhedron the line segment connecting them is also contained in the polyhedron.

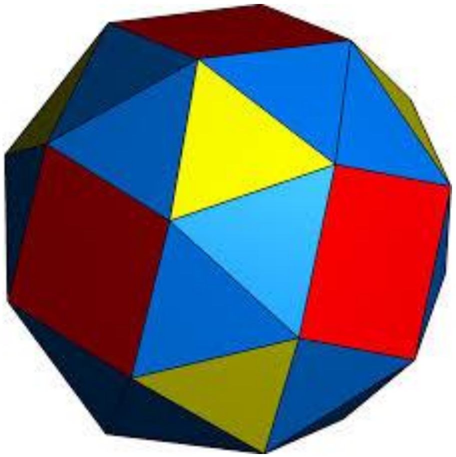
Examples

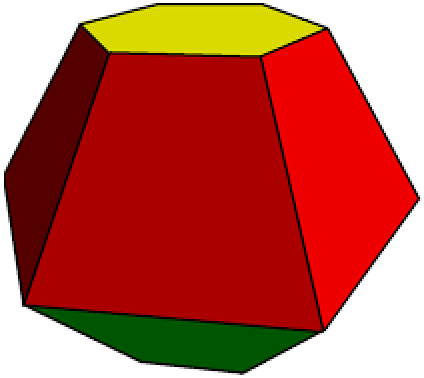












Polyhedral Graphs

Given a convex polyhedron, can turn it into a planar graph by projecting vertices/edges onto a sphere (which can then be flattened onto a plane).