Announcements

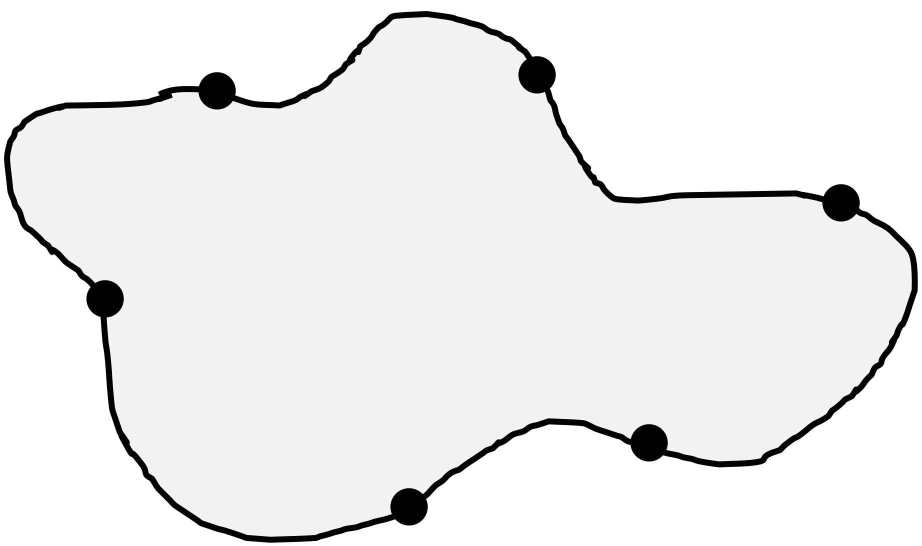
- Exam 1 Solutions online
- HW 4 Due on Sunday

Last Time

- Planar graphs
  - Can draw in the plane without crossing edges
- Faces
  - Regions bounded by edges
  - One infinite face
- Euler's Formula
  - For connected graphs, $v - e + f = 2$.

Sides to a Face

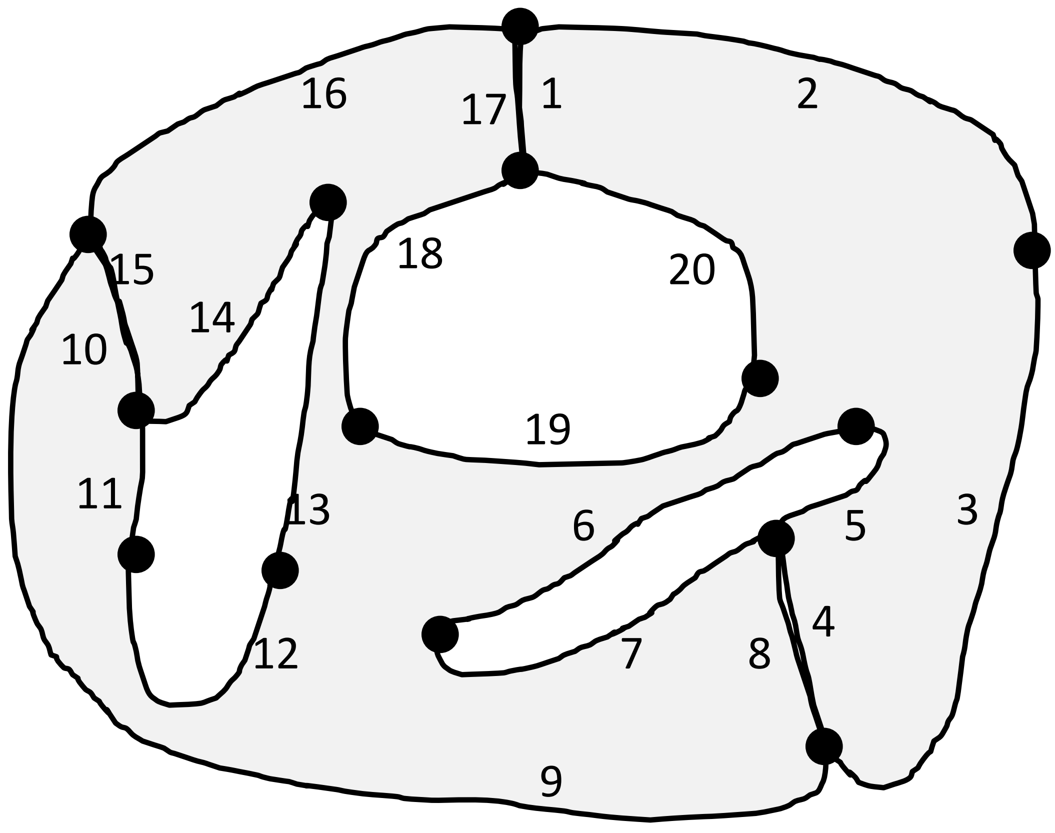
If G is a connected planar graph, any face (including the infinite one) will be bounded by a loop of edges.



The number of *sides* of the face is the number of edges in this loop.

Example

You can have weird examples like this:



Note that sides 1/17, 4/8, and 10/15 are really the same edge listed twice.

Today

- Dual Handshake Lemma
- Applications of Euler’s Formula
  - Edge Bounds
  - Non-planarity of K_5 and $K_{3,3}$
- Fary’s Theorem

Face Bounds

To really make use of Euler's Formula, it is important to get an idea of how many faces there are.

There is a way of counting these that is somewhat dual to the Handshake Lemma.

Dual Handshake Lemma

Lemma: For a connected, planar graph,

$$\sum_{\text{Faces } f} \text{Sides}(f) = 2|E|.$$

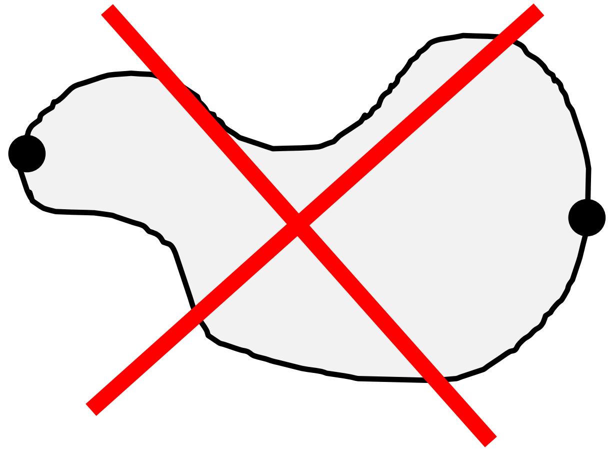
- Note similarity to Handshake Lemma. Sides of faces instead of degrees of vertices.
- Proof similar.

Proof

- Count the number of pairs of an edge as a side of a face (be careful to count edges that are double sides twice).
- Each edge on two faces (or one face twice).
- Each face f has Sides(f) edges.

A Key Observation

Every face has *at least three* sides.



(unless |V| =2)

$$2e = \sum_{\text{Faces } f} \text{Sides}(f) \geq 3f.$$

More Generally,

If G only has faces with at least k sides then

$$e \geq kf/2.$$

Edge Bound

Theorem (1.33): If G is a connected planar graph with $|V| \geq 3$, then

$$|E| \leq 3|V| - 6.$$

Proof

We know:

- $v - e + f = 2$.
- $e \geq 3f/2$.

So:

$$2 = v - e + f \leq v - e + 2e/3 = v - e/3$$

Rearranging, we find:

$$e \leq 3v-6.$$

Question: Side Bound

If each face has at least k sides, what is the maximum number of edges?

A) $v/(1-2/k)-6$
B) $k(v-2)$
C) $(v-2)/(1-2/k)$
D) $3v-6$

$$\begin{aligned} 2 &= v - e + f \\ &\le v - e + 2e/k \\ &= v - e(1-2/k) \end{aligned}$$

K_5 Non-Planar

Theorem (1.34): The K_5 is non-planar.

Proof: If it were, we would have

$$e \leq 3v-6 = 9$$

But e = 10. Contradiction!

$K_{3,3}$ Non-Planar

Theorem (1.32): $K_{3,3}$ is non-planar.

Proof: $K_{3,3}$ is bipartite, so it has no odd cycles. Therefore, if planar any face has at least 4 sides.

If planar,

$$e \leq 2v-4 = 8.$$

But e=9. Contradiction!

Minimum Degree

Theorem (1.35): If G is a finite, connected planar graph, its vertices have minimum degree at most 5.

Proof: Otherwise, each vertex has degree 6 or more.

Handshake Lemma implies

$$2e = \Sigma\, d(v) \geq 6v.$$

But then

$$3v - 6 \geq e \geq 3v.$$

Contradiction!

Triangulations

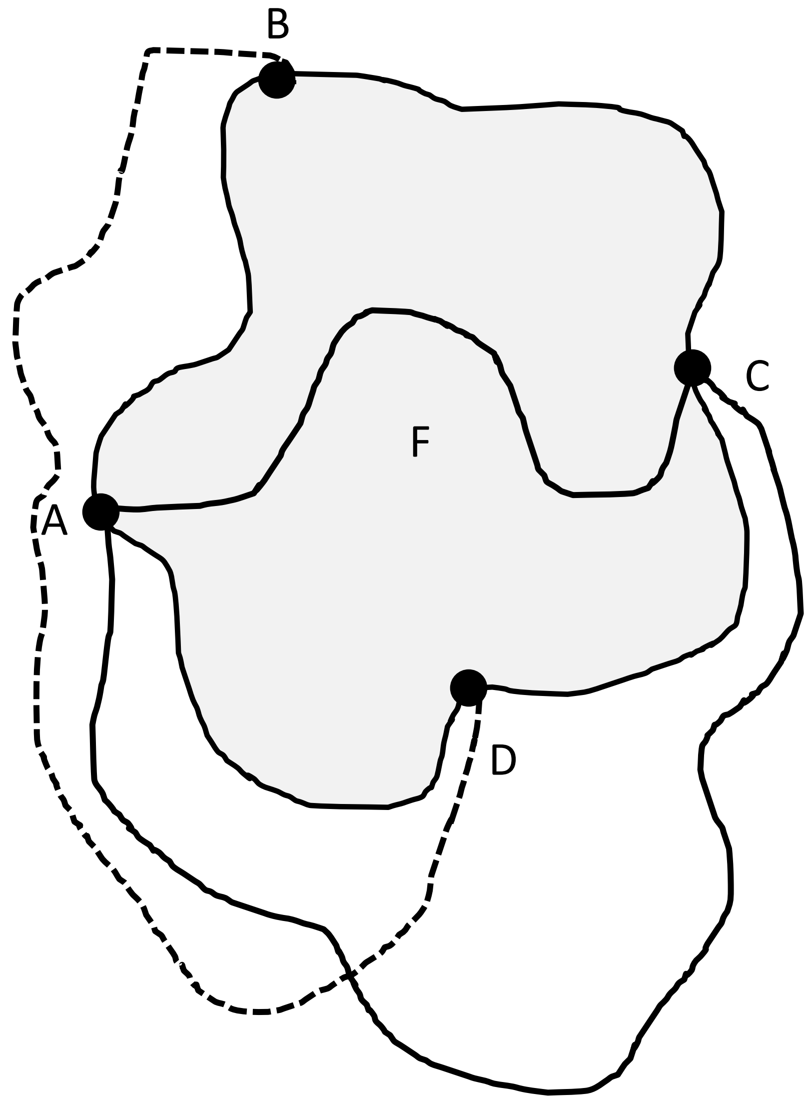
We note that our edge bound of 3v-6 has an equality case if and only if all faces are triangles.

We can always ensure that this is the case if we add more edges.

<u>Lemma:</u> For any planar embedding of a graph G there is a way to add more edges to G to get a new planar graph G’ in which all faces are triangles.

Proof

- Add edges until triangulated.
- Consider a face F with at least 4 sides.
- Add edge between non-adjacent vertices.
- Cannot if already outside edge.
- Cannot have on both ends.



Fary’s Theorem

Theorem (V. 7.4.2): Any finite (simple) planar graph G has a plane embedding where all of the edges are straight line segments.

Proof Strategy

- Induct on v.
  - If $v \leq 3$, easy to draw.
- Assume G is connected (ow/ draw each component separately)
- Find a vertex v of low degree.
- Draw G-v with straight lines.
- Re-insert v into drawing.