Announcements

- Exam 1 Friday
- No Homework this week
- Exam Instructions due before exam

Last Time

Menger's Theorem: If k is the minimum number of vertices you need to remove to separate u and v, then there are k vertex-disjoint paths from u to v.

What About Edge Cuts?

<u>Theorem 4.5.2:</u> The minimum number of edges you need to remove from G to separate u and v is the same as the maximum number of edge disjoint paths from u to v.

We'll prove this later when we talk about flows, under the much catchier name "The Maxflow-Mincut Theorem".

Today

Planar Graphs

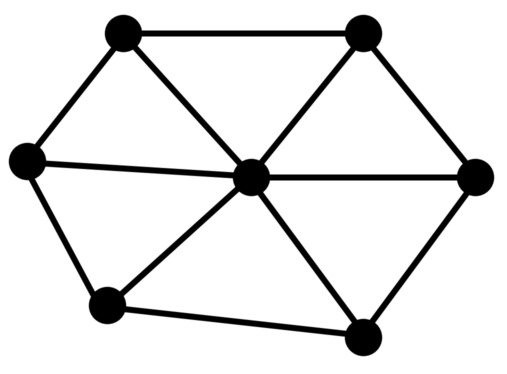
- Introduction
- Faces and Euler’s Formula

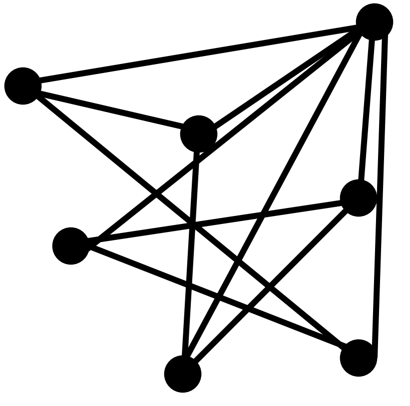
Ch 1.5: Planar Graphs

- Planarity Definition
- Faces and Euler’s Formula
- Platonic Solids
- Straight Line Embeddings
- Non-Planar Graphs

Graph Drawings

- When drawing a graph, it is convenient to draw it so that no two edges cross.
- When is this possible?





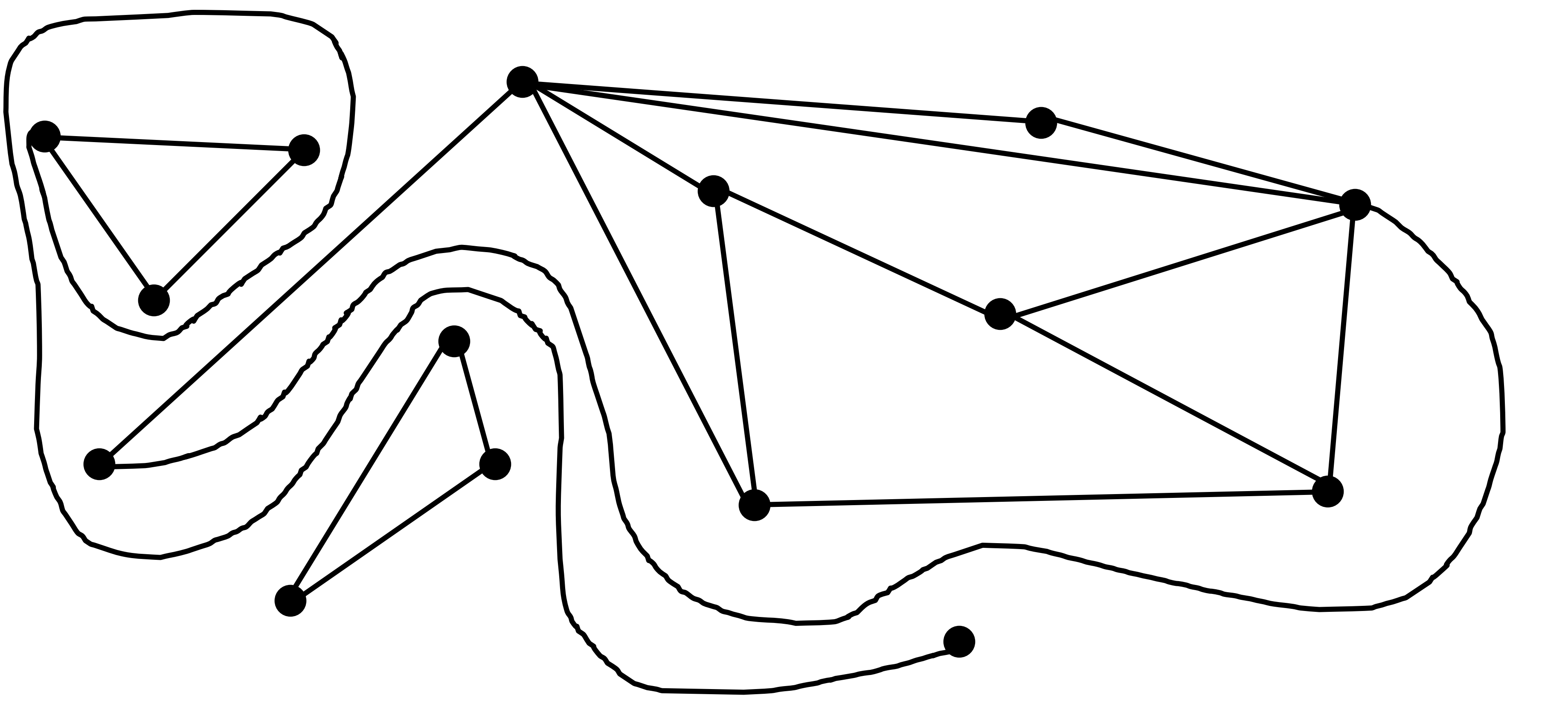
Planar Embeddings

Definition: A *planar embedding* of a graph G is a drawing of G so that

- Each vertex of G corresponds to a point in the plane.
- Each edge of G corresponds to a curve connecting its endpoints.
- No two edge-curves cross except at endpoints.

Planar Graphs

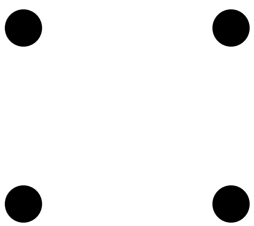
Definition: A graph is *planar* if it has a planar embedding.



Question: Planarity

Which of the graphs below are planar?

A



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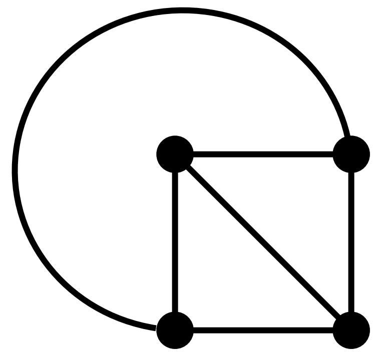

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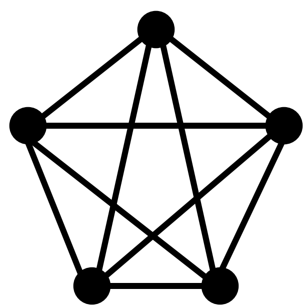

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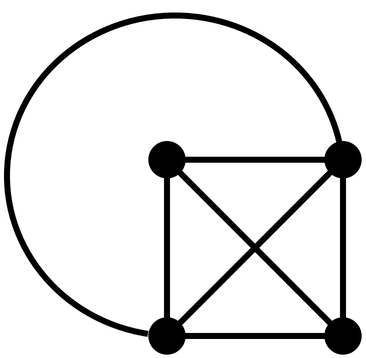
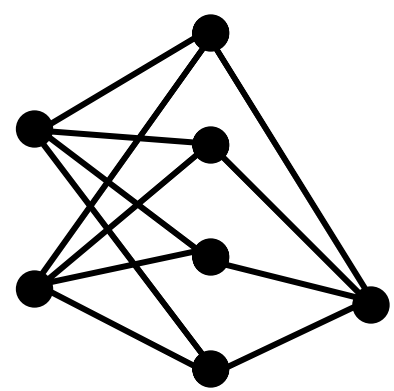


E

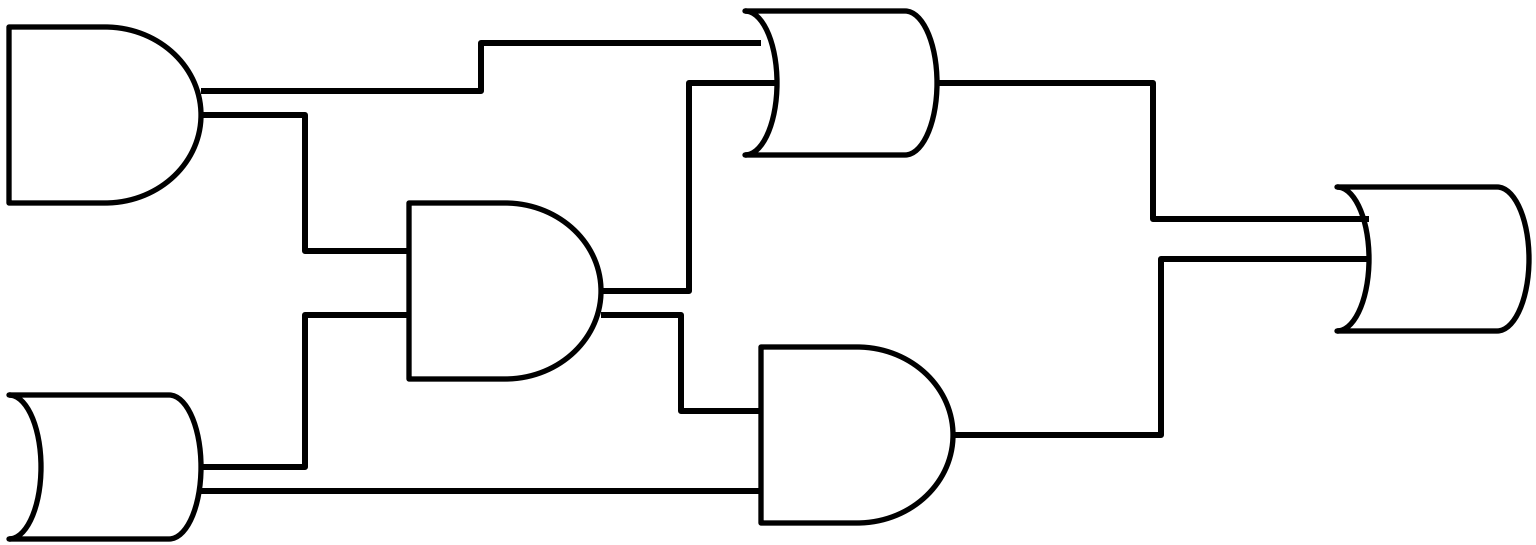


Notes

- Whether or not a graph is planar might depend on finding the right embedding.
- It is always possible to find a planar embedding of a planar graph with straight line edges (we'll show this later).
- Planarity game: http://planarity.net/

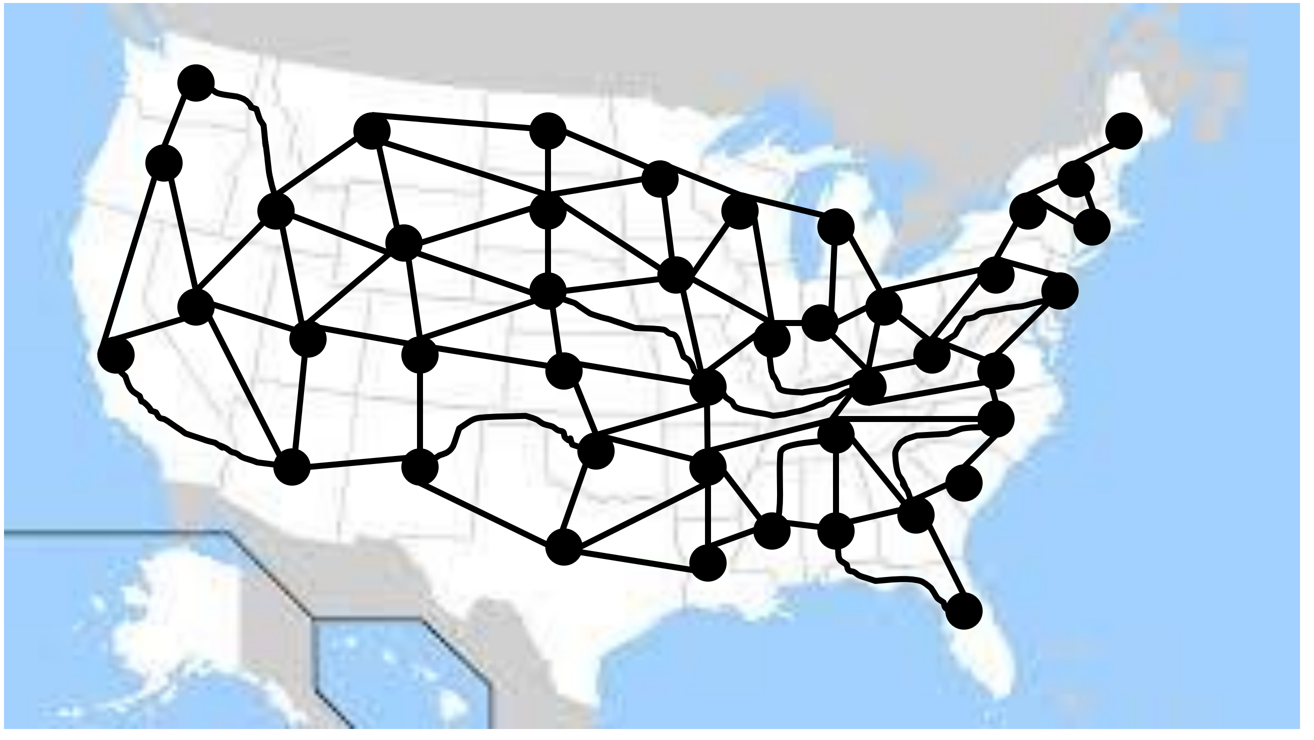
Application: Electrical Circuits

A circuit can be thought of as a graph with gates as vertices and wires as edges. With some technologies it is important to lay out circuits with few or no crossed wires.



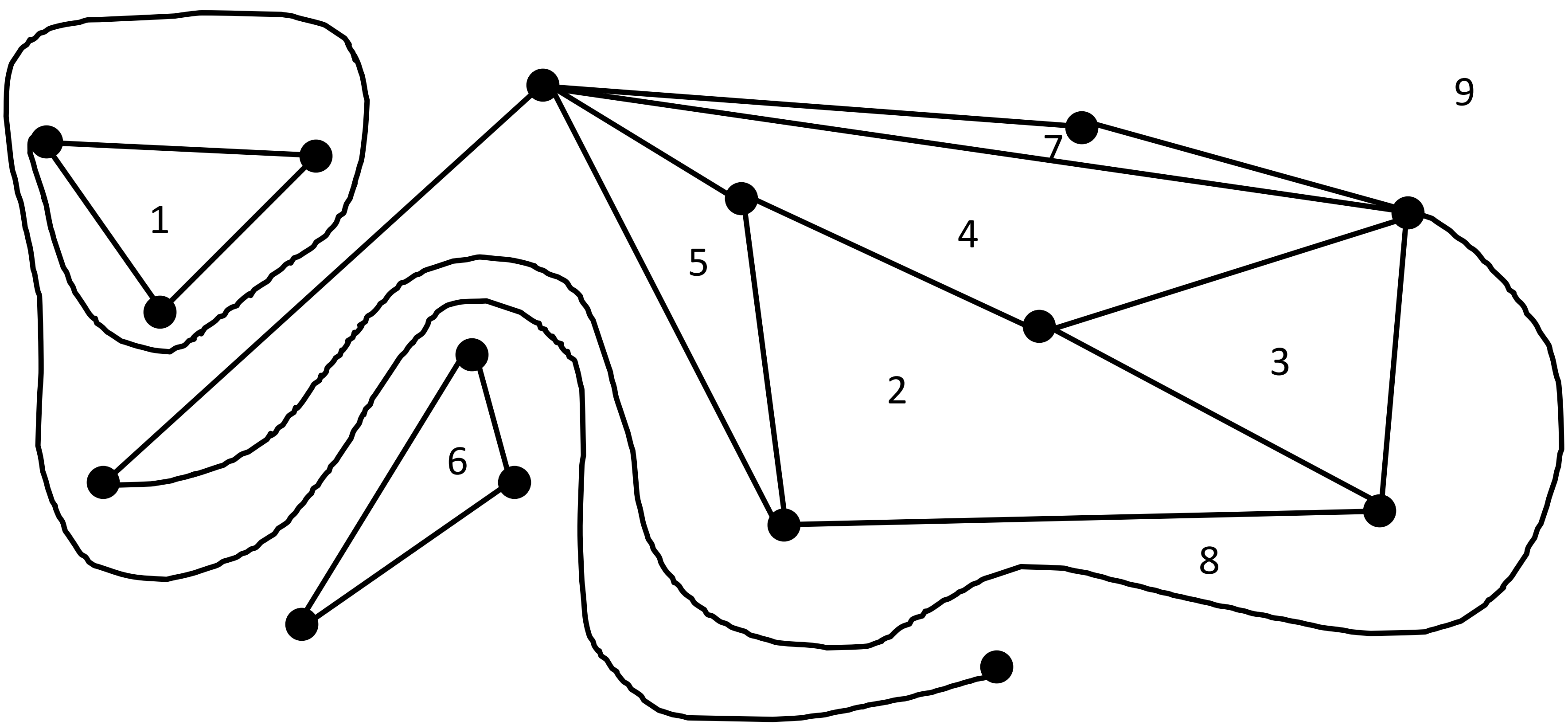
Application: Maps

Given a map with simply connected regions, the adjacency graph on regions is planar.



Faces

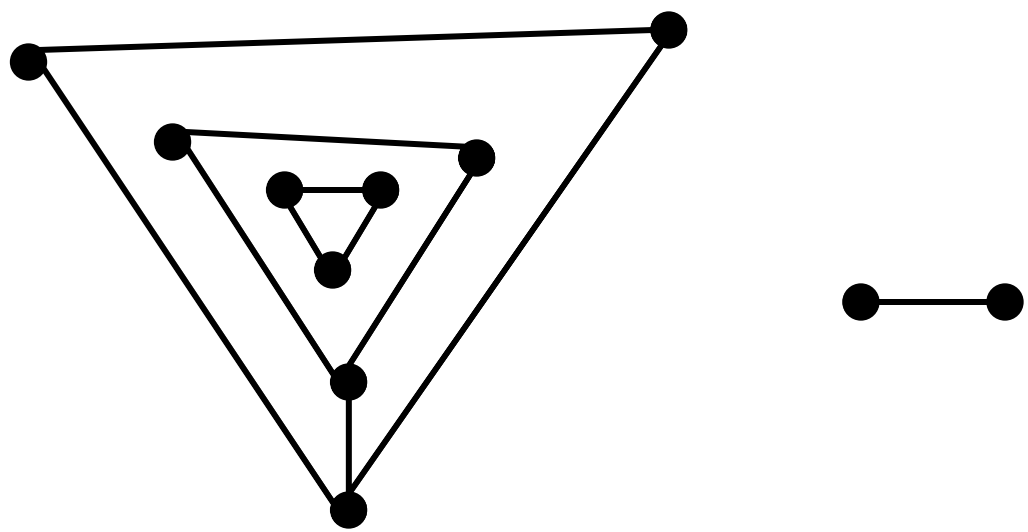
A planar embedding of a graph divides the plane into regions. These are called *faces*.



Question: Faces

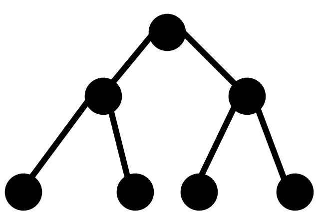
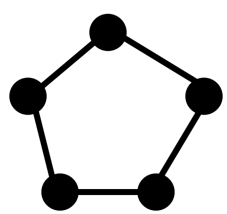
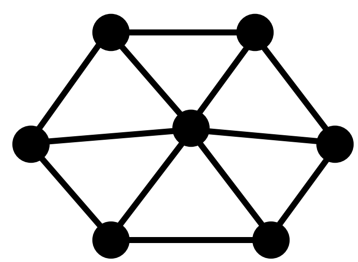
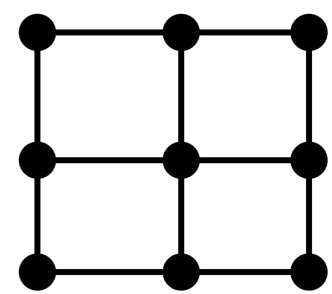
How many faces does the graph below have?

A) 1
B) 2
C) 3
D) 4
E) 5



Examples

- Tree: $v = n$ $e = n-1$ $f = 1$
- Cycle: $v = n$ $e = n$ $f = 2$
- Wheel: $v = n+1$ $e = 2n$ $f = n+1$
- Grid: $v = n^2$ $e = 2n(n-1)$ $f = (n-1)^2+1$

$v – e + f$ is always 2!

Euler's Formula

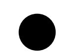
Theorem (1.31): For any planar embedding of a connected graph G with v vertices, e edges and f faces (including the infinite face)

$$v - e + f = 2$$

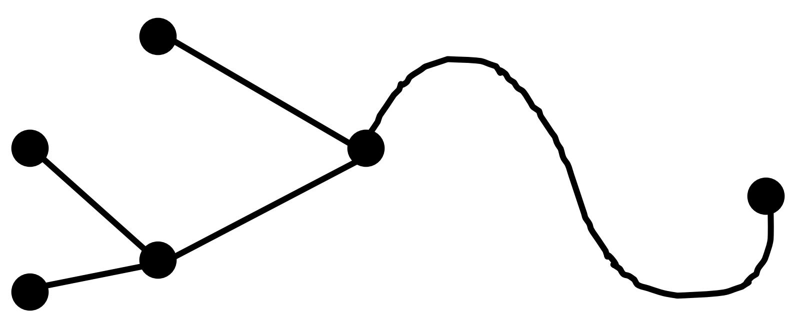
Trees

We begin by proving our result for trees. There e=v-1, so we need only show f=1.

- Use induction on v.
- If v=1, clearly true.

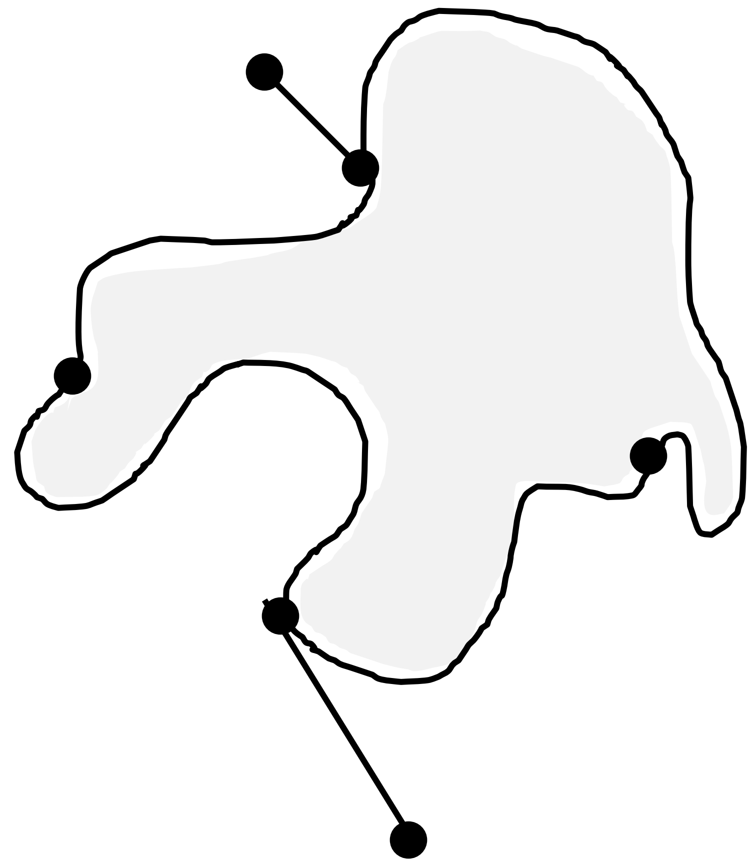


- For v>1, contracting a leaf into the tree doesn’t change number of faces.



General Graphs

- Use induction on e.
- Base case: G is a tree.
- Otherwise, G has a cycle
- Cycle separates plane into inside and outside.
- Remove an edge of cycle, decreases f by 1.
- IH => $v - (e-1) + (f-1) = 2$



Question: Euler's Formula

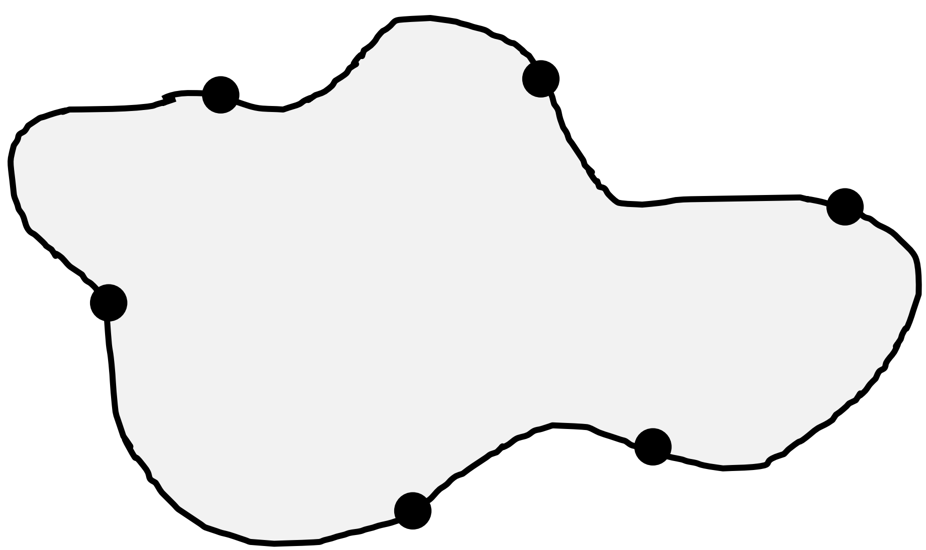
How many faces does a connected, planar graph with 12 vertices and 30 edges have?

A) 12
B) 15
C) 20
D) 25
E) 30

$$12-30+20 = 2$$

Sides to a Face

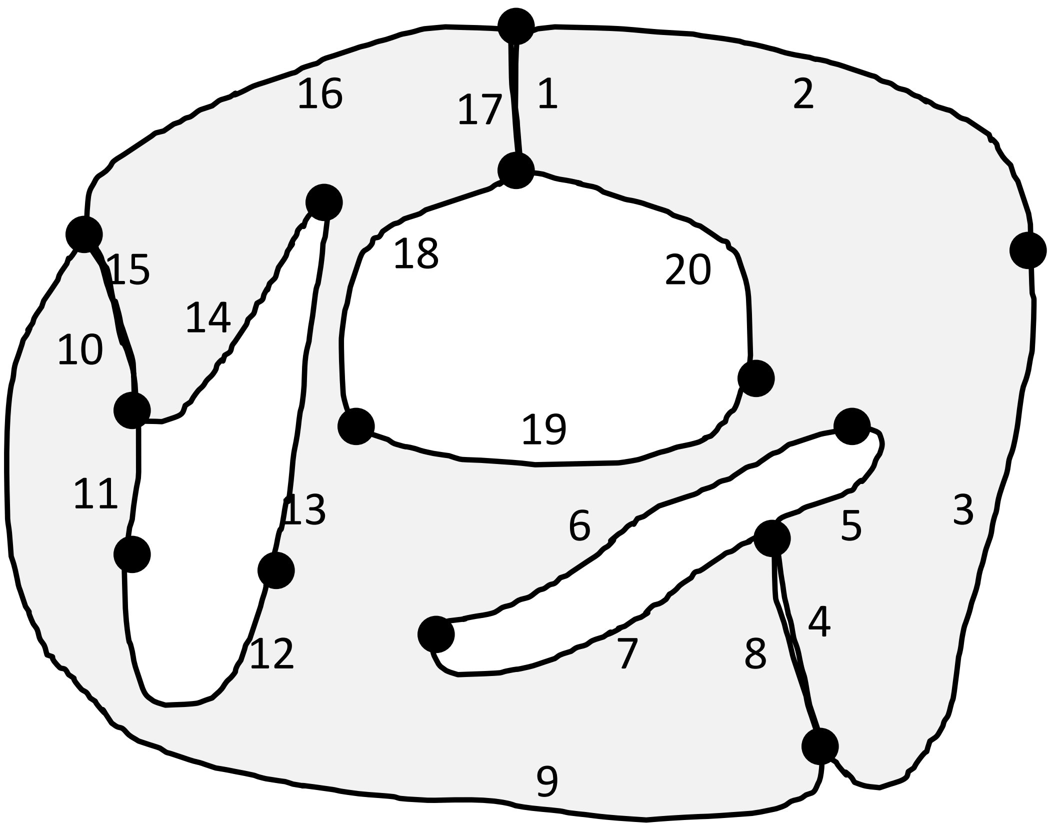
If G is a connected planar graph, any face (including the infinite one) will be bounded by a loop of edges.



The number of *sides* of the face is the number of edges in this loop.

Example

You can have weird examples like this:



Note that sides 1/17, 4/8, and 10/15 are really the same edge listed twice.