Announcements

- No HW Due this week
- Exam 1 on Friday
- Please complete exam 1 instructions assignment on gradescope before the exam.
- Exam Review Video on Course Webpage.

Last Time

- Structure of Blocks

Today

- Menger's Theorem
- Introduction to Planar Graphs (Ch 1.5)

More Vertices to Cut

We've studied when a graph can be disconnected by removing a single vertex. What about k vertices?

<u>**Definition:**</u> Given a connected graph G and vertices u and v, $\kappa(u,v)$ is the minimum number of vertices that one needs to remove from G to disconnect u and v.

Question: κ Example

What is κ(u,v) in the graph below?

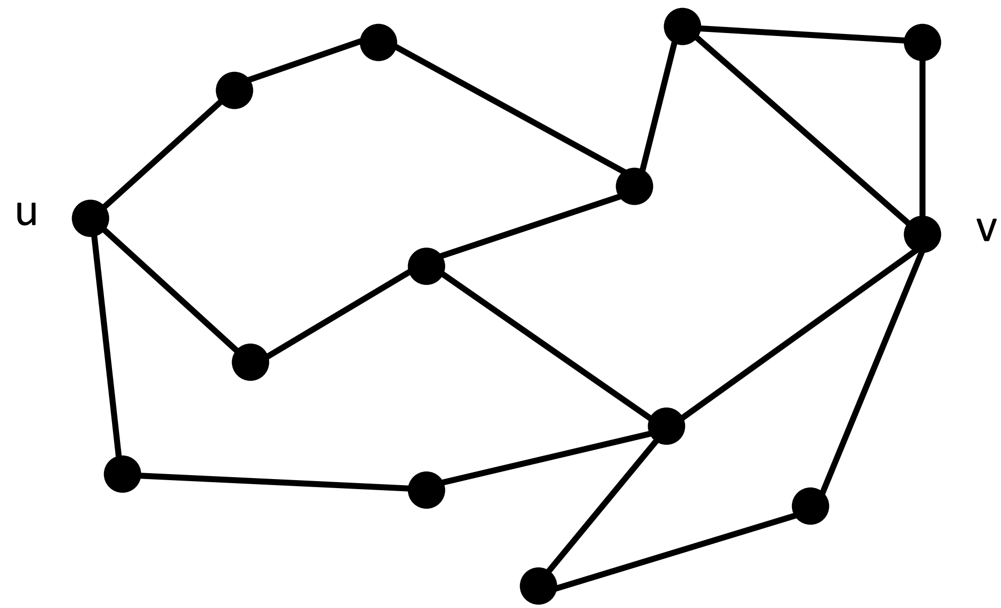
A) 1

B) 2

C) 3

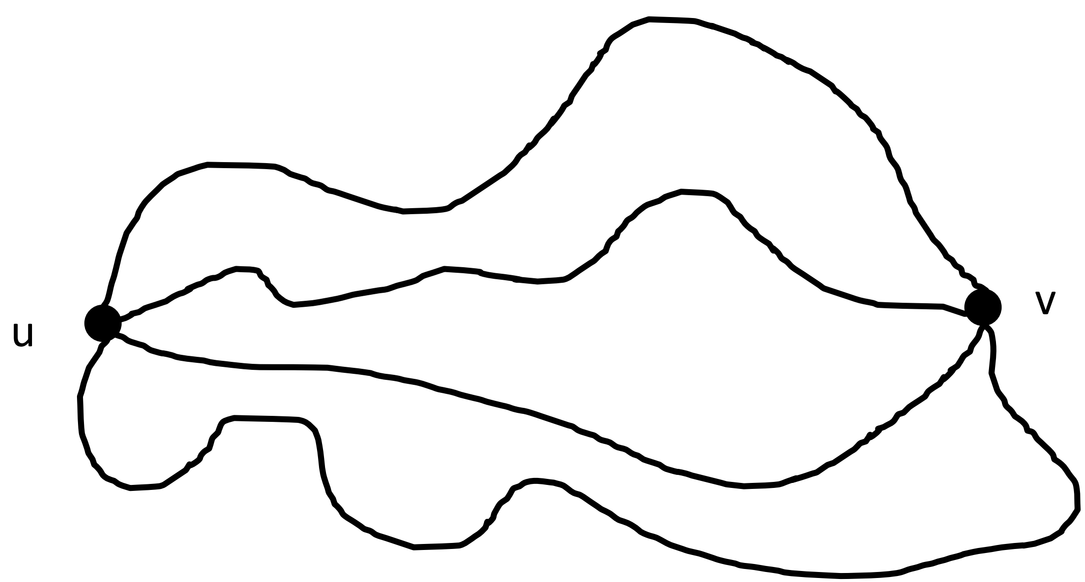
D) 4

E) 5



How Small Can κ(u,v) be?

- If k vertex-disjoint paths from u to v, κ(u,v) is at least k.
- Is this tight?



Menger's Theorem

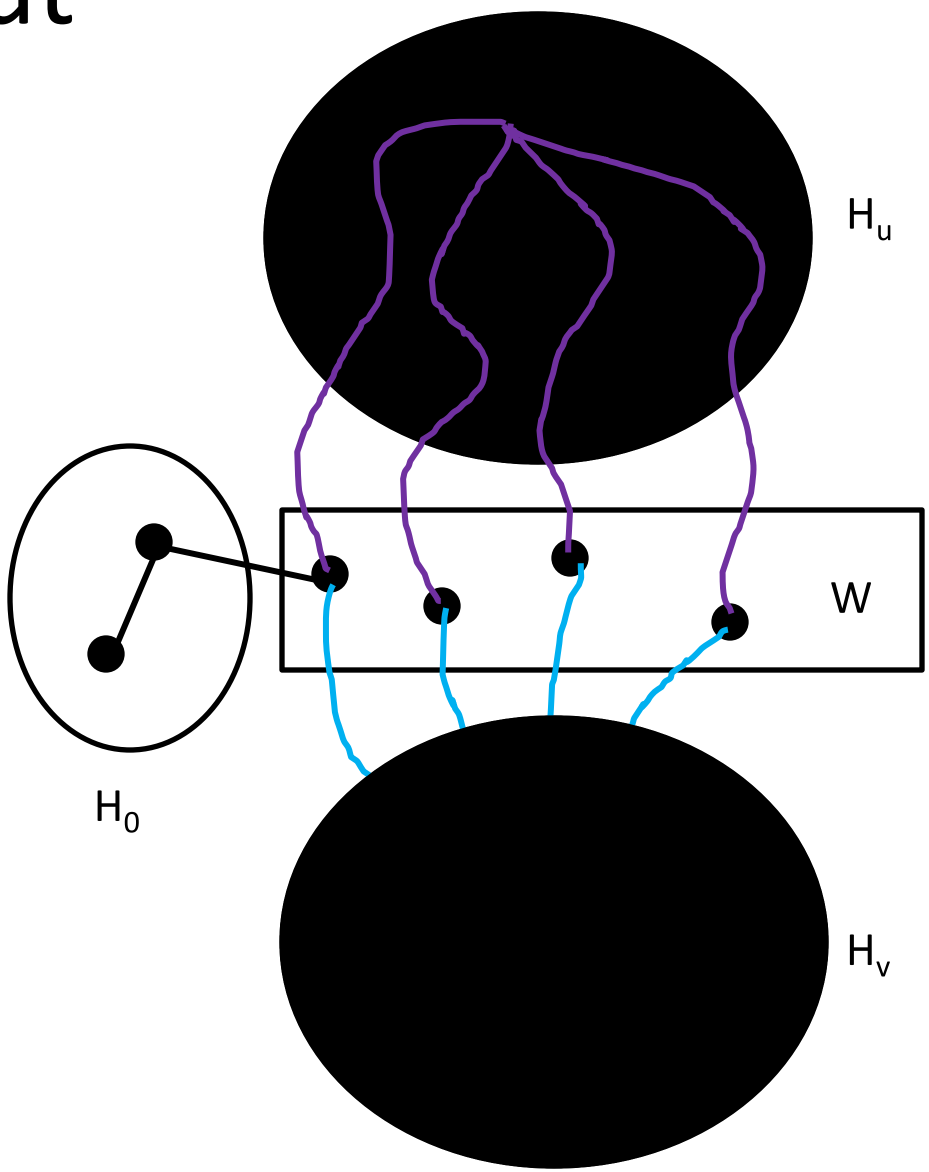
<u>Theorem 4.5.1:</u> There exist $\kappa(u,v)$ vertex-disjoint paths between u and v for any pair of vertices in any graph G.

Proof Strategy

- Use induction on the number of edges (assume true for all graph with fewer edges)
- Let W be a cut set of size $k = \kappa(u,v)$.
- Find k vertex-disjoint paths.

Cut

- Removing W splits graph into H_u,H_v,H_0.
- Create G_u replacing H_u by single vertex.
  - κ still = k.
  - IH => k vertex-disjoint paths.
- Do same for G_v
- Combine Paths



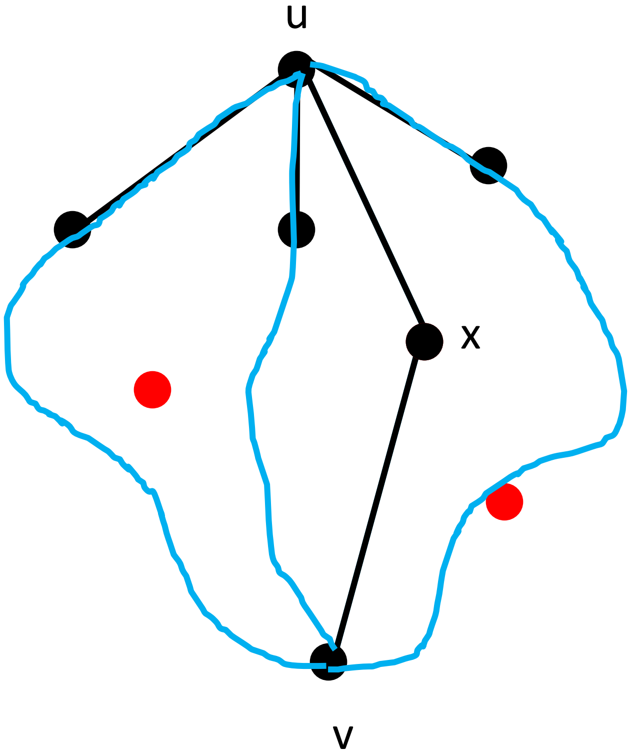
Doesn't Quite Work!

- Inductive hypothesis requires that G_u and G_v *smaller* than G.
- Need H_u and H_v to have more than one vertex.
- True unless $W \supset N(u)$ or $W \supset N(v)$.
  - Note that N(u), N(v) are cutsets. Only happens if W = N(u) or N(v).
- Done unless W = N(u) or N(v).
  - wlog W = N(u).

Case 1: Common Neighbor

N(u) ∩ N(v) non-empty.

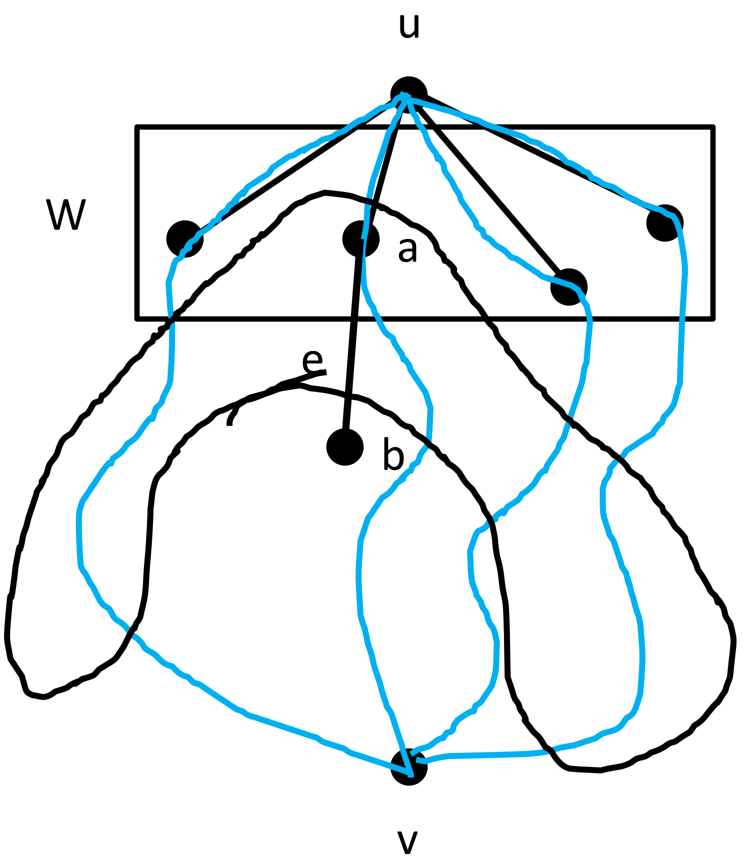
- Assume x in intersection.
- IH on G-x
  - cannot be (k-2)-cut
  - (k-1) disjoint paths
- Add u-x-v and done.



Case 2: No Common Neighbor

$N(u) \cap N(v) = \emptyset$

- Let e = (a,b) be an edge out of W.
- IH on G-e
  - If κ = k, find k paths
  - If κ = k-1, W'+a, W'+b another cut set. One won't consist of just N(u) or N(v)
  - Apply result here



Summary

- If N(u) and N(v) overlap, recurse.
- If W is neither N(u) nor N(v), induct on each side and glue together.
- If W = N(u), push off N(u) to find new W that isn't.

Question: Algorithm

Does this proof of Menger's Theorem naturally lead to an algorithm for *finding* these k disjoint paths?

A) Yes
B) No

Algorithm would require finding this set W.