Announcements

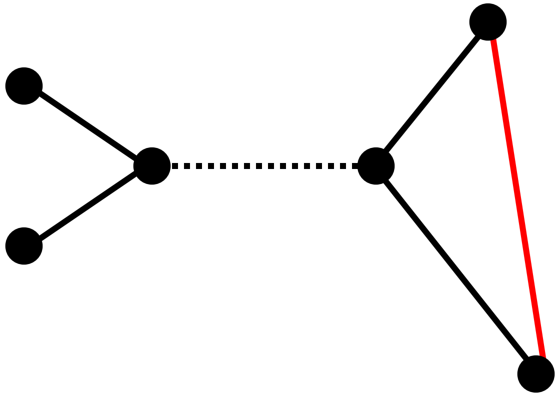
- Homework 3 Due Sunday
- Exam 1 Instructions
  - 4Qs in 1hour
  - During class time (problems released at 3, due at 4)
  - Material through the end of this chapter
  - No cooperation
  - Instructions on webpage & gradescope assignment

HW2 Q1

- Most people got that given T you could get a bipartition of T.
- However, you need to show that this is *also* the bipartition for G.

HW2 Q3

- Common mistake: If you have a tree T and remove an edge and add a new one, it does *not* always create a new tree.



Last Time

Block: Maximal Subgraph of G with no cut vertex.

Block Graph: A tree describing how G can be decomposed into blocks.

Today

Structure of Blocks:

- Ear Decomposition
- Theta-less graphs
- Vertex Cycles

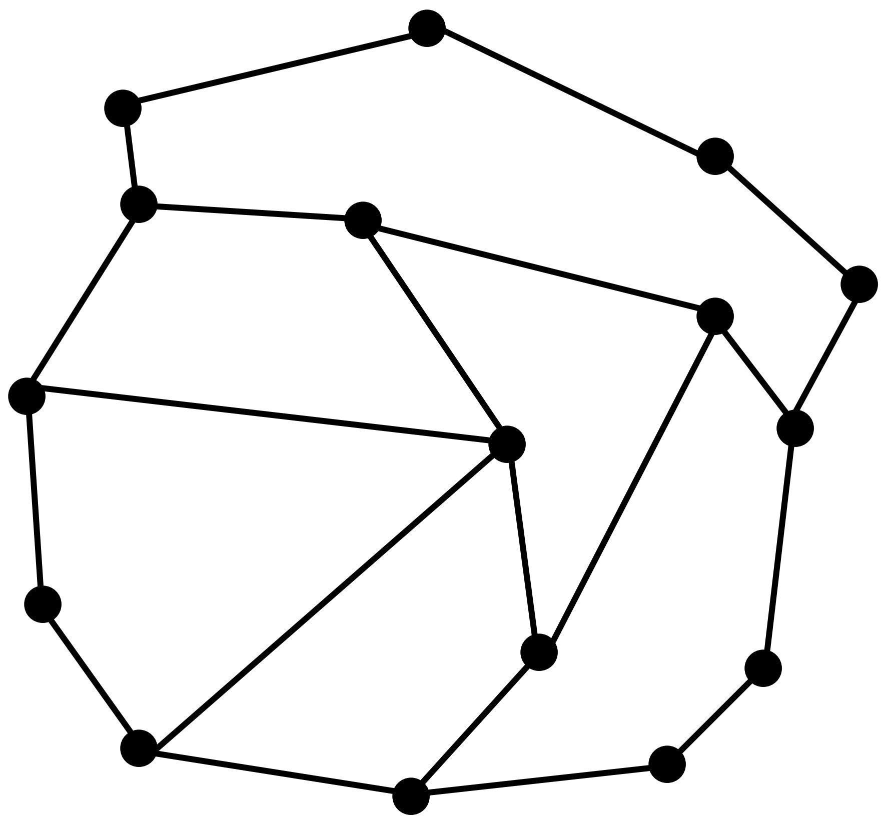
What Does a Block Look Like?

Lemma: Any block is either a K_2 or contains a cycle.

Proof:

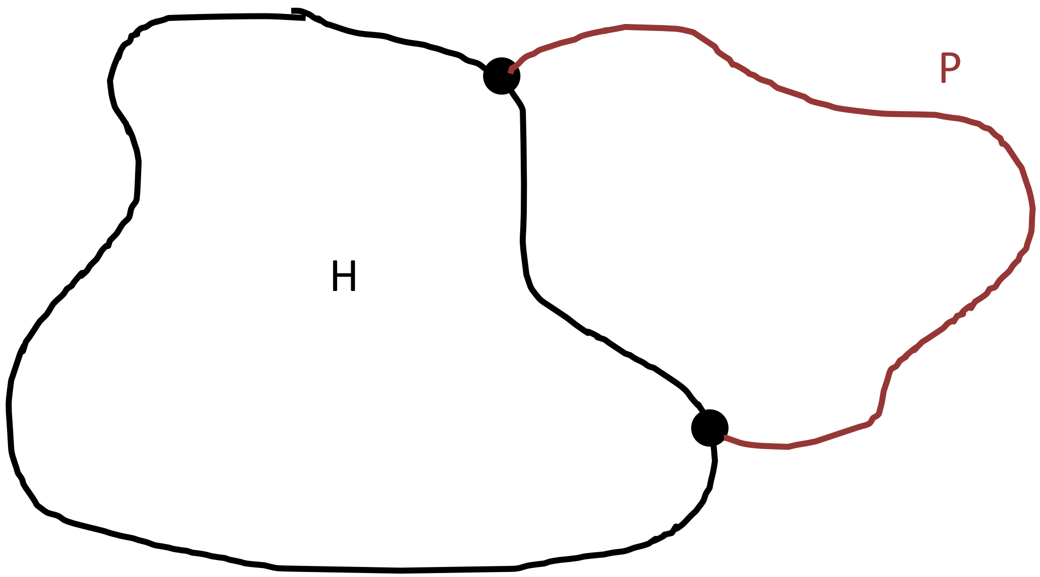
- If it doesn't have a cycle, it's a tree.
- Any non-leaf would be a cut vertex.
- All vertices degree-1
- Handshake Lemma => $|V|=2|E|$.
- Can only happen with two vertices.

Build off a Cycle



Ears

Definition: Given a subgraph H of G an *ear* is a path P in G starting and ending at vertices of H but with no intermediate vertices in H.

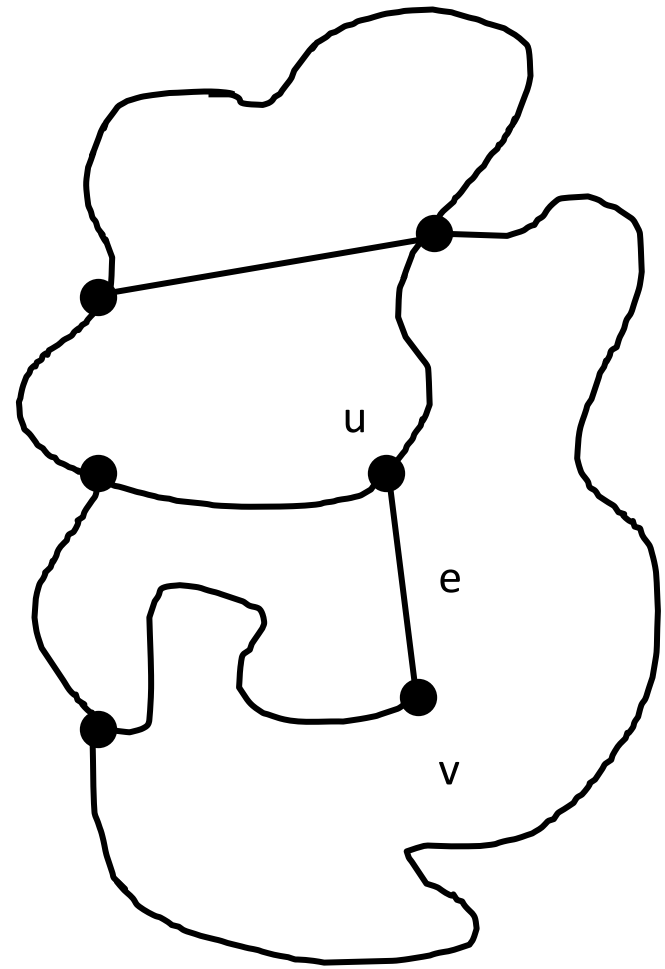


Decomposition

<u>Theorem 4.2.1:</u> Any block is either a K_2 or can be obtained by starting with a cycle and adding ears.

Proof

- Start with a cycle
- If all of block, done.
- Some edge e=(u,v) leading out.
  - u is not a cut vertex
  - Some path from v back
  - Add ear
- Repeat

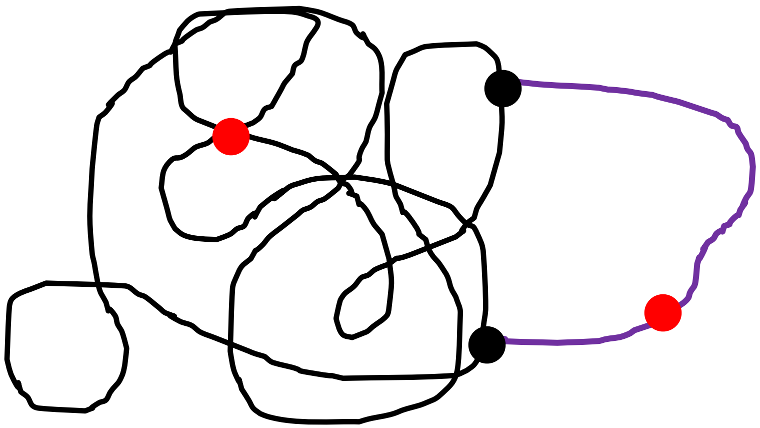


Question: If and only if?

Is every graph with such an ear decomposition free of cut vertices?

A) Yes

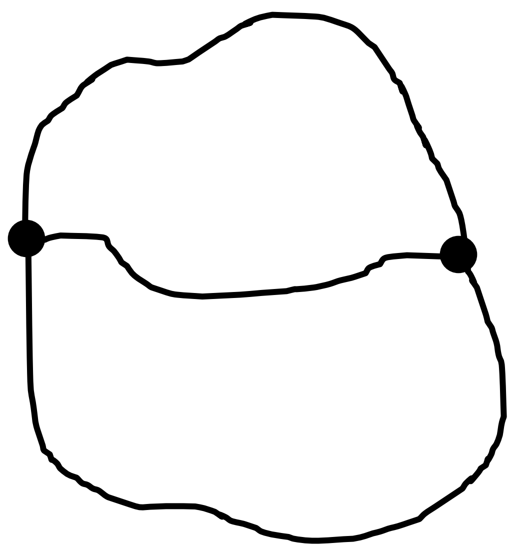
B) No



Adding an ear can't create cut vertices.

Theta Graphs

Definition: A *theta* in a graph is a pair of vertices with three vertex-disjoint paths between them.



Theta-Less Blocks

Lemma: Any block without a theta as a subgraph is either a K_2 or a cycle.

Proof:

- Either a K_2 or cycle + ears.
- Adding an ear to a cycle gives a theta!
- Either a K2 or a cycle.

Conclusion

Proposition 4.1.2: Any connected, theta-less graph G is a tree of cycles and K_2s.

Another Theorem on Blocks

Theorem 4.2.4: For a finite graph G with at least 3 vertices, the following are equivalent:

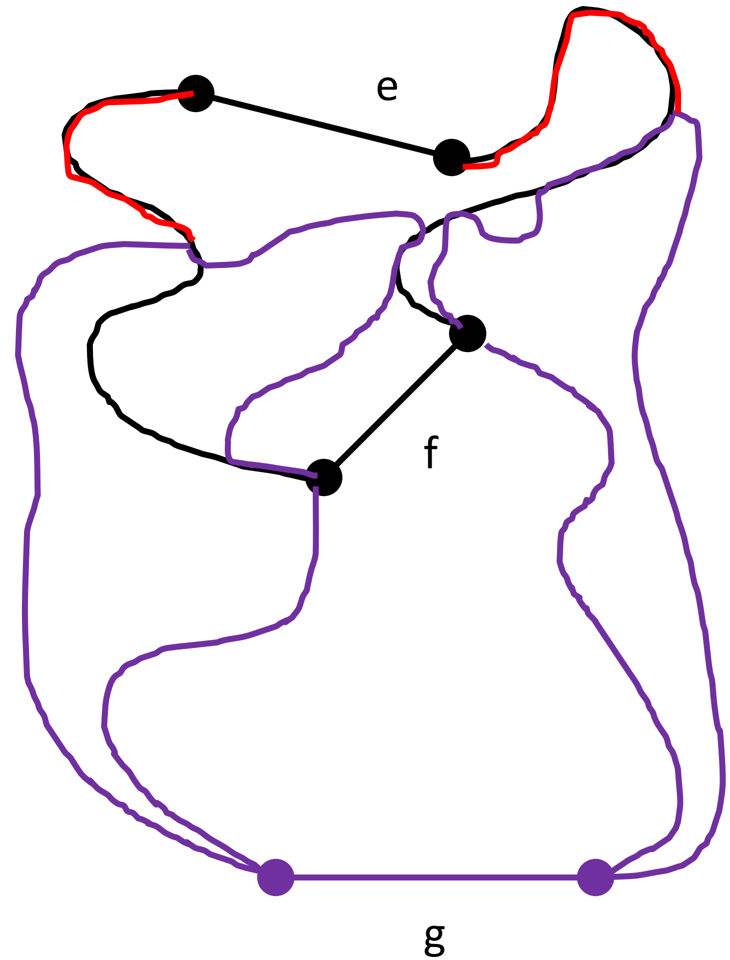
1) G is a single block
2) Any two edges of G are in a common cycle
3) Any two vertices of G are in a common cycle

Proof Strategy

- Show (1) => (2) => (3) => (1).
- For (1) => (2)
  - Want to know when two edges have common cycle
  - Say e ~ f if e and f have common cycle
  - Claim: ~ is an equivalence relation.
    - In particular if e ~ f and f ~ g then e ~ g.

Equivalence Relation

- e and f have common cycle
- f and g have common cycle
- Want cycle for e and g.
- Could be more complicated.
- Connect from first intersections



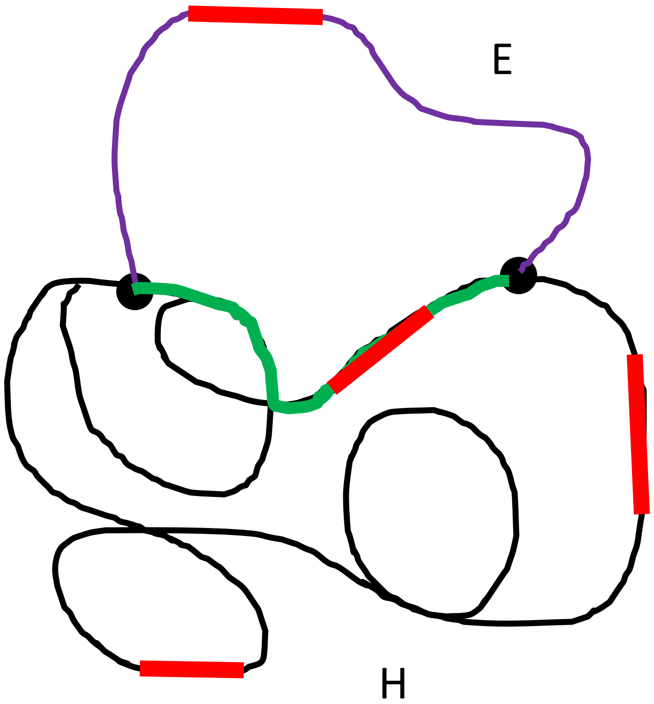
(1) => (2)

Recall, we want to show that if G is a single block that any two edges share a cycle (i.e. e ~ f).

- Prove by induction on the number of ears.
  - Base Case: G is a cycle: all edges in same cycle.
  - Inductive step: G = H + ear. Any two edges of H share a cycle.

Inductive Step

- G = H + ear E
- Any two edges of H share cycle.
- Show e~f for any e,f.
  - Both in H by IH.
  - e in E in cycle.
  - e ~ f for some f in H.
  - f ~ g for all g in H.
  - e ~ g by transitivity.
  - Any two share a cycle.



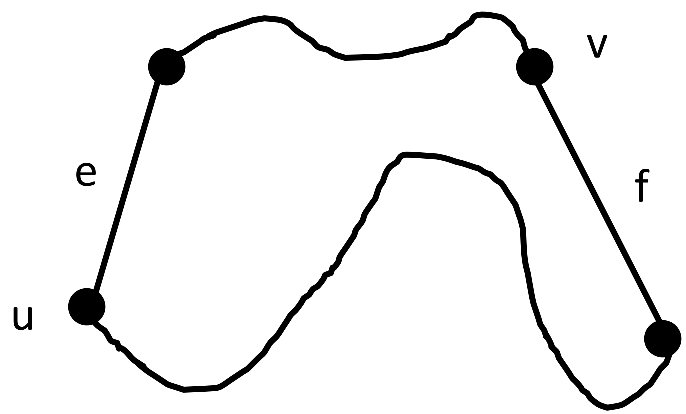
(2) => (3)

Any two *edges* share a cycle. Want same for any two *vertices*.

Vertices u and v.

On edges e and f.

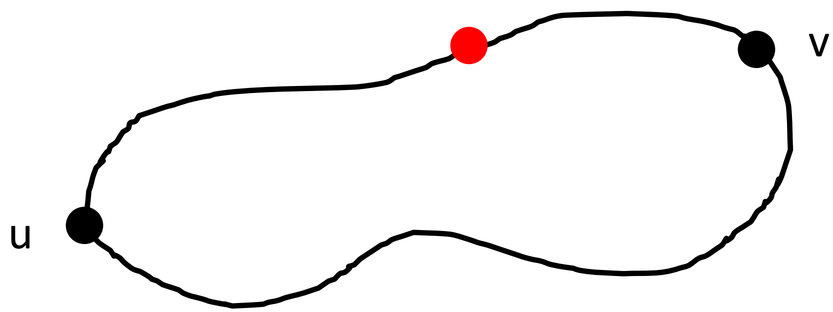
Edges share cycle, so vertices do too.



(3) => (1)

Any two vertices share a cycle. Show that there are no cut vertices.

- Show no cut vertex separates u and v.
- Share a cycle.
- Single cut vertex cannot disconnect cycle.



Summary

The following are equivalent:

1) G has no cut vertices
2) Any two edges share a cycle
3) Any two vertices share a cycle.