Announcements

- Homework 3 Due Sunday
- If you cannot make exam 1 during class time, please email me by tomorrow.

Today

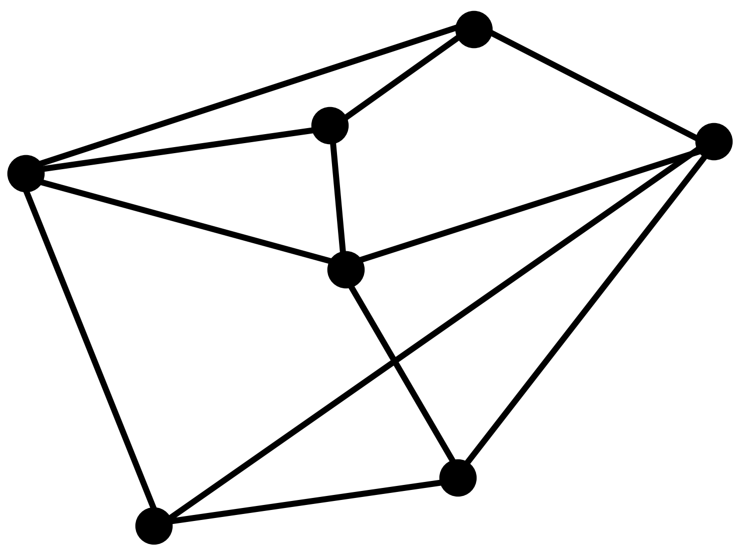
Hamiltonian Graphs

- Definition
- Complexity
- Some conditions

Hamiltonian Graphs

<u>Definition:</u> A *Hamiltonian path/cycle* in a graph G is a path/cycle that uses every vertex of G exactly once.

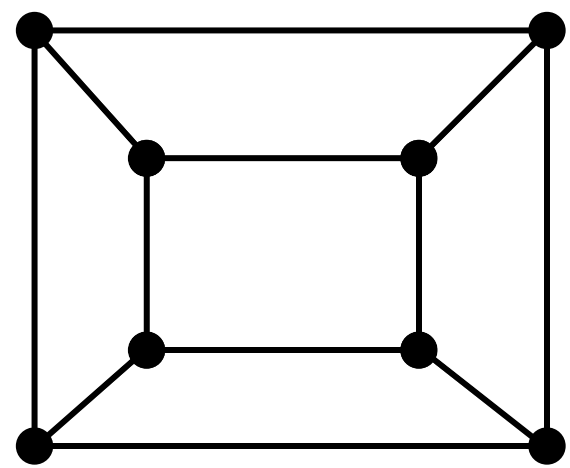
A graph is *Hamiltonian* if it has a Hamiltonian cycle



Question: Hamiltonian Cycles

Which of the following graphs has a Hamiltonian cycle?

A) C_{21}
B) P_{20}
C) K_5
D) $K_{7,9}$
E) The graph shown



Comparison

Eulerian Graphs:

- Is there a circuit that uses every *edge* exactly once?
- Easy characterization based on number of odd degree vertices
- Simple algorithms to construct

Hamiltonian Graphs:

- Is there a circuit that uses every *vertex* exactly once?
- No characterization
- NP-Hard to construct

Focus on partial characterizations.

More Edges

Adding more edges to a graph might make it either easier or harder to find an Eulerian circuit.

More edges can only make it easier to find a Hamiltonian cycle.

Idea: Any graph with enough edges should be Hamiltonian.

How many is “enough”?

Minimum Degrees

Theorem (1.22): If a graph G on n > 2 vertices has minimum degree at least n/2, it is Hamiltonian.

Warmup

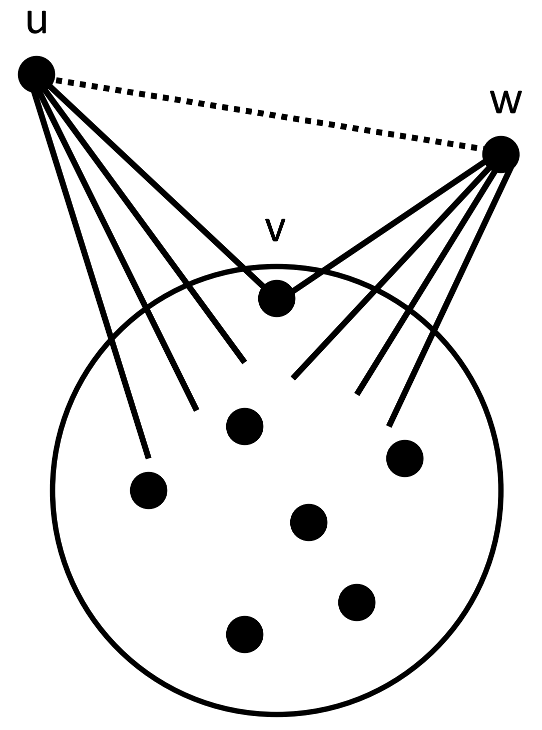
How do we even know that G is connected?

Lemma: If a graph G on n vertices has minimum degree $\delta(G) \geq (n-1)/2$, then G is connected.

Proof

NTS can get between any vertices u, w.

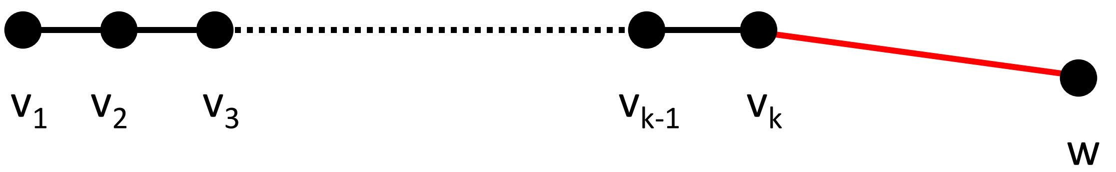
- If edge (u,w), done.
- Else, each has (n-1)/2 edges to the other n-2 vertices
- Must both connect to some common vertex v.
- Have path u → v → w



Longest Path

Back to our main proof. Let $\delta(G) \geq n/2$.

Let $v_1, v_2, \ldots, v_k$ be the *longest* path in G.

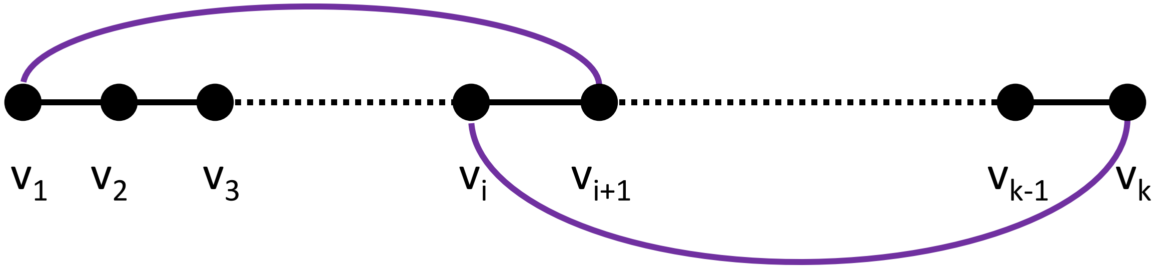


Note that v_1 and v_k cannot connect to vertices outside of this path.

Claim: G has a cycle of length k.

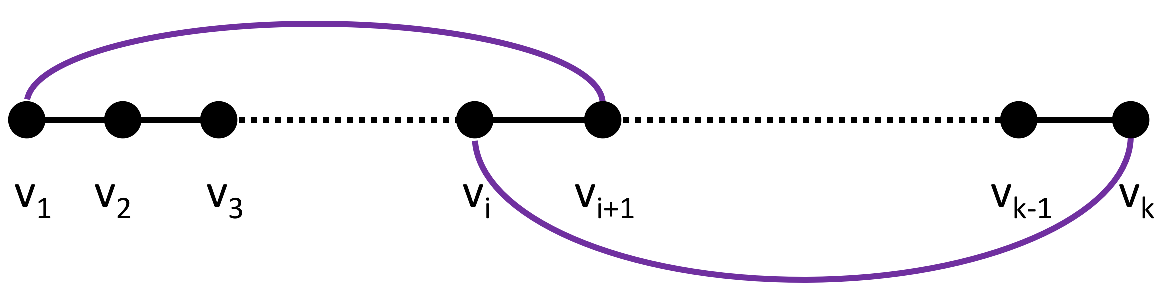
Cycle

To get a cycle, we want to find an i so that there is an edge from v_1 to v_{i+1} and an edge from v_k to v_i.



Counting

- There are ≥ n/2 i's so that v_k connects to v_i.
- There are ≥ n/2 i's so that v_1 connect to v_{i+1}.
- There are only k-1 < n total i's.
- The two lists must have some index in common.

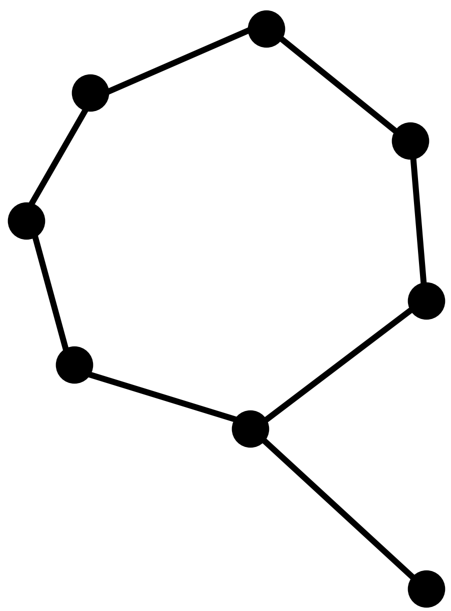


Cycle

Have a cycle of length k.

G is connected so either:

- k=n.
  - Cycle is Hamiltonian
- k<n
  - Some vertex in cycle connects to some other vertex.
  - We have a longer path.



In Summary

Given any path, can either:

- Extend path ⇒ longer path.
- Turn into cycle that connects to another vertex ⇒ longer path.
- Turn into Hamiltonian cycle.

If we start with the *longest* path, it must lead to a Hamiltonian cycle.

Question: Algorithm

Does the proof we've presented give an algorithm for finding Hamiltonian cycles in graphs with $\delta(G) \geq n/2$?

A) Yes

B) No

Generalization

Actually, you can get slightly more out of the same argument.

Theorem (1.23): If G is a graph on at least 3 vertices and if $d(u)+d(v) \geq n$ for all pairs of non-adjacent vertices u and v, then G is Hamiltonian.

Counting

- Either v_1 adjacent to v_k, can use edge between them
- Or
  $\#\{i: v_k \text{ adjacent to } v_i\} + \#\{i: v_1 \text{ adjacent to } v_{i+1}\} \geq n$
  - So there must be some i for which both hold.