Final Exam Review

Math 154

Note

This review will only cover material beyond what was on the two midterm review videos. If you want a comprehensive review of all course material that may be on the final, make sure to review the old videos as well.

Matchings and Flows (Ch 1.7)

- Bipartite Matching
  - Hall’s Theorem
  - Konig’s Theorem
- Flows
  - Maxflow-Mincut & Applications
- Perfect Matchings in General
  - Tutte’s Theorem

Matchings

Definition: A *matching* in a graph G is a set of edges of G no two of which share an endpoint. The *size* of a matching is the number of edges. A matching is *maximum* if its size is as large as possible.

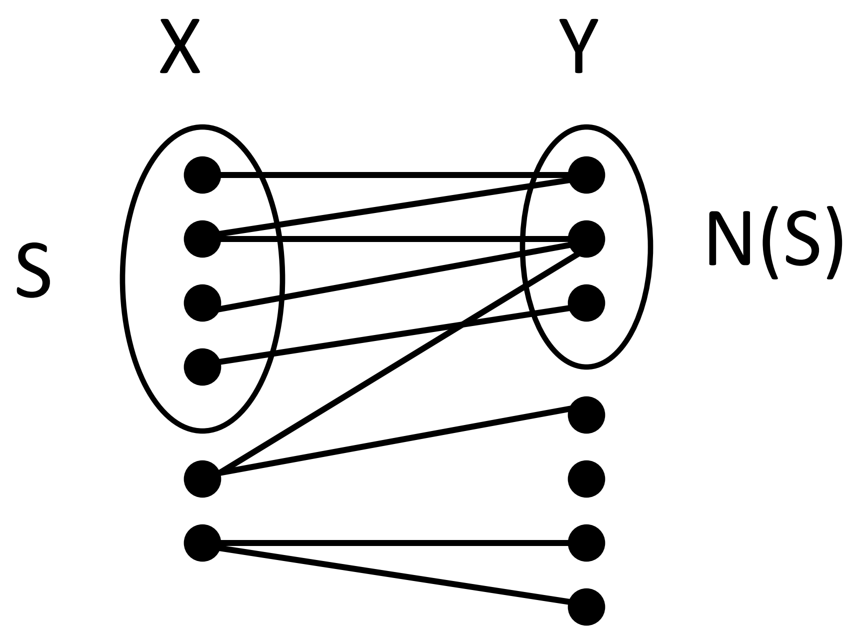
The Marriage Lemma

Theorem (1.51): Let G be a bipartite graph with parts X and Y. There is a matching of G using all the vertices in X if and only if for every subset $S \subseteq X$, $|N(S)| \geq |S|$, where N(S) is the set of neighbors of elements of S.

Easy Direction

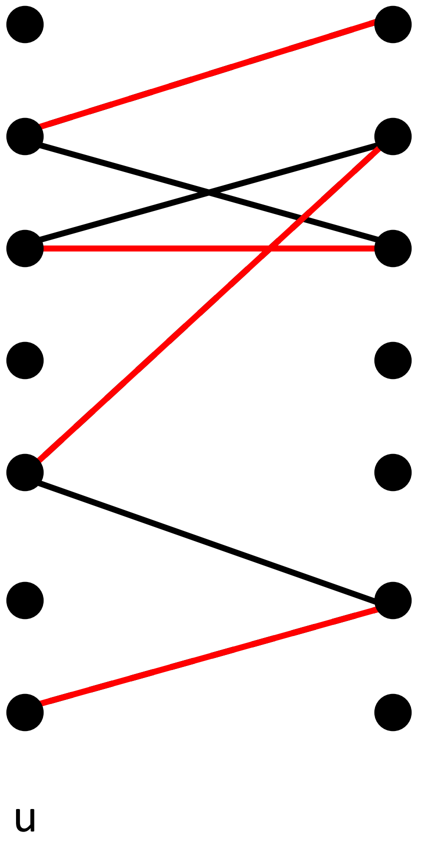
Suppose there is an S with |N(S)| < |S|.

Each element of S can only match to an element of N(S), and there are not enough for each to get a different one.



Augmenting Paths

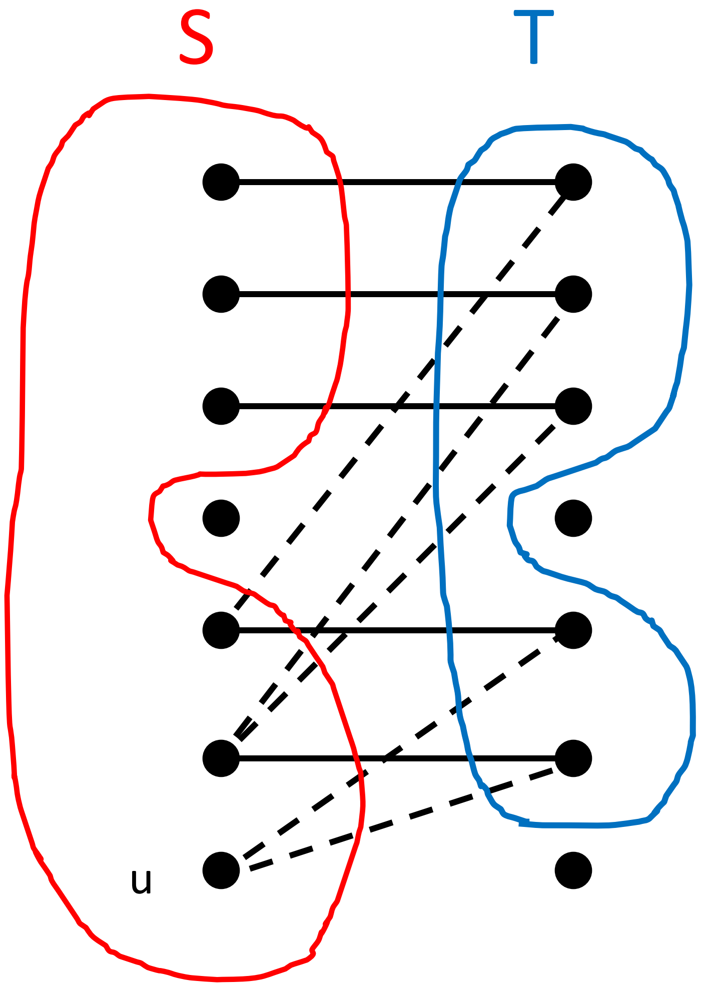
- Consider paths starting at u that take any edge of the graph from X to Y, and take matching edges back.
- If you can reach an unmatched vertex, can increase matching.



Set S

If you cannot find one ending on unmatched vertex:

- Consider all vertices you can reach with such a path.
- Let S be the set of X-vertices you can reach.
- T set of Y-vertices.



Application: Regular Bipartite Graphs

Proposition (1.57): Any regular, bipartite graph has a perfect matching (i.e. a matching that uses all of the vertices).

Bipartite Graph Degrees

Note that every edge of a bipartite graph attaches to each side of the graph.

Lemma: Let G be a bipartite graph with parts X and Y, then

$$\Sigma\ d(x) = |E| = \Sigma\ d(y)$$

Corollary: For regular bipartite G, $|X| = |Y|$.

Application: Edge Coloring Bipartite Graphs

Recall, that given a graph G, the edge coloring number was either $\Delta(G)$ or $\Delta(G)+1$.

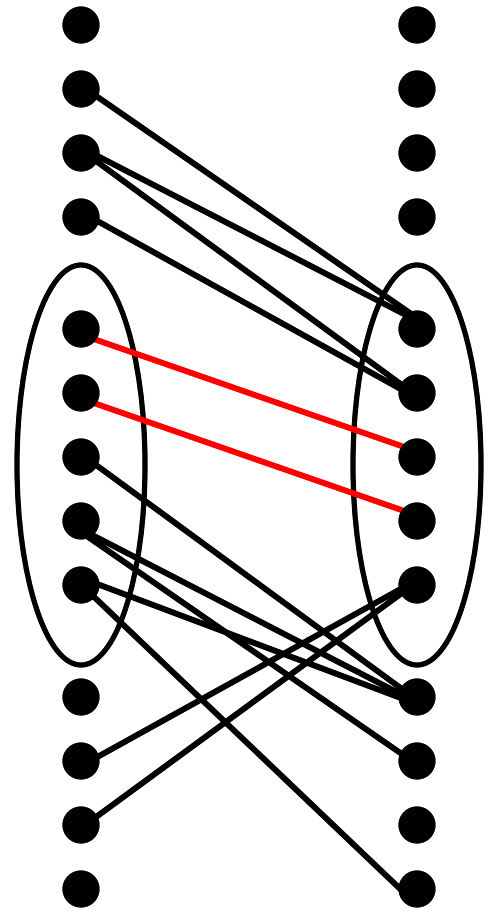
Theorem (V 6.1.1): Every finite bipartite graph G can be edge colored with $\Delta(G)$ colors.

Proof Idea

- Find a matching M which includes all of the vertices of G of maximum degree.
- Color all edges in M one color, inductively color G-M.
  - Since M includes an edge from each vertex of max degree $\Delta(G\text{-}M)=\Delta(G)-1$, can be colored with $\Delta(G)-1$ colors.

In General

- Consider all degree k vertices
- Find matching M_1 among max degree vertices.
- Remaining degree k vertices don't connect.
- Match degree k vertices on each side.



Vertex Cover

Definition: A *vertex cover* is a set C of vertices so that every edge is incident on some vertex of C.

Lemma: The size of the maximum matching is at most the size of the minimum vertex cover.

Proof: Each edge of M uses different vertex of C.

Is this tight?

Konig’s Theorem

Theorem (1.53): For any finite, bipartite graph G the size of the maximum matching equals the size of the minimum vertex cover.

Note: This is a generalization of Hall’s Theorem.

Definitions

A *network* is a directed graph G with designated *source* and *sink* vertices s and t.

A *flow* is a subgraph of G so that for each vertex v other than s and t $d_{in}(v) = d_{out}(v)$.

The *size* of a flow is $d_{out}(s) - d_{in}(s)$.

Augmenting Paths

Definition: Given a graph G and flow F an *augmenting path* is an s-t path that uses either edges of G unused by F in the forwards direction, or edges used by F in the backwards direction.

Lemma: Given an augmenting path, you can add it to F to get a path with 1 more unit of flow.

Cuts

Definition: A *cut* is a partition of the vertices into two sets S and T, which contain s and t, respectively.

The *size* of a cut is the total number of edges from vertices in S to vertices in T.

Cuts and Flows

Lemma (V 8.3.1): For a network G a flow F and a cut (S,T) it is the case that

Size(F) = #{edges in F from S to T} - #{edges in F from T to S}

Proof

Consider the sum over all v in S of $d_{out}(v)-d_{in}(v)$.

On the one hand this is 0 except for v = s, where it is Size(F).

On the other hand, each edge contributes to one in degree and one out degree. This makes its total contribution 0 unless it crosses the cut. This gives 1 for each edge from S to T and -1 for each edge from T to S.

Maxflow-Mincut

Theorem (V. 8.3.2): For any network G the size of a maximum flow in G is the same as the size of a minimum cut.

Maxflow ≤ Mincut

Let F be a flow and C be a cut.

By Lemma:

Size(F) = #{edges in F from S to T} - #{edges in F from T to S} ≤ Size(C)

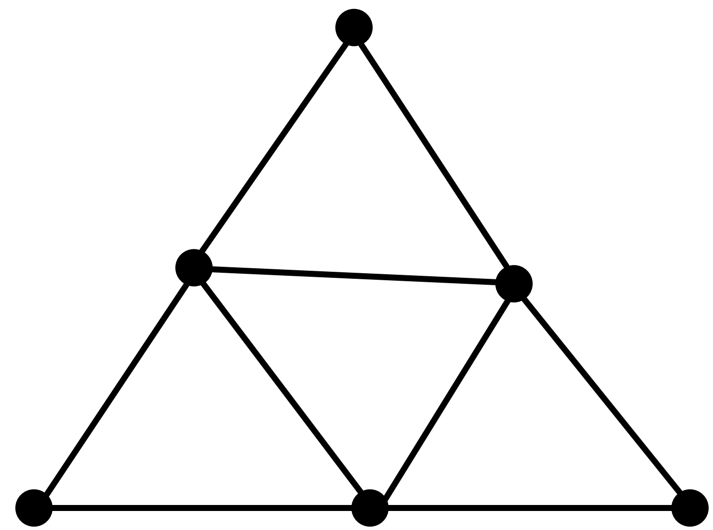
Any flow is smaller than any cut, so the maximum flow size is at most than the minimum cut size.

Maxflow ≥ Mincut

- Let F be a maximum flow.
- No augmenting paths.
- Let S be the set of vertices v you can reach from s using unused forward edges or used backwards edges.
- F uses all edges out of S, no edges into S.
- Lemma says: Size(F) = Number of edges out of S = Size(C).
- Maxflow ≥ Mincut.

Perfect Matchings

- Given graph G
- Find a set of edges that uses each vertex once.



In General

Odd vs. Even is important.

Definition: For a graph G let $\Omega(G)$ denote the number of connected components of G with an odd number of vertices.

Lemma: If there is a set S of vertices of G with $|S| < \Omega(G-S)$, then G has no perfect matching.

Tutte’s Theorem

Surpsingly, this is the only thing that can go wrong.

Theorem (1.59): If G is a finite graph so that for every set S of vertices $|S| \geq \Omega(G-S)$, then G has a perfect matching.

A Lemma

Lemma: If G has an even number of vertices, then S-Ω(G-S) is always even.

Note: If G has an odd number of vertices, then for S = ∅, |S| < Ω(G-S).

Proof: The number of vertices of G equals |S| plus the sum of the sizes of the connected components of G-S. If a collection of numbers adds to an even number, it must contain an even number of odd numbers.

Proof Overview

- Let S be *maximal* so that |S| = Ω(G-S).
- Claim 1: All components of G-S are odd.
- Claim 2: For any component C of G-S and v in C, C-v has perfect matching.
  - Use IH and maximality of S.
- Claim 3: There is a perfect matching between points in S and components of G-S
  - Use Hall's Theorem
- Match S and components. Then find perfect matching of remaining.

Ramsey Theory (Ch 1.8)

- Introduction
- Definitions
- Existence of Ramsey Numbers
- Derivation of Small Ramsey Numbers
- Lower Bounds
- Other Ramsey Problems

Ramsey's Theorem

Theorem: For any positive integers p and q there exists a number N so that for any $n \geq N$ and any red-blue coloring of the edges of a K_n, there is either a red K_p or a blue K_q.

Definition: The smallest such number N is called the *Ramsey Number*, R(p,q).

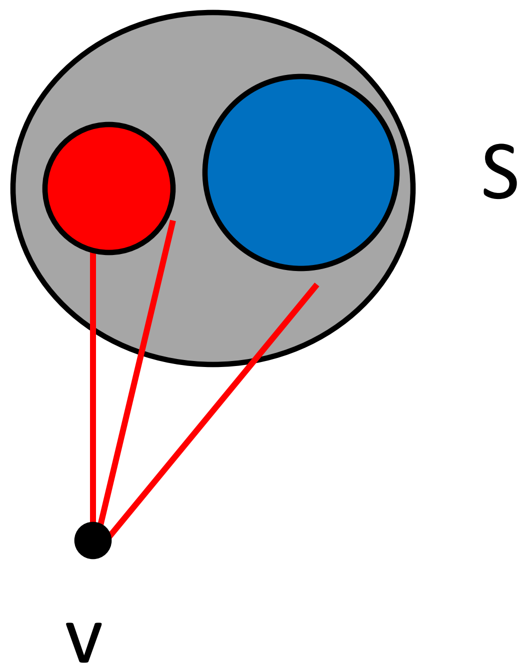
Furthermore: $R(p,q) \leq R(p-1,q)+R(p,q-1)$.

Proof I

- Take $n \geq R(p-1,q)+R(p,q-1)$. Color K_n.
- Consider vertex v, has at least $R(p-1,q)+R(p,q-1)-1$ edges out of it.
- Either has $R(p-1,q)$ red edges or $R(p,q-1)$ blue edges.
- WLOG v has $R(p-1,q)$ red edges to other vertices S.

Proof II

- Have v, S
  - All edge v to S red.
  - |S| ≥ R(p-1,q).
- IH: Color of S has either:
  - Red K_{p-1}.
    - v + clique is red K_p
  - Blue K_q
    - Gives blue K_q
- Either way we're done.



Known Ramsey Numbers

- $R(1,n) = 1$
- $R(2,n) = n$
- $R(3,3) = 6$
- $R(3,4) = 9$
- $R(4,4) = 18$
- $R(3,5) = 14$
- $R(3,6) = 18$
- $R(3,7) = 23$
- $R(3,8) = 28$
- $R(3,9) = 36$
- $R(4,5) = 25$

Upper Bound

Theorem: $R(p,q) \le 2^{p+q}$.

Proof: By induction on p+q.

- If p=1 or q=1, $R(p,q) = 1 < 2^{p+q}$.
- Assume the inequality holds for smaller p+q.
  - $R(p,q) \le R(p-1,q)+R(p,q-1)$
    $\le 2^{p+q-1}+2^{p+q-1} \le 2^{p+q}$.

Lower Bound

Theorem (1.66): If $n \geq 3$, $R(n,n) \geq 2^{n/2}$.

Random Construction

Color the edges of a K_N randomly. On average how many monochromatic K_ns?

- $\approx N^n$ many collections of n vertices.
- Each has a $\approx 2^{-n(n-1)/2}$ probability of being monochromatic
- Average number of monochromatic Kns is roughly $N^n/(2^{n(n-1)/2}) \approx [N/2^{(n-1)/2}]^n$.
- If N much smaller than $2^{n/2}$, this is less than 1, so *some* coloring must have none.

Graph Ramsey Numbers

Definition: For graphs G and H, we define the graph Ramsey number R(G,H) to be the minimum n so that any red-blue coloring of K_n has either a red copy of G or a blue copy of H.

Finiteness

Note that G and H are contained in complete graphs, so this is finite.

Theorem (1.67):

$R(G,H) \leq R(|V_G|, |V_H|)$

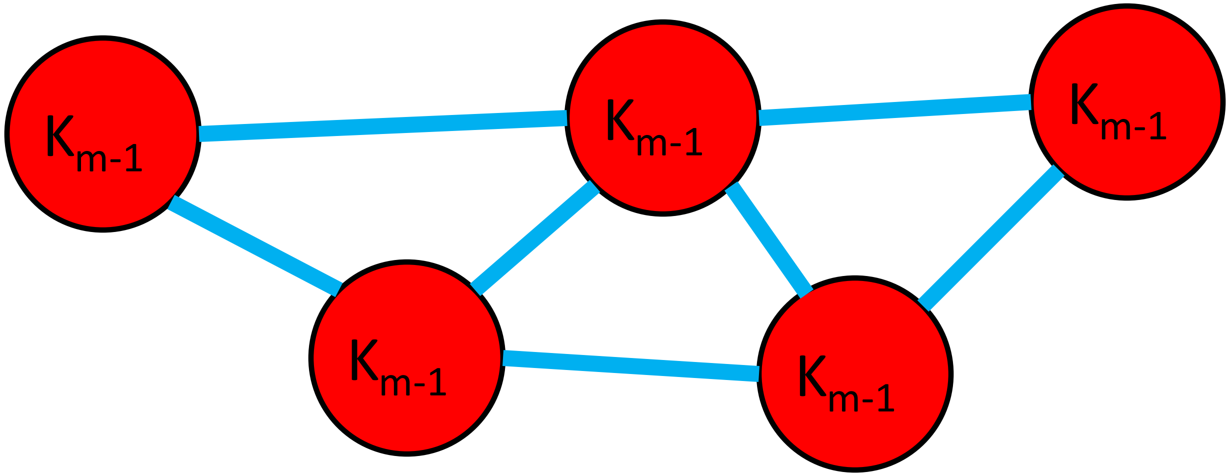
Proof: Let $m = R(|V_G|, |V_H|)$. Any red-blue coloring of K_m has either a monochromatic complete red graph on $|V_G|$ or monochromatic blue complete graph on $|V_H|$. These contain a red copy of G or blue copy of H.

Example

Theorem (1.70): If m and n are integers with m-1 dividing n-1 and T_m is a tree with m vertices then

$$R(T_m, K_{1,n}) = m+n-1.$$

Coloring



- Blue K_{m-1}s connected by red edges.
- No Red T_m: All but CCs size m-1.
  - No Blue K1,n:Each vertex has blue degree (m+n-3) – (m-2) = n-1.

Upper Bound

- Need to show that any red-blue coloring of a K_{n+m-1} has either a red T_m or a blue $K_{1,n}$.
- If any vertex has n or more blue edges, have blue $K_{1,n}$.
- Otherwise, consider G_r, graph of red edges.
  - Note that $\delta(G_r) \geq m-1$.

Lemma

Lemma (1.16): Let T be any tree on k vertices and G a graph with $\delta(G) \geq k-1$. Then G contains a copy of T.

Apply to G_r and T_m to get final result.

Proof by Induction on k

- Base case: k = 1
  - Can embed single point
- Assume can embed any tree on k-1 vertices.
- Let v be a leaf of T. Removing edge (u,v) gives T’
- By IH, embed T’ in G.
- Need new neighbor of u to be v.
- u has k-1 neighbors, only k-2 are used.