Exam 2 Review

Math 154 Spring 2020

Ch 1.5: Planar Graphs

- Planarity Definition
- Faces and Euler's Formula
- Platonic Solids
- Straight Line Embeddings
- Non-Planar Graphs

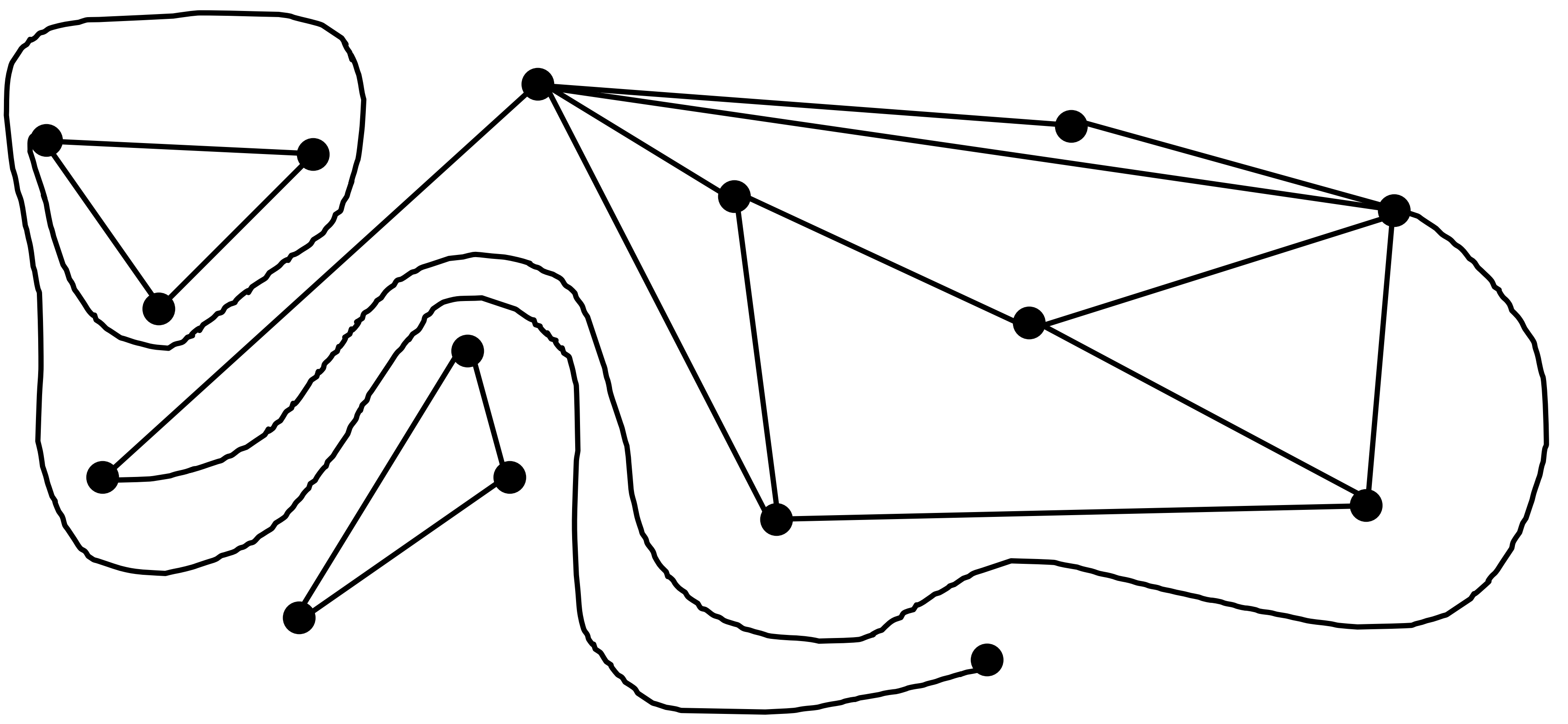
Planar Embeddings

Definition: A *planar embedding* of a graph G is a drawing of G so that

- Each vertex of G corresponds to a point in the plane.
- Each edge of G corresponds to a curve connecting its endpoints.
- No two edge-curves cross except at endpoints.

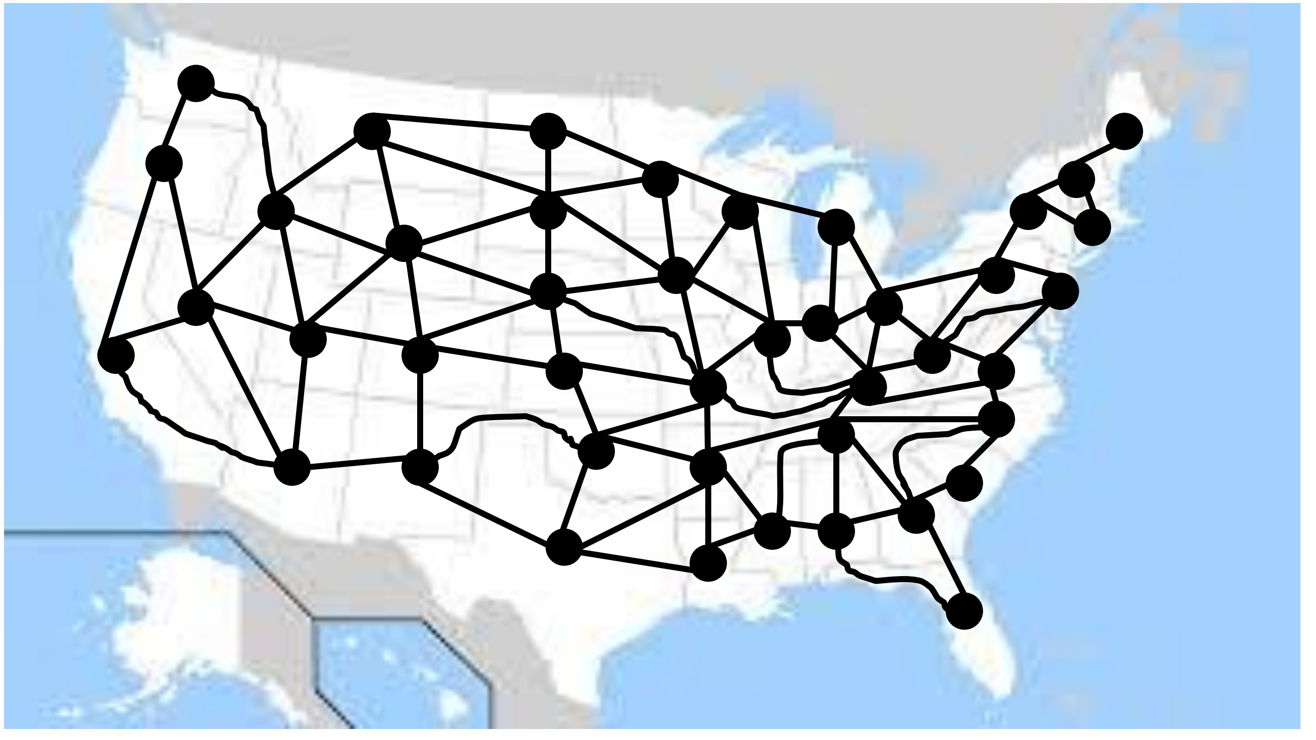
Planar Graphs

<u>Definition:</u> A graph is *planar* if it has a planar embedding.



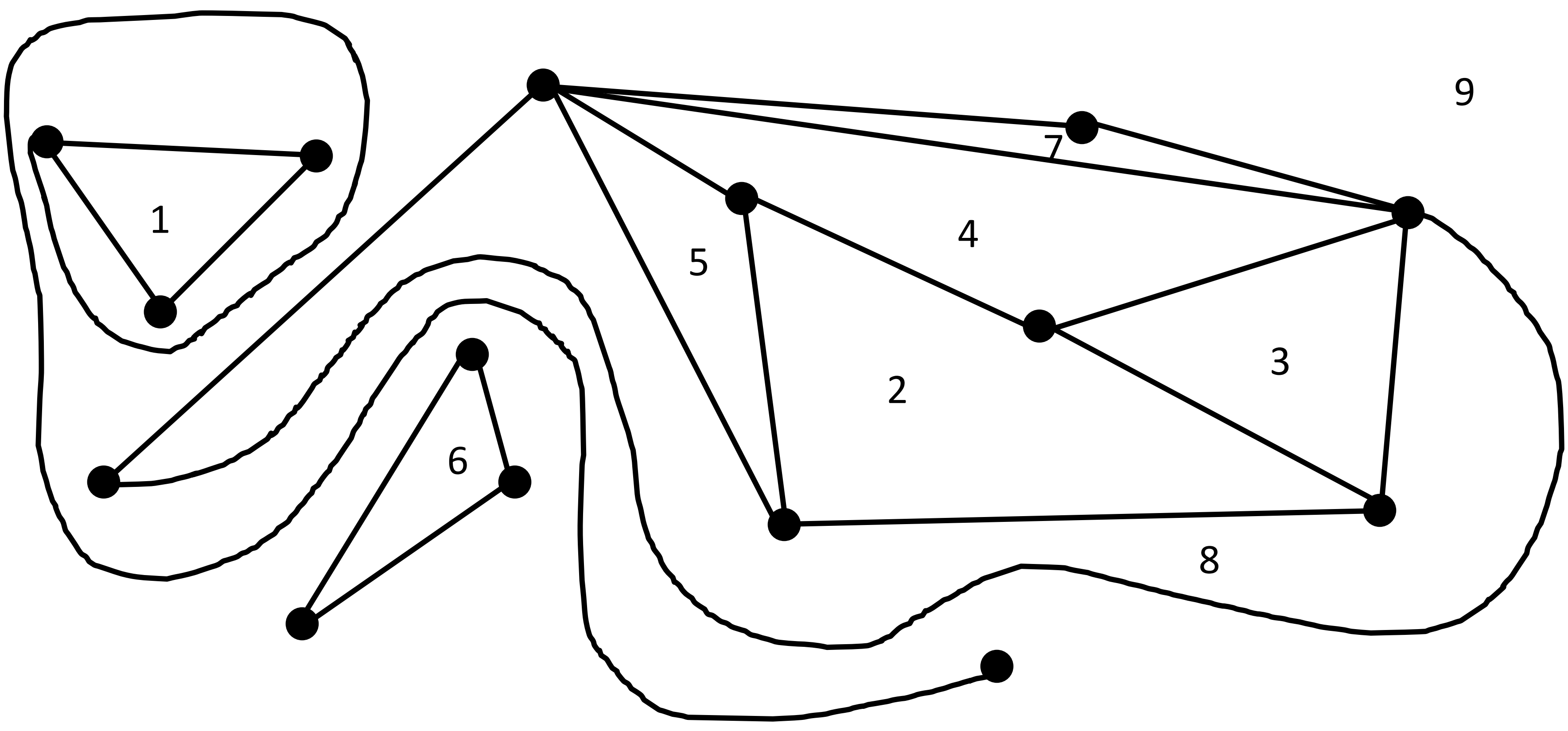
Application: Maps

Given a map with simply connected regions, the adjacency graph on regions is planar.



Faces

A planar embedding of a graph divides the plane into regions. These are called *faces*.



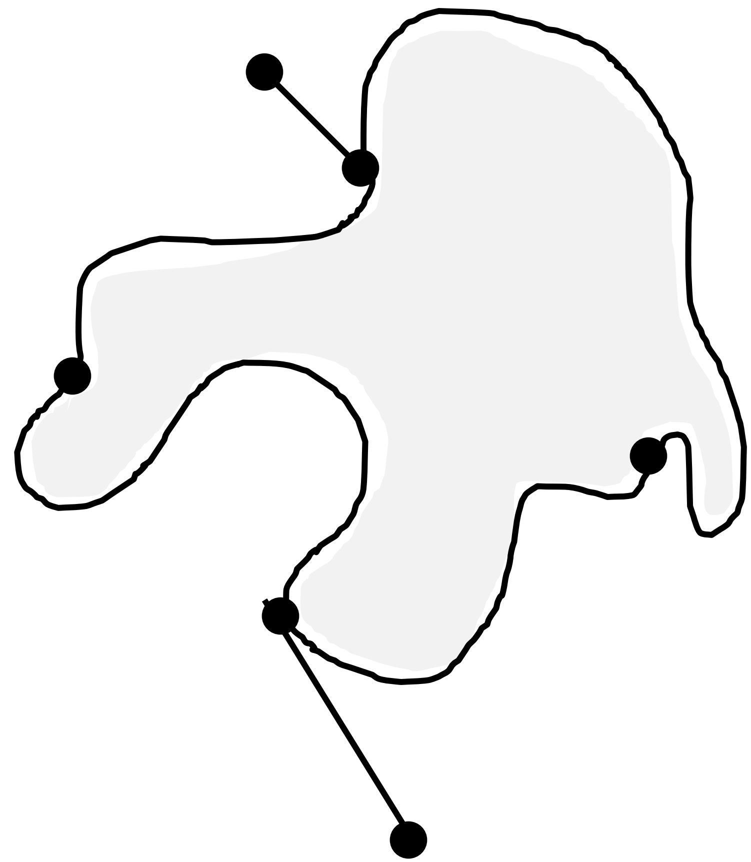
Euler’s Formula

Theorem (1.31): For any planar embedding of a connected graph G with v vertices, e edges and f faces (including the infinite face)

$$v - e + f = 2$$

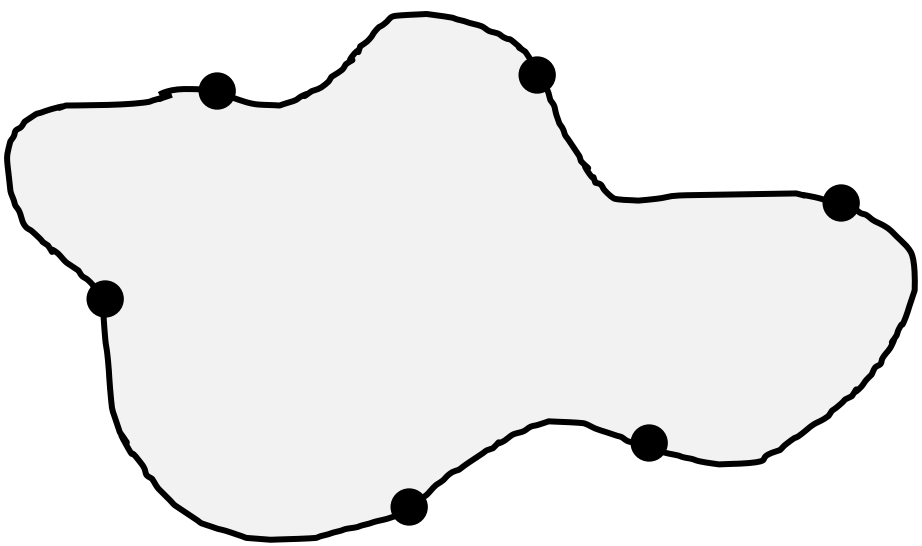
General Graphs

- Use induction on e.
- Base case: G is a tree.
- Otherwise, G has a cycle
- Cycle separates plane into inside and outside.
- Remove an edge of cycle, decreases f by 1.
- IH => $v - (e-1) + (f-1) = 2$



Sides to a Face

If G is a connected planar graph, any face (including the infinite one) will be bounded by a loop of edges.



The number of *sides* of the face is the number of edges in this loop.

Dual Handshake Lemma

Lemma: For a connected, planar graph,

$$\sum_{\text{Faces } f} \text{Sides}(f) = 2|E|.$$

- Note similarity to Handshake Lemma. Sides of faces instead of degrees of vertices.
- Proof similar.

Edge Bound

Theorem (1.33): If G is a connected planar graph with $|V| \geq 3$, then

$$|E| \leq 3|V| - 6.$$

K_5 Non-Planar

Theorem (1.34): The K_5 is non-planar.

Proof: If it were, we would have

$$e \leq 3v-6 = 9$$

But e = 10. Contradiction!

$K_{3,3}$ Non-Planar

Theorem (1.32): $K_{3,3}$ is non-planar.

Proof: $K_{3,3}$ is bipartite, so it has no odd cycles. Therefore, if planar any face has at least 4 sides.

If planar,

$$e \leq 2v-4 = 8.$$

But e=9. Contradiction!

Minimum Degree

Theorem (1.35): If G is a finite, connected planar graph, its vertices have minimum degree at most 5.

Proof: Otherwise, each vertex has degree 6 or more.

Handshake Lemma implies

$$2e = \Sigma\, d(v) \geq 6v.$$

But then

$$3v - 6 \geq e \geq 3v.$$

Contradiction!

Triangulations

We note that our edge bound of 3v-6 has an equality case if and only if all faces are triangles.

We can always ensure that this is the case if we add more edges.

Lemma: For any planar embedding of a graph G there is a way to add more edges to G to get a new planar graph G' in which all faces are triangles.

Fary's Theorem

<u>Theorem (V. 7.4.2):</u> Any finite (simple) planar graph G has a plane embedding where all of the edges are straight line segments.

Proof Strategy

- Induct on v.
  - If $v \leq 3$, easy to draw.
- Assume G is connected (ow/ draw each component separately)
- Find a vertex v of low degree.
- Draw G-v with straight lines.
- Re-insert v into drawing.

Lemma

Lemma: Given any polygon P in the plane with at most 5 sides, there is a point v inside of P with straight line paths to each of P's vertices.

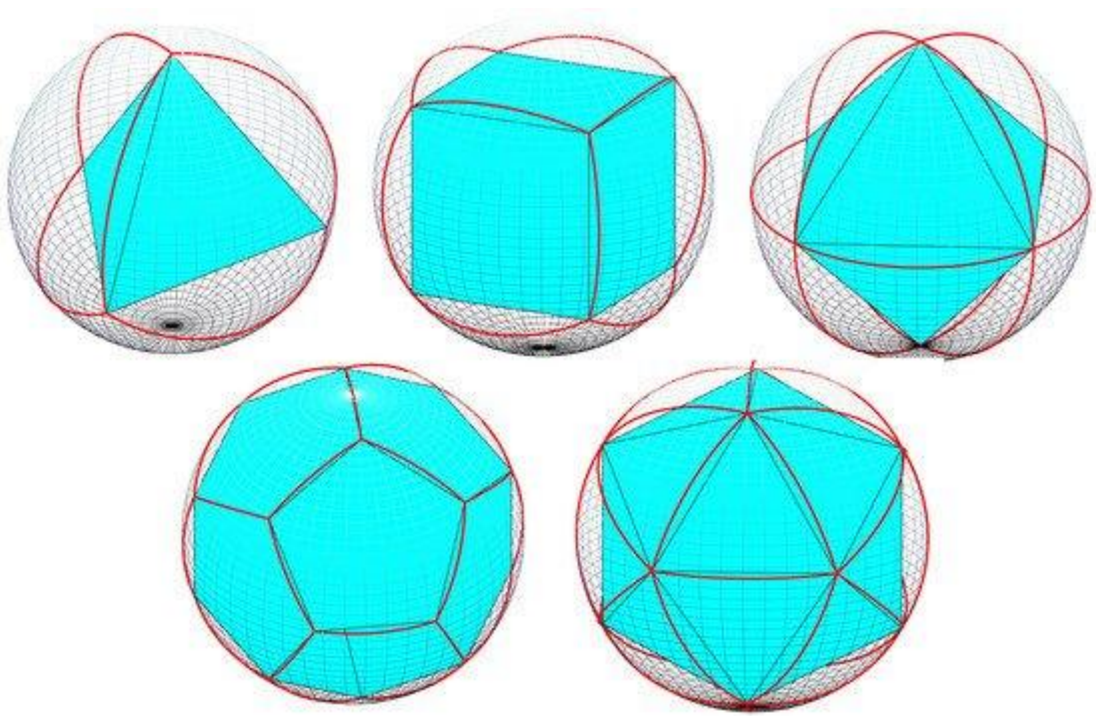
Polyhedra

Definition: A *polyhedron* is a 3 dimensional figure bounded by finitely many flat *faces*. Two faces meet at an *edge* and edges meet at *vertices*.

A polyhedron is *convex* if for any two points in the polyhedron the line segment connecting them is also contained in the polyhedron.

Polyhedral Graphs

Given a convex polyhedron, can turn it into a planar graph by projecting vertices/edges onto a sphere (which can then be flattened onto a plane).



Euler’s Formula

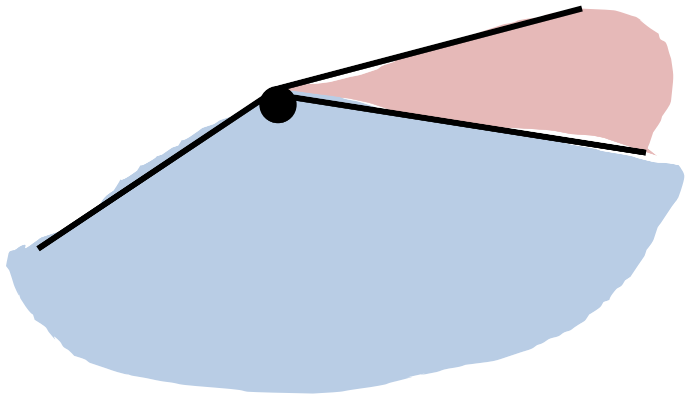
Euler’s Formula applies directly:

For any polyhedron:

#Faces - #Edges + #Vertices = 2

Degrees

Note that in any polyhedron, each vertex has degree at least 3.



Handshake Lemma implies

$$2e = \Sigma\, d(v) \geq 3v$$

$$e \geq 3v/2$$

Edges and Vertices

Remember we also saw that if each face of a planar graph had at least k edges then

$$e \leq (v-2)/(1-2/k)$$

If k = 6, we have that

$$3v/2 \leq e \leq (v-2)/(2/3) = 3(v-2)/2.$$

Contradiction!

Corollary: Every polyhedron has a face with at most 5 sides.

Regular Polyhedra

A regular polyhedron is a highly symmetric polyhedron (like a cube). In particular, it has the following properties:

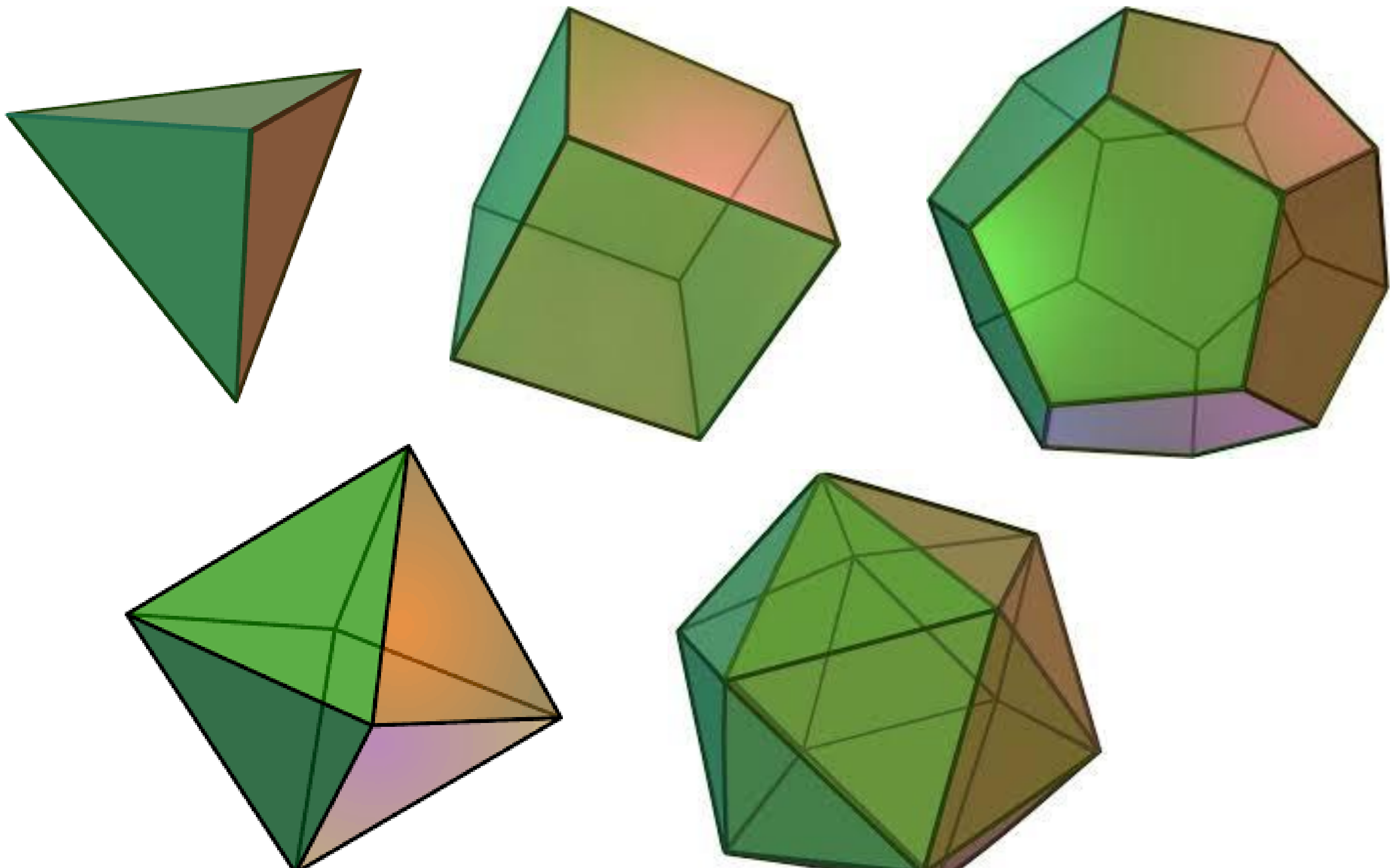
- All edges are the same length.
- All faces are regular polygons with the same number – s of sides.
- The same number of faces, d, meet at each vertex.

Counting Continued

Therefore, we must have:

- $5 \geq d, s \geq 3$
- $2/d + 2/s > 1$
- $e = 2/(2/d+2/s-1)$
- $v = 2e/d$
- $f = 2e/s$

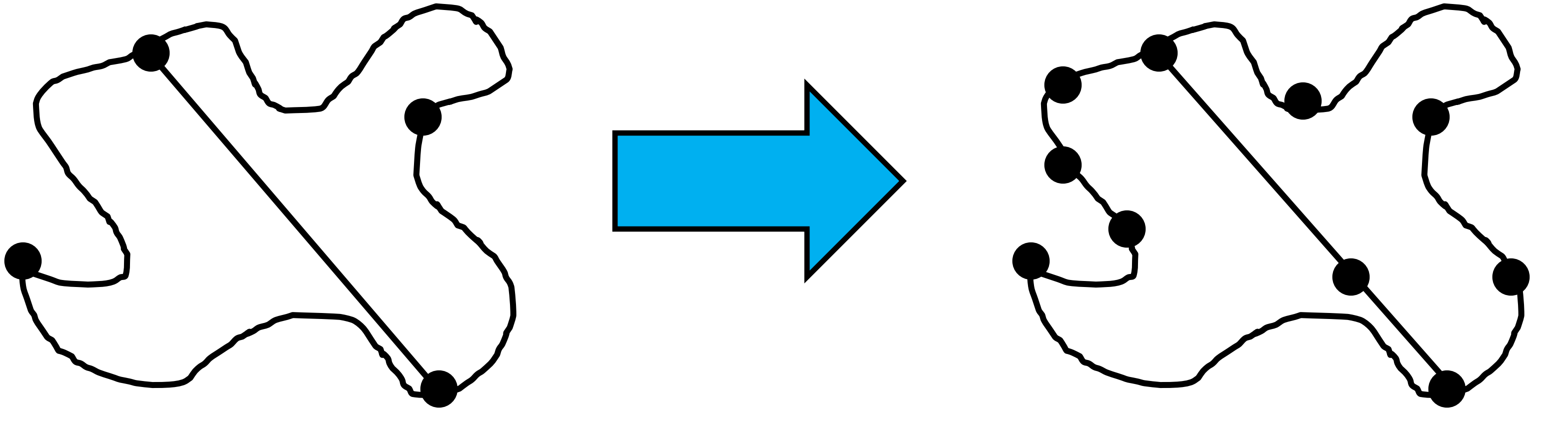
Only 5 Platonic Solids



Subdivisions

Here's one thing that doesn't much affect planarity:

<u>**Definition:**</u> A *subdivision* of a graph G is obtained by placing vertices in the middle of some of its edges.



Subdivisions II

Lemma: If G’ is a subdivision of G, then G’ is planar if and only if G is.

Proof:

- Given a plane embedding of G add vertices in the middle of edges to get embedding of G’.
- Given embedding of G’, remove vertices and join edges, to get embedding of G.

Kuratowski's Theorem

<u>Theorem (V. 7.0.1):</u> A finite graph G is planar if and only if it has no subdivision of a K_5 or $K_{3,3}$ as a subgraph.

Coloring Problems (Ch 1.6)

- Introduction and Definitions
- Basic Results
- Brooke’s Theorem
- Colorings of Planar Graphs
- Edge Colorings

Colorings

Definition: A (vertex) *coloring* of a graph G is an assignment of a color to each vertex of G so that no two adjacent vertices have the same color. This is an *n-coloring* if only n different colors are used.

Definition: The *Chromatic Number*, $\chi(G)$, of a graph G is the smallest number n so that G has an n-coloring.

Basic Facts

- A graph has $\chi(G) = 1$ if and only if G has no edges.
- A graph has $\chi(G) \leq 2$ if and only if G is bipartite.
- Determining $\chi(G)$ for more complicated graphs is difficult. For 2-colorings once you color a vertex, there is only one possible choice for its neighbors. For 3-colorings, you have 2.

Cliques

Definition: The *clique number*, $\omega(G)$, of a graph G is the largest n so that K_n is a subgraph of G.

Corollary: For any graph G, $\chi(G) \geq \omega(G)$.

Note: This bound is far from tight.

Upper Bound

Lemma: For G a graph on n vertices, then $\chi(G) \leq n$.

Proof: Give each vertex a different color.

Greedy Coloring

Coloring Strategy: Color vertices one at a time, giving each a color that doesn't conflict.

Max Degree

Definition: For a graph G, let $\Delta(G)$ denote the maximum degree of any vertex of G.

Lemma: For any graph G, $\chi(G) \leq \Delta(G)+1$.

Proof: Use the greedy coloring.

Note: This bound is again often far from tight.

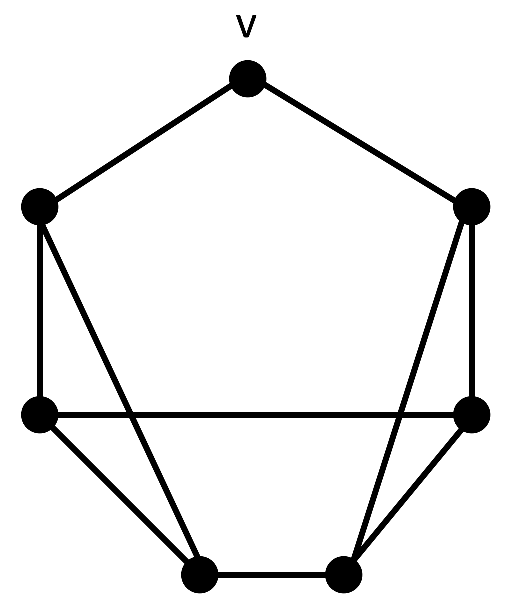
This usually isn't tight

Theorem 1.43 (Brook's Theorem): If G is a finite connected graph that is neither an odd cycle nor a complete graph,
$\chi(G) \leq \Delta(G)$.

Non-Regular Graphs

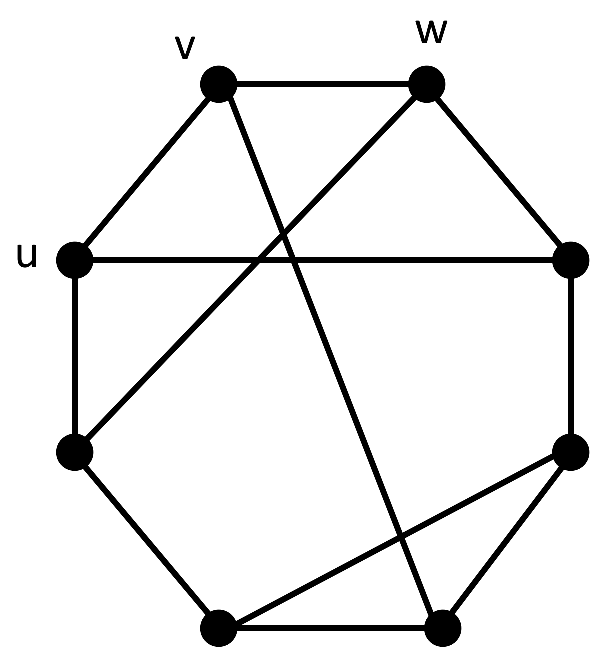
If G not regular

- Some v, d(v) < Δ(G)
- If you can color G-v, greedily color v
- But v's neighbors in G-v have smaller degree
- Recurse



Not a Cut Set

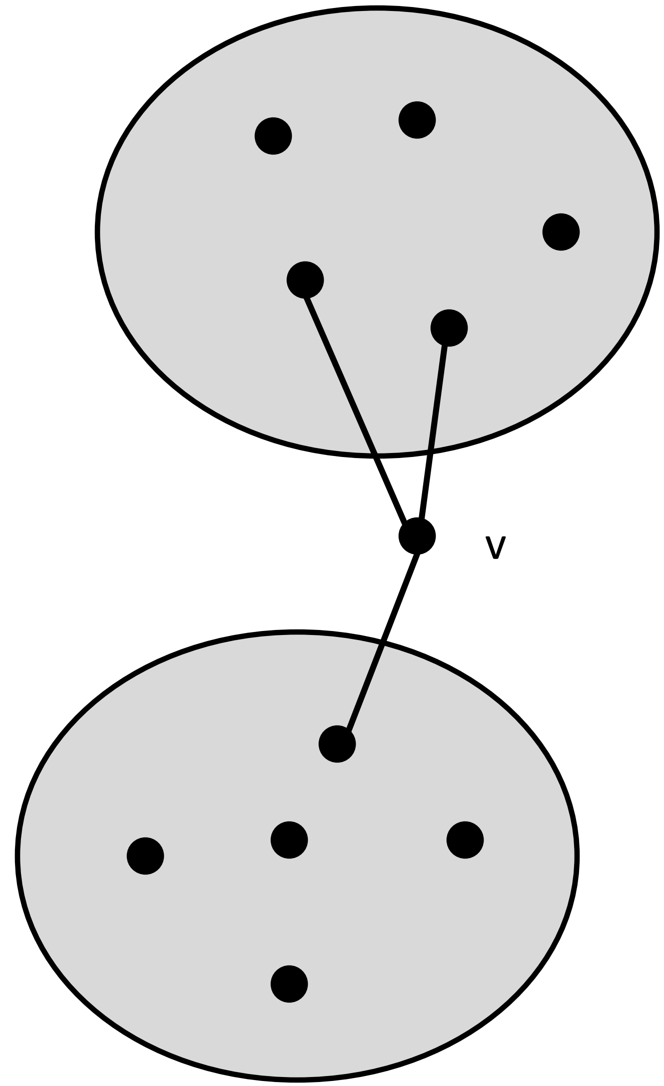
If {u,w} is not a cut set, we can do our recursive coloring, assigning colors to u and w first.



Case 1: Single Cut Vertex

Suppose v is a cut vertex.

- Inductively color each component of G-v.
- For each component can change colors in that component.
- Arrange so two neighbors of v are same color
- Color v



Case 2a

Try to make u, w different colors on both sides.

- Use greedy coloring, choosing colors for u and w last.
- Unless they each connect to $\Delta(G)$-1 vertices on same side, can pick different colors.
- Make colors on top same as on bottom.

Case 2b

Suppose that u, w can only be the same in any coloring of top component.

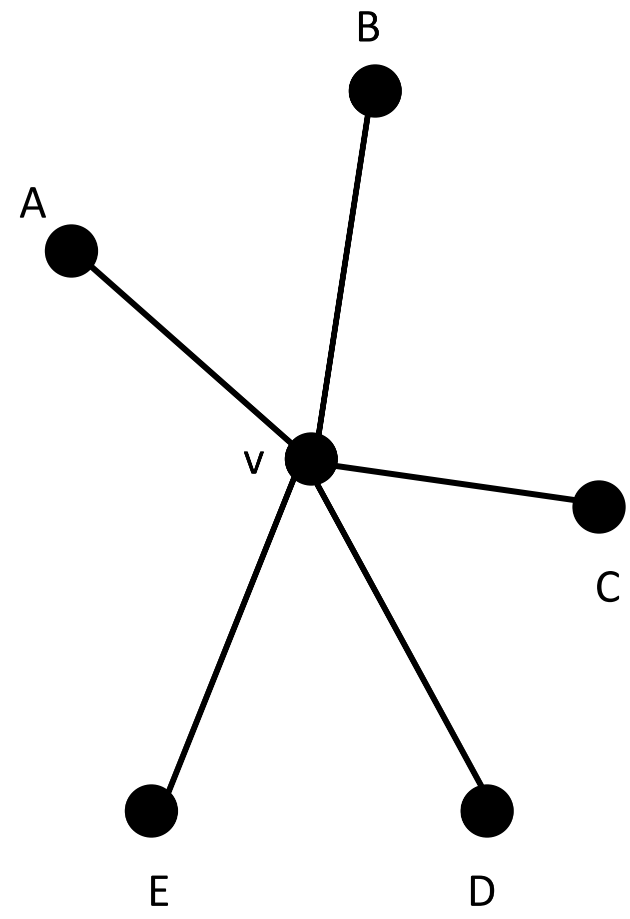
- Must each connect to $\Delta(G)-1$ on top
- Only connect to 1 each on bottom.
- Color bottom component, can pick u, w same color (only 2 disallowed options).

The 5-Color Theorem

Theorem 1.47 (Kemp): Every planar graph is 5-colorable.

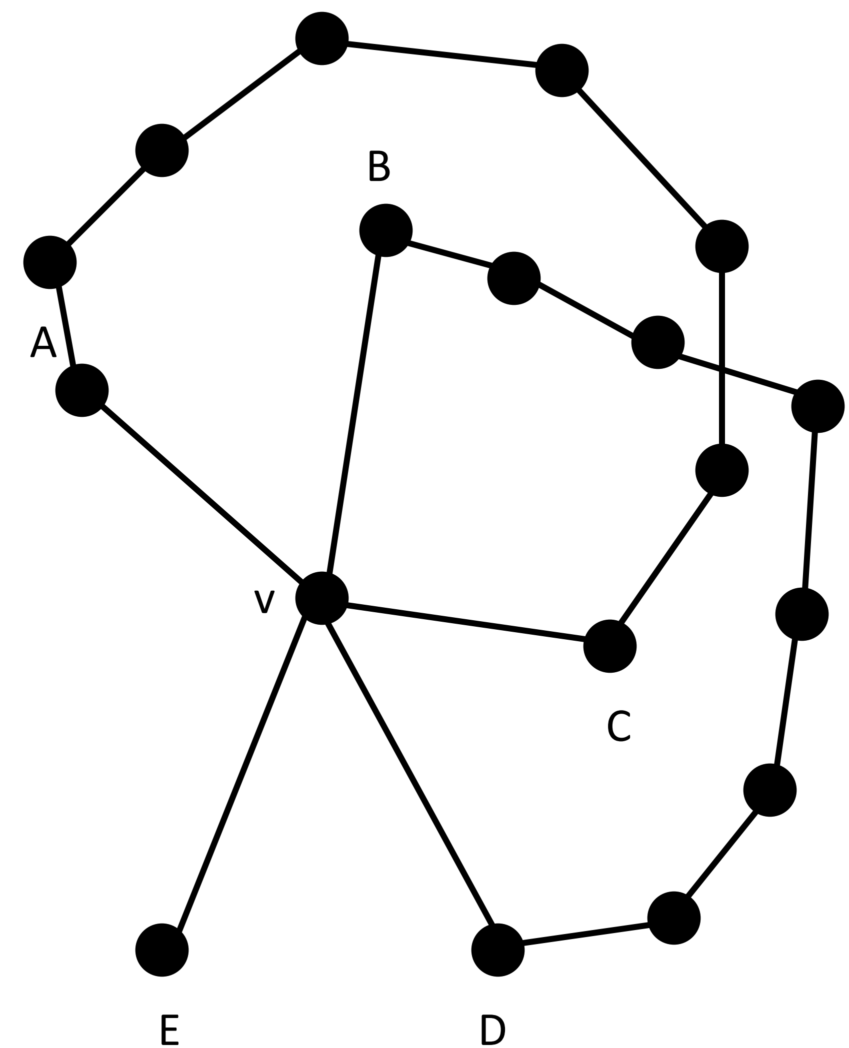
Proof

- Induct on |V|
  – Base case easy
- Take v with d(v) ≤ 5
- Color G-v
  – OK *unless* v's neighbors use all 5 colors
  – Try to recolor them



Kemp Chains

- Can recolor unless red-green chain from A to C
- Can recolor unless blue-yellow chain from B to D
- Cannot have both!
- Always a way to recolor and add v



The Four Color Map Theorem

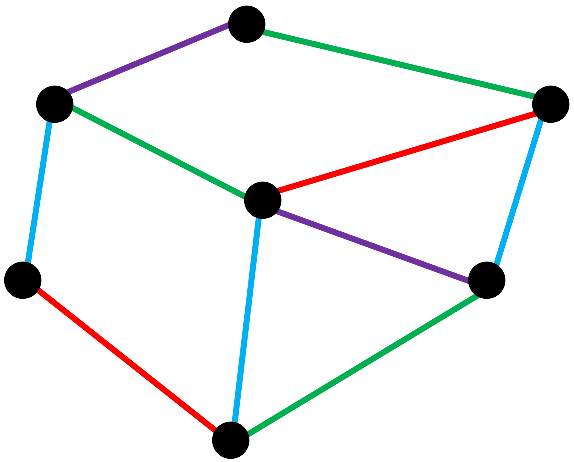
Theorem 1.46: Every planar graph is 4-colorable.

Notes:

- Optimal
- Proof along the same lines as above- add one vertex by recoloring some nearby ones
- Too many cases to check by hand. All known proofs are computer assisted.

Edge Colorings

Definition: An *edge coloring* of a graph is an assignment of a color to each edge so that no two edges incident on the same vertex are the same color.



How Many Colors are Needed

Lemma: Any edge coloring of a graph G requires at least $\Delta(G)$ colors.

Vizing's Theorem

Theorem (V. 6.2.1): Any finite graph G has an edge coloring with at most $\Delta(G)+1$ colors.

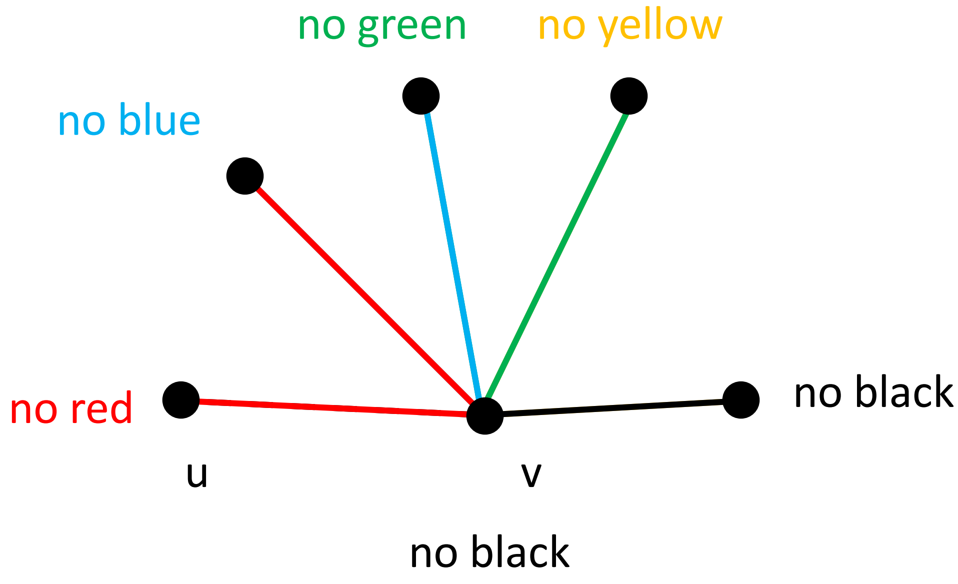
- The minimum number of colors is either $\Delta(G)$ or $\Delta(G)+1$.
- Both are possible. C_n requires 2 colors ($\Delta(G)$) when n is even and 3 colors ($\Delta(G)+1$) when n is odd.

Proof Idea

- Proof by induction on |E|
- Color G-e, show how to insert last edge
- This might require some recoloring of its neighbors

Case 1

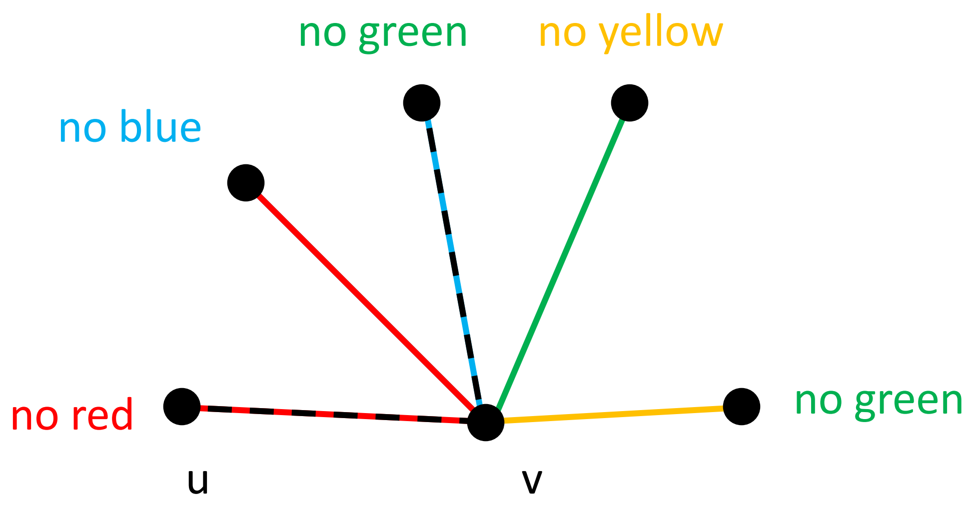
If the chain ends eventually, you can recolor all of the affected edges, inserting the new one.



Case 2

Otherwise, the chain must eventually loop back.

Recolor everything up to the loop.



Recoloring the Cycle

- u, v, w all have degree 1 in H
- One must be in own component
- Recolor that component & add edge