Math 154 Homework Solution

Fall 2021

Solution to Homework 1
Haixiao Wang

This homework is due on gradescope by Friday October 1st at 11:59pm pacific time. Remember to justify your work even if the problem does not explicitly say so. Writing your solutions in \texttt{LaTeX} is recommend though not required.

Please cite any other students with whom you collaborated on any problems.

Question 1 (2-Regular Graphs, 25 points). Show that any finite, 2-regular graph $G$ is a disjoint union of cycles. In particular, show that $G$ has a number of induced subgraphs that are cycles and so that:

1. Each vertex is in exactly one of these induced subgraphs.
2. No edges connect these subgraphs to each other.

**Proof.** Let $G = (V, E)$ be a finite 2-regular graph with $n$ vertices. For any vertex $v_1 \in V$, there are 2 edges incident to $v_1$. Let $e_1$ be one of them. Following $e_1$, we would move to another vertex $v_2 \in V$, which has 2 edges as well since $G$ is 2-regular. Starting at vertex $v_2$, one edge $e_1$ connects it back to $v_1$. The other edge $e_2$ will connect $v_2$ to some other vertex $v_3$. By repeating the process above, we would obtain a path $\{v_1, \cdots, v_n\}$, which must eventually reach a vertex we have seen before. This must be $v_1$ (since any other $v_k$ is adjacent only to $v_{k-1}$ and $v_{k+1}$), which gives one cycle $C_n$, as an induced subgraph of $G$. For any vertex $v_k \in C_n$, if it is connected to an outside vertex $v \in G \setminus C_n$, the degree of $v_k$ would be 3, since $v_k$ has already been adjacent to $v_{k-1}$ and $v_{k+1}$, which violates the 2-regularity assumption. Hence $C_n$ is a connected component of $G$. Repeating this procedure for each connected component of $G$, we find that $G$ is a disjoint union of cycles. \qed

Question 2 (Properties Inherited by Subgraphs, 25 points). For which of the following graph properties $P$ does the following hold: If $G$ is a graph satisfying $P$ and $H$ is an induced subgraph of $G$, then $H$ must also satisfy $P$. For each property, either give a proof or a counter-example.

(a) $G$ is a complete graph [5 points]
(b) $G$ is a bipartite graph [5 points]
(c) $G$ is a cycle [5 points]
(d) $G$ is a simple graph [5 points]
(e) $G$ is a path [5 points]

**Proof.** Remember that a subgraph $H$ is an induced subgraph if it contains all the edges of $G$ connecting two vertices in $V(H)$.

(a) If $G = (V(G), E(G))$ is complete, then for any $v \in V(G)$, $v$ is connected to all $u \in V(G) \setminus \{v\}$. As a result, for any $v \in V(H) \subset V(G)$, $v$ is connected to all $u \in V(H) \setminus \{v\}$. Hence $H = (V(H), E(H))$ is complete.
(b) If \( G = (V(G), E(G)) \) is bipartite, then the vertex set \( V(G) \) can be partitioned into two disjoint subsets \( U(G) \) and \( W(G) \), i.e., \( V(G) = U(G) \cup W(G) \) and \( U(G) \cap W(G) = \emptyset \), such that for any \((u, w) = e \in E(G)\), we have \( u \in U(G) \) and \( w \in W(G) \). Let \( H = (V(H), E(H)) \) be an induced subgraph of \( G \). Let \( U(H) = U(G) \cap V(H) \) and \( W(H) = W(G) \cap V(H) \). For any edge \((u, w)\) of \( H \) must also be an edge of \( G \) and therefore, one of the vertices (say \( u \)) is in \( U(G) \) and the other (\( w \)) in \( W(G) \). However, since both are in \( V(H) \), this means that \( u \in U(H) \) and \( w \in W(H) \). Thus, \( H \) is bipartite.

(c) The induced subgraph \( H \) may not be cycle. The counter-example can be seen in Figure 1.

![Figure 1: The induced subgraph \( H \) of \( C_6 \) is marked red, which is not a cycle.](image)

(d) A simple graph \( G \) has neither multiedges, nor self loops. The induced subgraph \( H \) doesn’t contains any multiedges or self loops. Hence \( H \) is simple.

(e) The counter-example can be seen in Figure 2, where \( H \) has 2 disjoint components.

![Figure 2: The induced subgraph \( H \) of this path is marked red, which is not a path.](image)

Question 3 (Hypergraph Handshake Lemma, 25 points). Suppose that you have a hypergraph where each edge is incident on exactly \( k \) vertices. Formulate and prove a version of the Handshake Lemma for this type of graph.

Proof. Remember that in graph \( G = (V, E) \) the degree of vertex \( v \in V \), denoted by \( d(v) \), is the number of vertices adjacent to \( v \), i.e., the number of edges containing \( v \). A hypergraph \( H = (V, E) \) is called \( k \)-uniform if each edge \( e \in E \) is incident on exactly \( k \) vertices. The degree of \( v \), denoted by \( d(v) \), can be similarly defined as the number of edges containing \( v \). The analogous Handshake Lemma is

\[
\sum_{v \in V} d(v) = k|E|.
\]
To prove this, we are going to count the number of vertex-edge incidence pairs in two different ways. On left hand side, each vertex \( v \) is incident on \( d(v) \) hyper-edges, thus the total number of pairs in this hypergraph is \( \sum_{v \in V} d(v) \). On right hand side, each hyper-edge \( e = (v_1, v_2, \cdots, v_k) \) contains \( k \) incident vertices \( v_1, v_2, \cdots, v_k \), thus the total number of pairs is \( k|E| \).

![Figure 3: An example of 3-uniform hypergraph \( H \) with 9 vertices and 5 hyperedges.](image)

**Question 4** (3-Regular Graphs, 25 points). Show that for every even integer \( n \geq 4 \) that there is a 3-regular graph with exactly \( n \) vertices. What happens if \( n \) is odd?

**Proof.** 1. Let \( n = 2k \) for some \( k \geq 2 \). Consider the cycle \( C_{2k} \) on vertices \( \{1, 2, \cdots, 2k - 1, 2k\} \). The desired graph is obtained by connecting vertex \( i \) and \( i + k \) for all \( 1 \leq i \leq k \). The degree of vertex \( i \) (\( 1 \leq i \leq k \)) is 3 since \( i \) is connected to \( i - 1 \), \( i + 1 \) and \( i + k \). The degree of vertex \( i \) (\( k + 1 \leq i \leq 2k \)) is also 3 since \( i \) is connected to \( i - 1 \), \( i + 1 \) and \( i - k \). Here vertex 1 is connected to \( 2k \).

![Figure 4: Examples for \( n = 4 \) and \( n = 6 \).](image)

2. There is no such graph. The Handshake Lemma, \( \sum_{v \in V} d(v) = 2|E| \), indicates that the sum of degrees should be even, however \( 3n \) is odd when \( n \) is odd.

**Question 5** (Extra credit, 1 point). Approximately how much time did you spend on this homework?

**Solution to Homework 2**