Announcements

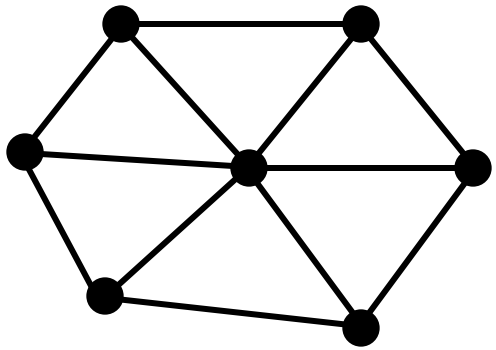
- HW 4 Due Today
- HW 5 Online Due Friday

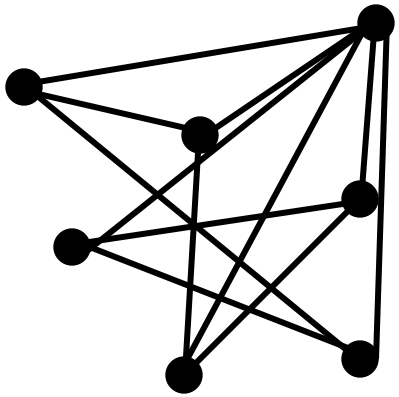
Last Time

- Planarity Definition
- Faces and Euler's Formula
- Platonic Solids

Graph Drawings

- When drawing a graph, it is convenient to draw it so that no two edges cross.
- When is this possible?





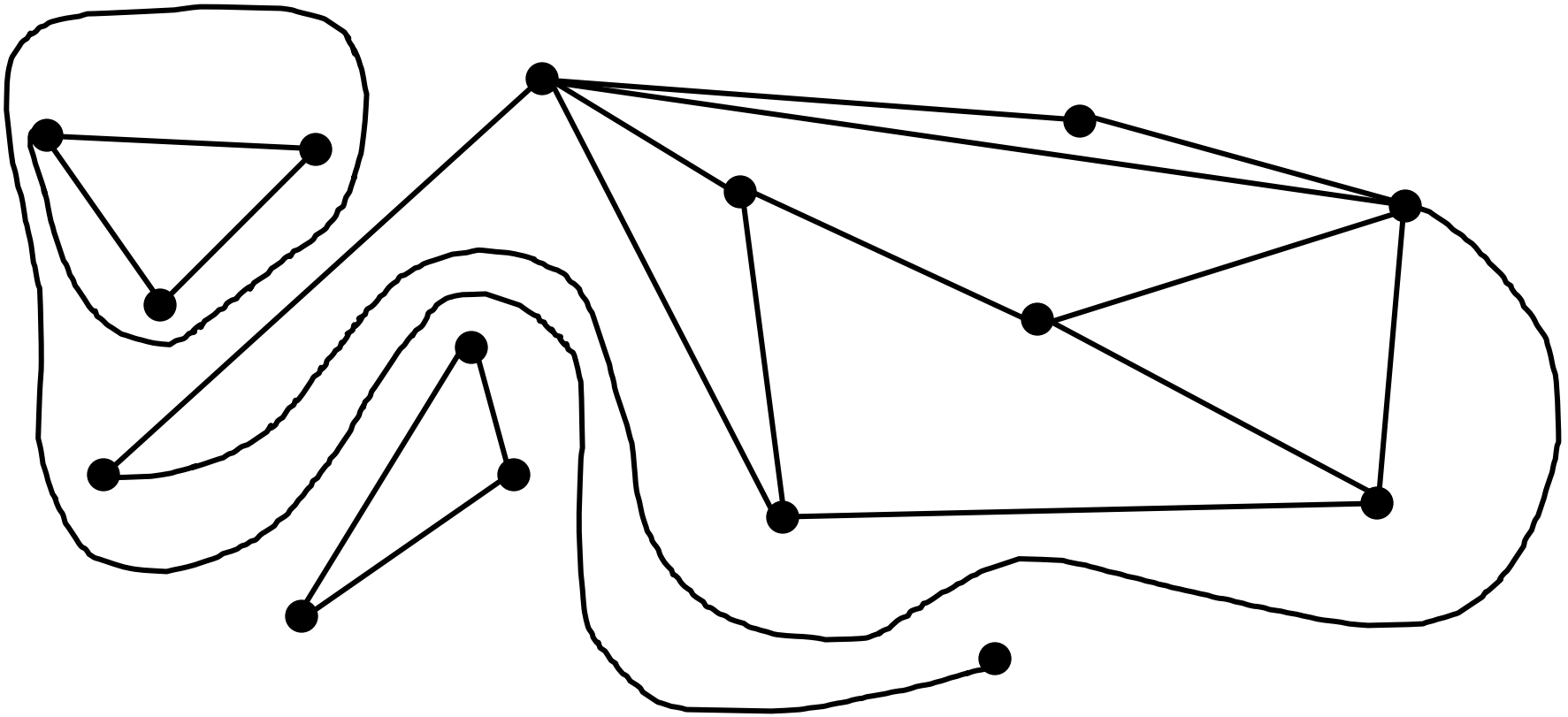
Planar Embeddings

Definition: A *planar embedding* of a graph G is a drawing of G so that

- Each vertex of G corresponds to a point in the plane.
- Each edge of G corresponds to a curve connecting its endpoints.
- No two edge-curves cross except at endpoints.

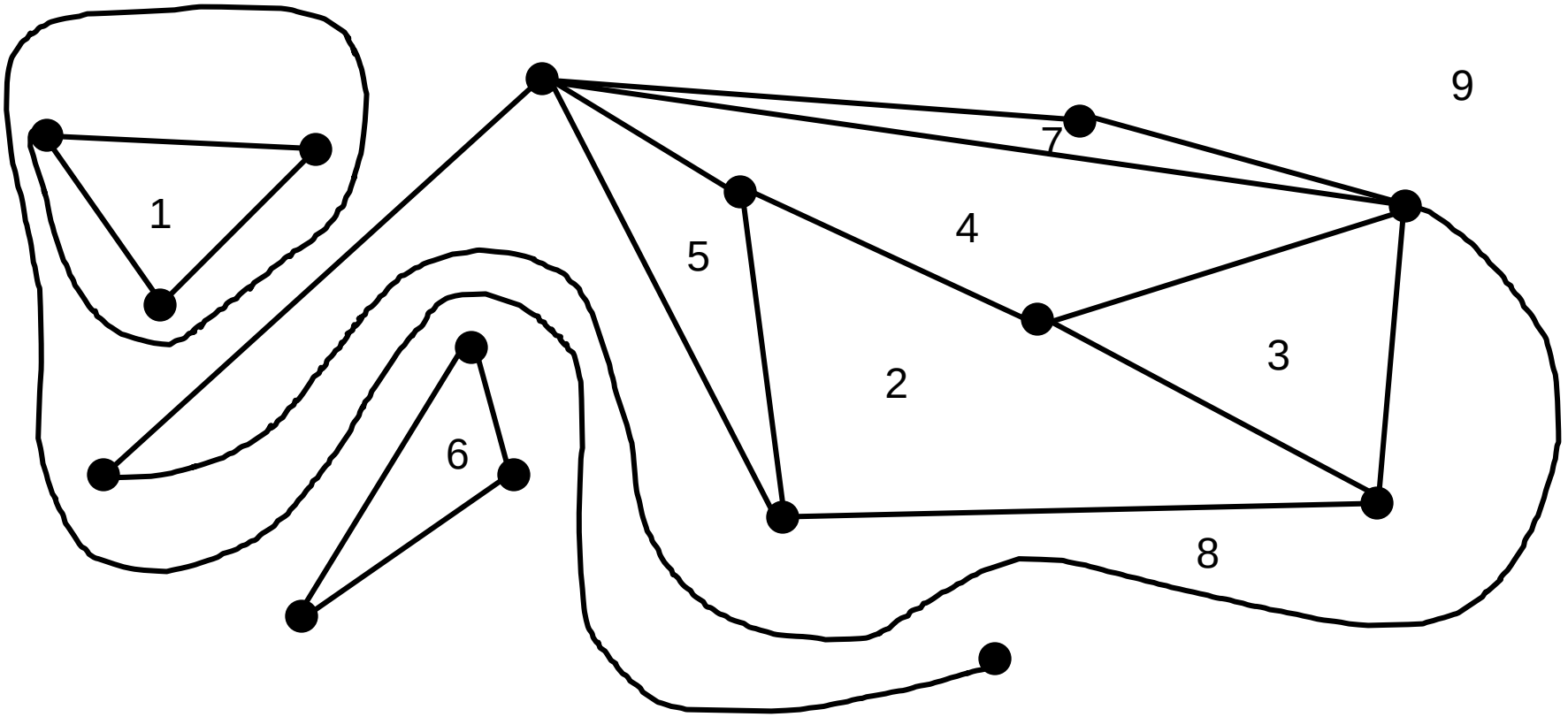
Planar Graphs

Definition: A graph is *planar* if it has a planar embedding.



Faces

A planar embedding of a graph divides the plane into regions. These are called *faces*.



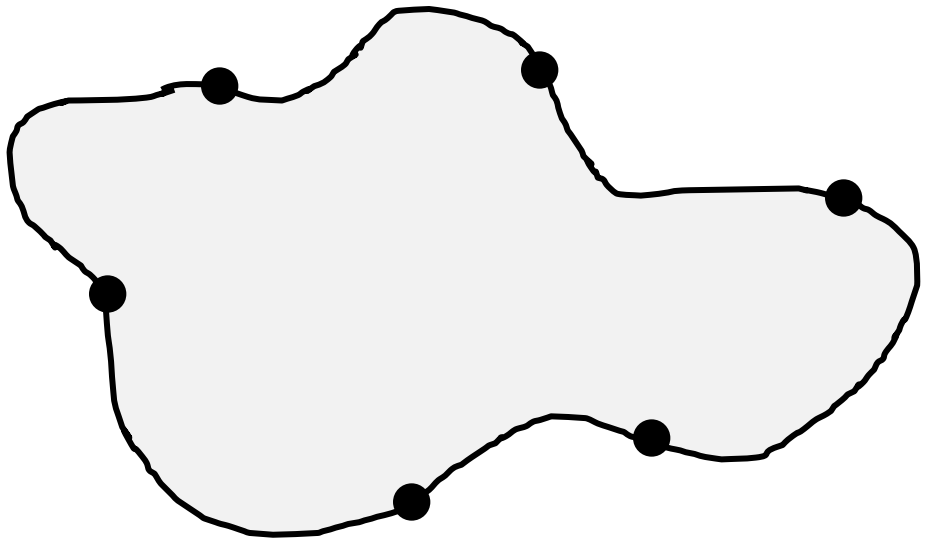
Euler's Formula

Theorem (1.31): For any planar embedding of a connected graph G with v vertices, e edges and f faces (including the infinite face)

$$v - e + f = 2$$

Sides to a Face

If G is a connected planar graph, any face (including the infinite one) will be bounded by a loop of edges.



The number of *sides* of the face is the number of edges in this loop.

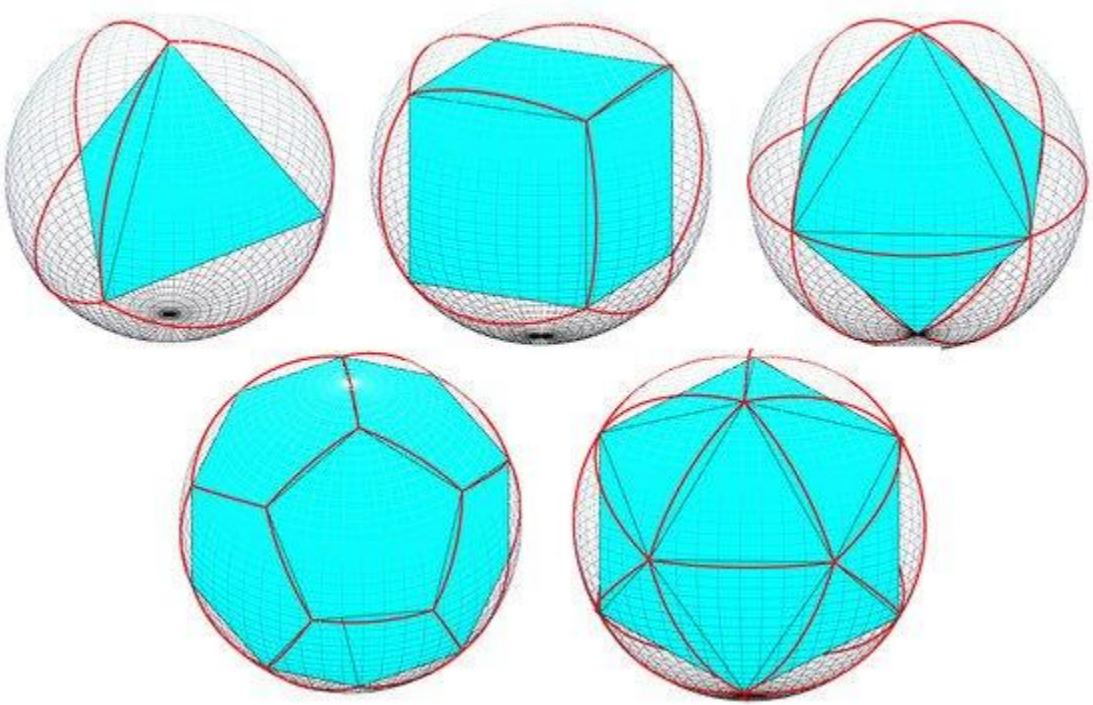
Polyhedra

Definition: A *polyhedron* is a 3 dimensional figure bounded by finitely many flat *faces*. Two faces meet at an *edge* and edges meet at *vertices*.

A polyhedron is *convex* if for any two points in the polyhedron the line segment connecting them is also contained in the polyhedron.

Polyhedral Graphs

Given a convex polyhedron, can turn it into a planar graph by projecting vertices/edges onto a sphere (which can then be flattened onto a plane)



Regular Polyhedra

A regular polyhedron is a highly symmetric polyhedron (like a cube). In particular, it has the following properties:

- All edges are the same length.
- All faces are regular polygons with the same number – s of sides.
- The same number of faces, d, meet at each vertex.

Today

- Dual Handshake Lemma
- Applications of Euler's Formula
  - Platonic Solids
  - Edge Bounds
  - Non-planarity of K_5 and $K_{3,3}$
- Fary's Theorem

Face Bounds

To really make use of Euler's Formula, it is important to get an idea of how many faces there are.

There is a way of counting these that is somewhat dual to the Handshake Lemma.

Dual Handshake Lemma

Lemma: For a connected, planar graph,

$$\sum_{\text{Faces } f} \text{Sides}(f) = 2|E|.$$

- Note similarity to Handshake Lemma. Sides of faces instead of degrees of vertices.
- Proof similar.

Proof

- Count the number of pairs of an edge as a side of a face (be careful to count edges that are double sides twice).
- Each edge on two faces (or one face twice).
- Each face f has Sides(f) edges.

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- The same number of faces, d, meet at each vertex.

Counting

Note that each vertex has degree-d. By the Handshake Lemma:

$$2e = \Sigma\, d(v) = dv.$$

By the dual handshake lemma:

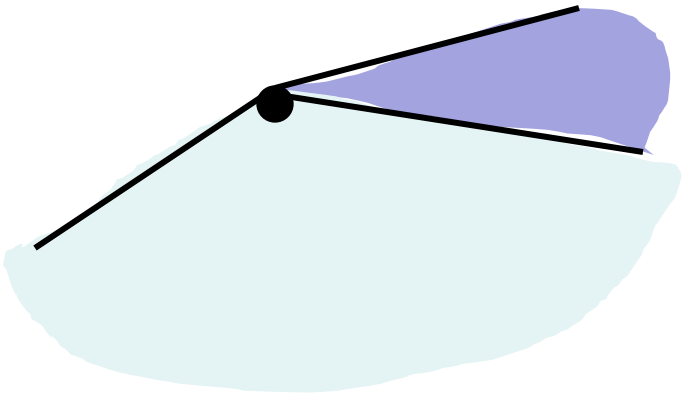
$$2e = \Sigma\, s(f) = sf.$$

By Euler's Formula:

$$2 = v - e + f = e(2/d+2/s-1).$$

Degrees

Note that in any polyhedron, each vertex has degree at least 3.



Counting Continued

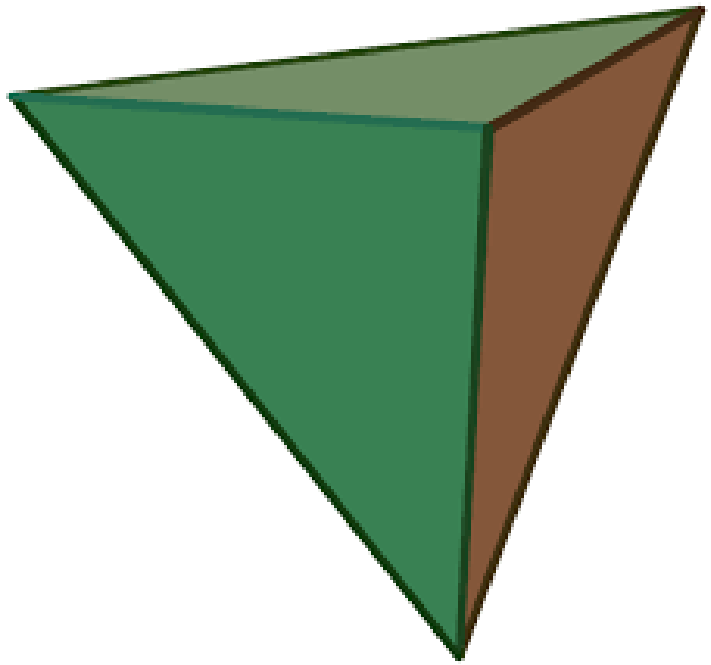
Therefore, we must have:

- $d, s \geq 3$
- $2/d + 2/s > 1$
- $e = 2/(2/d+2/s-1)$
- $v = 2e/d$
- $f = 2e/s$

Case 1: d = s = 3

v = 4, e = 6, f = 4

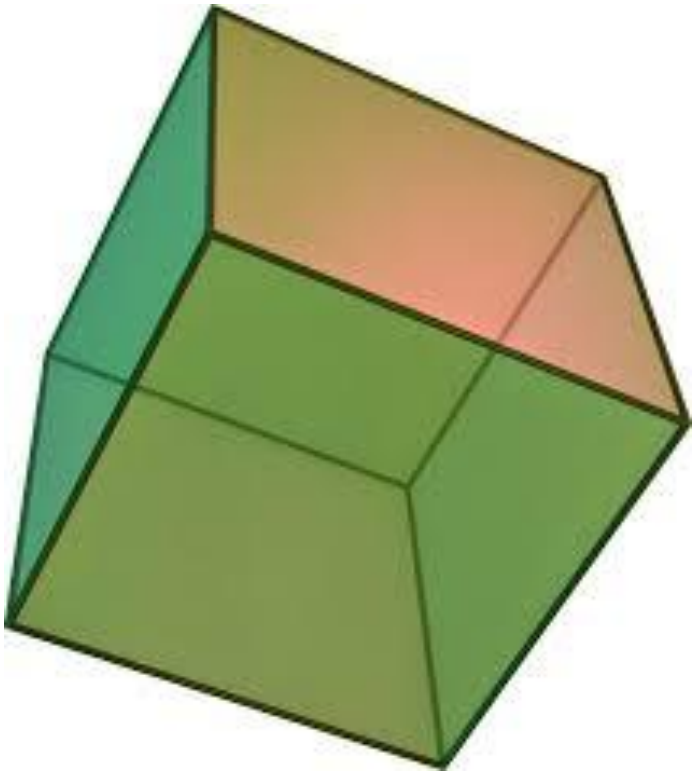
Regular tetrahedron



Case 2: d = 3, s = 4

v = 8, e = 12, f = 6

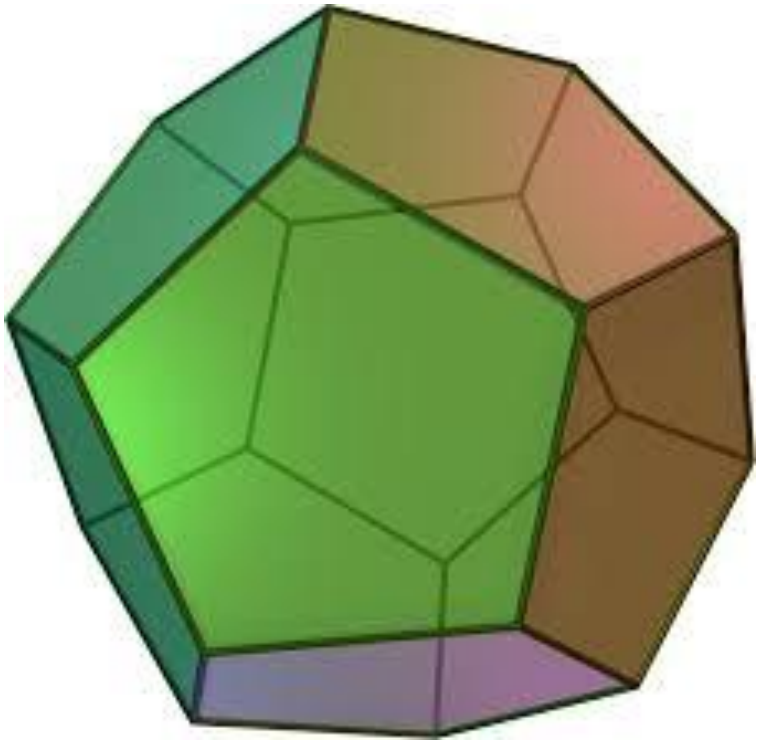
Cube



Case 3: d = 3, s = 5

v = 20, e = 30, f = 12

Regular dodecahedron



$d = 3,\ s \geq 6$

Doesn’t work!

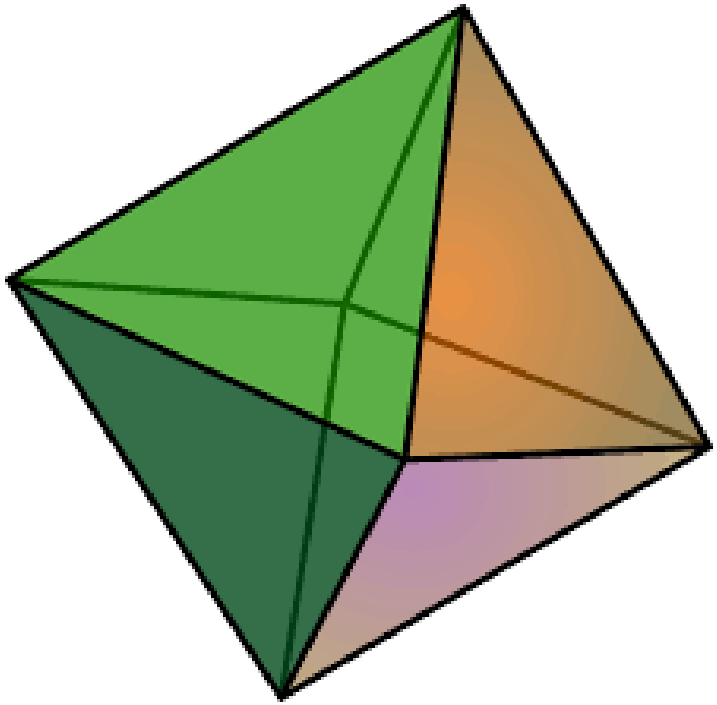
$2/d + 2/s \leq 2/3 + 2/6 = 1.$

3 hexagons at a vertex tessellate.

Case 4: d = 4, s = 3

v = 6, e = 12, f = 8

Regular Octahedron



$d = 4, s \geq 4$

Doesn't work!

$2/d + 2/s \leq 2/4 + 2/4 = 1.$

4 squares at a vertex tessellate.

Case 5: d = 5, s = 3

v = 12, e = 30, f = 20

Regular icosahedron



$d = 5, s \geq 4$

Doesn't work!

$2/d + 2/s \leq 2/5 + 2/4 = 9/10 < 1.$

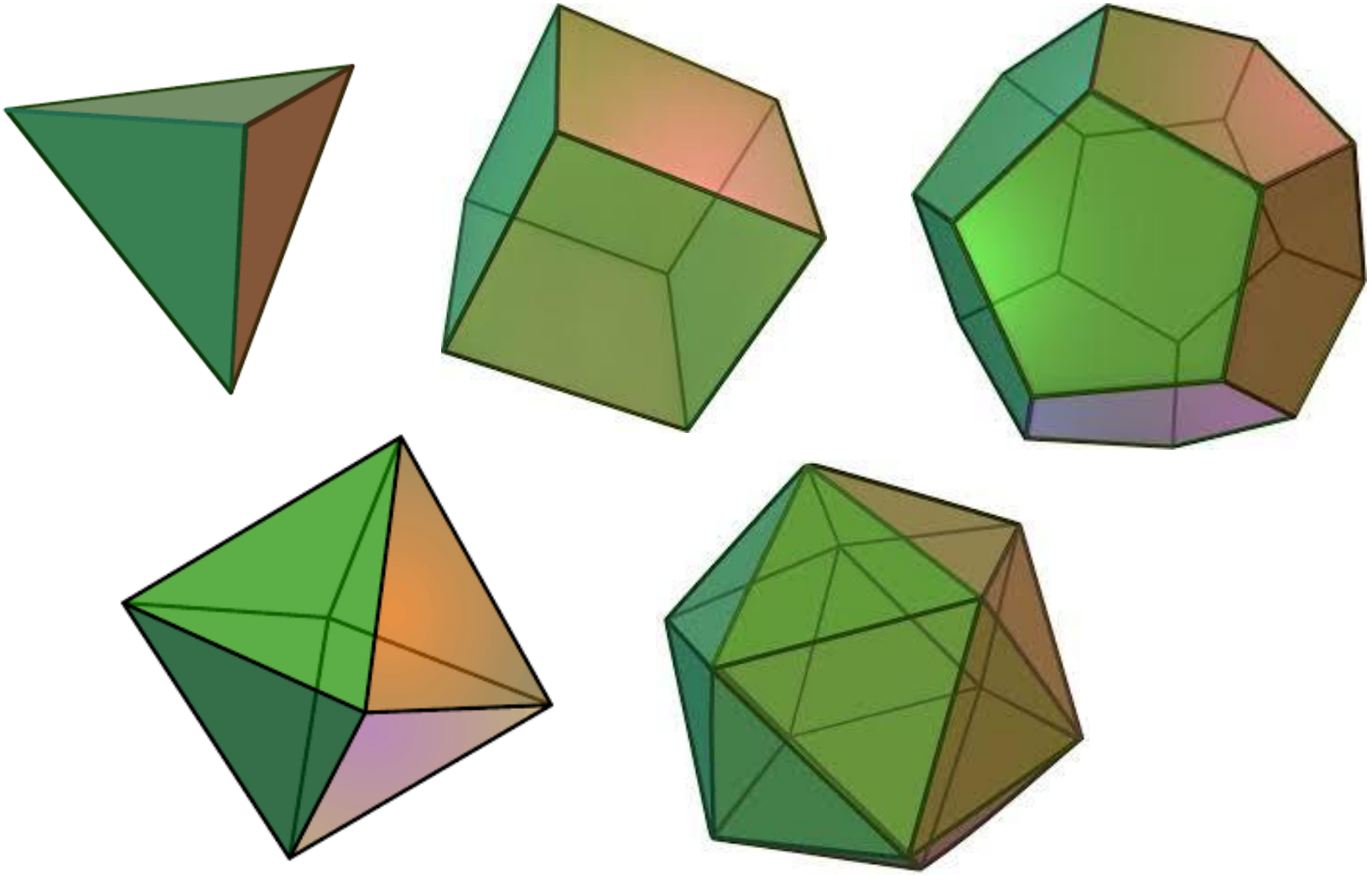
$d \geq 6,\ s \geq 3$

Doesn’t work!

$2/d + 2/s \leq 2/6 + 2/3 = 1.$

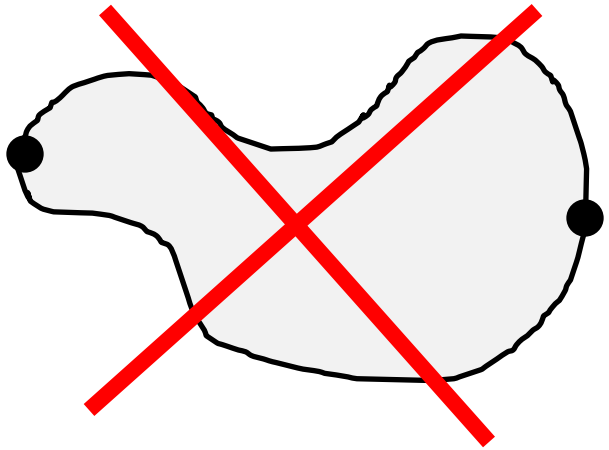
6 triangles at a vertex tessellate.

Only 5 Platonic Solids



A Key Observation

Every face has *at least three* sides.



(unless |V| =2)

$$2e = \sum_{\text{Faces } f} \text{Sides}(f) \geq 3f.$$

More Generally,

If G only has faces with at least k sides then

$$e \geq kf/2.$$

Edge Bound

Theorem (1.33): If G is a connected planar graph with $|V| \geq 3$, then

$$|E| \leq 3|V| - 6.$$

Proof

We know:

- $v - e + f = 2$.
- $e \geq 3f/2$.

So:

$$2 = v - e + f \leq v - e + 2e/3 = v - e/3$$

Rearranging, we find:

$$e \leq 3v-6.$$

Question: Side Bound

If each face has at least k sides, what is the maximum number of edges?

A)v/(1-2/k)-6

B)k(v-2)

C)(v-2)/(1-2/k)

D)3v-6

$$2 = v - e + f$$
$$\leq v - e + 2e/k$$
$$= v - e(1-2/k)$$

K_5 Non-Planar

Theorem (1.34): The K_5 is non-planar.

Proof: If it were, we would have

$$e \leq 3v-6 = 9$$

But $e = 10$. Contradiction!

$K_{3,3}$ Non-Planar

Theorem (1.32): $K_{3,3}$ is non-planar.

Proof: $K_{3,3}$ is bipartite, so it has no odd cycles. Therefore, if planar any face has at least 4 sides.

If planar,

$$e \leq 2v-4 = 8.$$

But e=9. Contradiction!

Minimum Degree

Theorem (1.35): If G is a finite, connected planar graph, its vertices have minimum degree at most 5.

Proof: Otherwise, each vertex has degree 6 or more.

Handshake Lemma implies

$$2e = \Sigma\, d(v) \geq 6v.$$

But then

$$3v - 6 \geq e \geq 3v.$$

Contradiction!

Edges and Vertices

If each face of a planar graph had at least k edges then

$$e \leq (v-2)/(1-2/k).$$

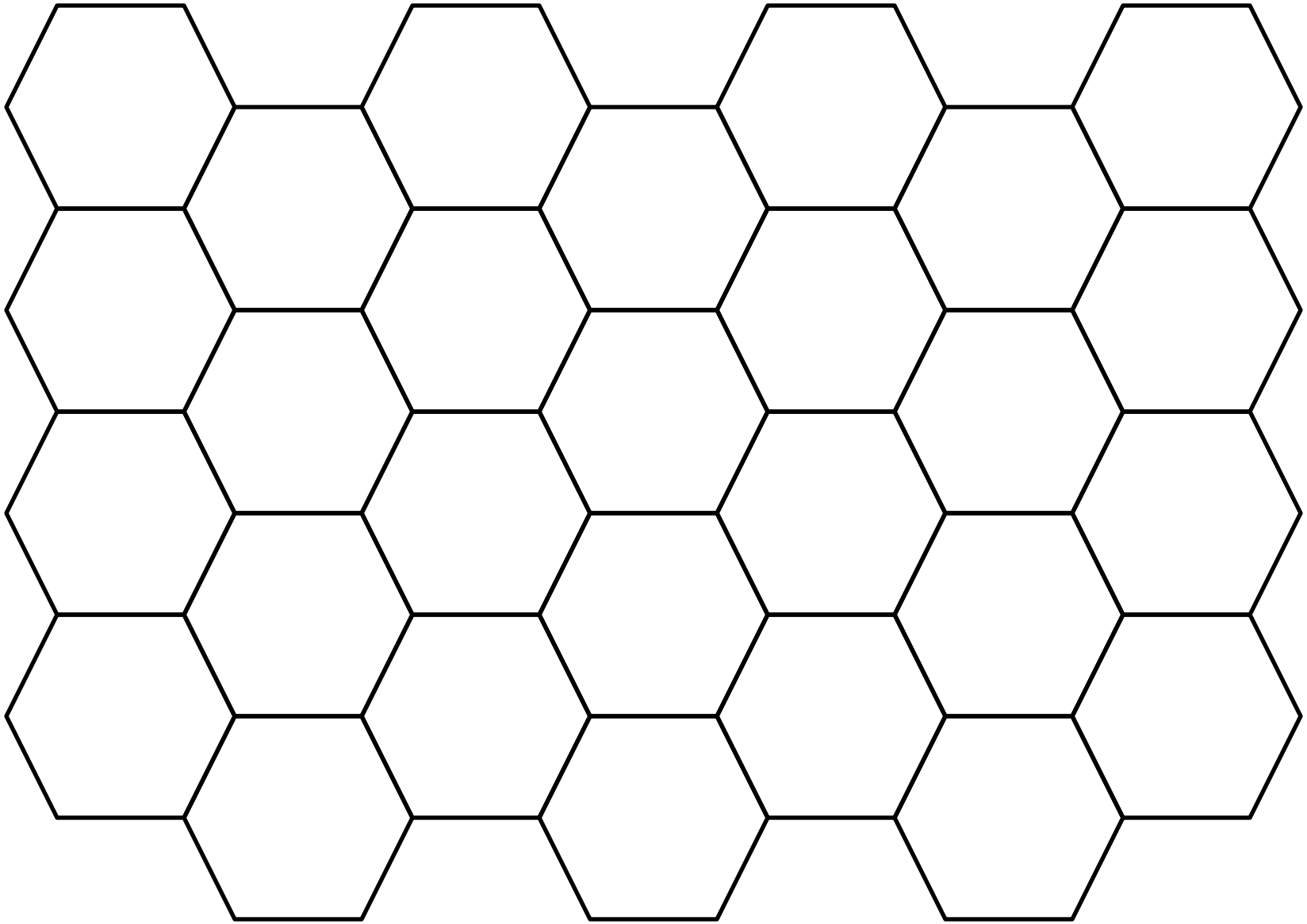
If k = 6, we have that

$$3v/2 \leq e \leq (v-2)/(2/3) = 3(v-2)/2.$$

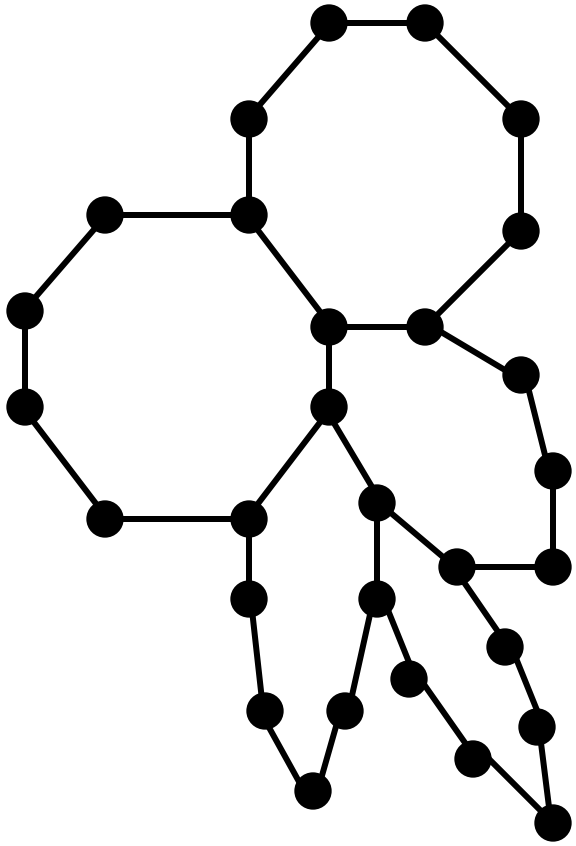
Contradiction!

Corollary: Every polyhedron has a face with at most 5 sides.

Discussion



Worse with More Sides



Triangulations

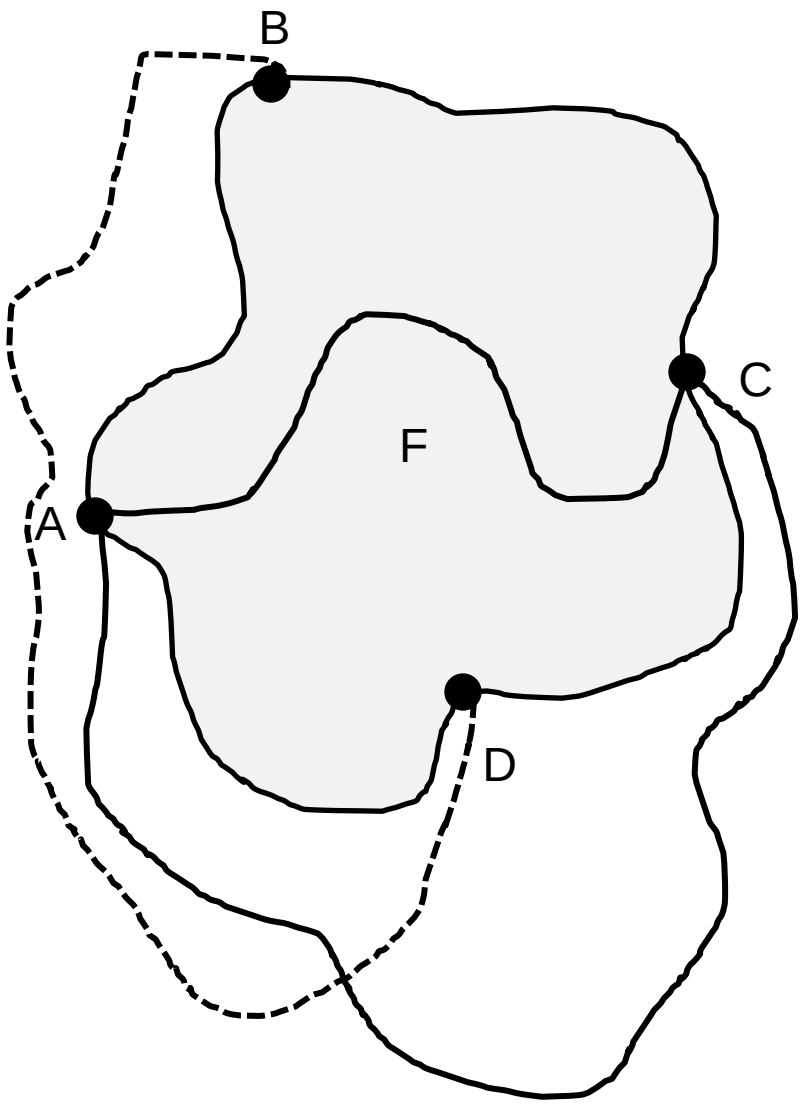
We note that our edge bound of 3v-6 has an equality case if and only if all faces are triangles.

We can always ensure that this is the case if we add more edges.

Lemma: For any planar embedding of a graph G there is a way to add more edges to G to get a new planar graph G' in which all faces are triangles.

Proof

- Add edges until triangulated.
- Consider a face F with at least 4 sides.
- Add edge between non-adjacent vertices.
- Cannot if already outside edge.
- Cannot have on both ends.



Fary's Theorem

Theorem (V. 7.4.2): Any finite (simple) planar graph G has a plane embedding where all of the edges are straight line segments.