Announcements

- HW 5 Due F

Last Time

- Dual Handshake Lemma
- Minimum Degree
- Non-Planarity Results (K_5, $K_{3,3}$)
- Triangulations

Dual Handshake Lemma

Lemma: For a connected, planar graph,

$$\sum_{\text{Faces } f} \text{Sides}(f) = 2|E|.$$

Edge Bound

Theorem (1.33): If G is a connected planar graph with $|V| \geq 3$, then

$$|E| \leq 3|V| - 6.$$

In fact if each face has at least k sides,

$$|E| \leq (v-2)/(1-2/k).$$

K_5 Non-Planar

Theorem (1.34): The K_5 is non-planar.

Proof: If it were, we would have

$$e \leq 3v-6 = 9$$

But $e = 10$. Contradiction!

$K_{3,3}$ Non-Planar

Theorem (1.32): $K_{3,3}$ is non-planar.

Proof: $K_{3,3}$ is bipartite, so it has no odd cycles. Therefore, if planar any face has at least 4 sides.

If planar,

$$e \leq 2v-4 = 8.$$

But e=9. Contradiction!

Minimum Degree

Theorem (1.35): If G is a finite, connected planar graph, its vertices have minimum degree at most 5.

Proof: Otherwise, each vertex has degree 6 or more.

Handshake Lemma implies

$$2e = \Sigma\, d(v) \geq 6v.$$

But then

$$3v - 6 \geq e \geq 3v.$$

Contradiction!

Triangulations

Lemma: For any planar embedding of a graph G there is a way to add more edges to G to get a new planar graph G' in which all faces are triangles.

Today

- Proof of Farey's Theorem
- Classification of planar graphs and Kuratowski's Theorem
- Introduction to coloring problems

Fary's Theorem

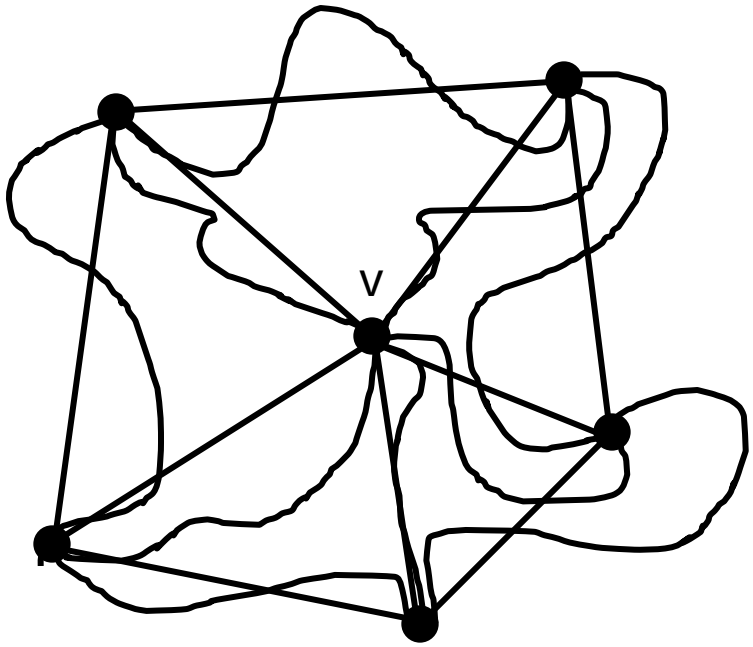
Theorem (V. 7.4.2): Any finite (simple) planar graph G has a plane embedding where all of the edges are straight line segments.

Proof Strategy

- Induct on v.
  - If $v \leq 3$, easy to draw.
- Assume G is connected (ow/ draw each component separately)
- Find a vertex v of low degree.
- Draw G-v with straight lines.
- Re-insert v into drawing.

Idea

- G triangulated and $d(v) \leq 5$
- Draw G-v w/ lines.
- Find a place for v with straight lines to neighbors.



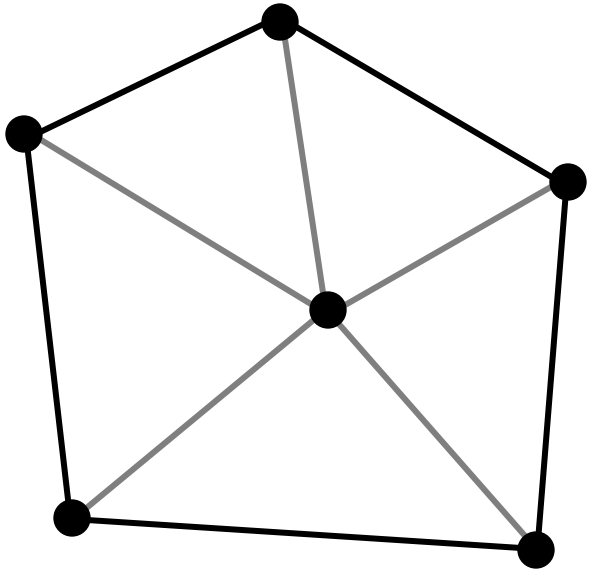
Lemma

Lemma: Given any polygon P in the plane with at most 5 sides, there is a point v inside of P with straight line paths to each of P's vertices.

Idea: Consider cases based on the locations of the non-convex (i.e. > 180 degree) angles.

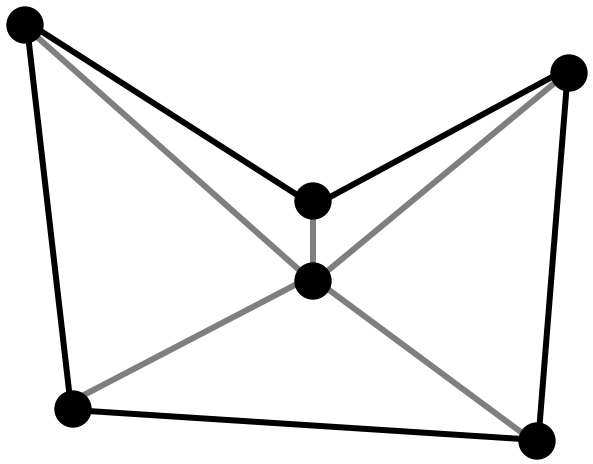
Case 1: No non-convex angles

Any point on the interior works.



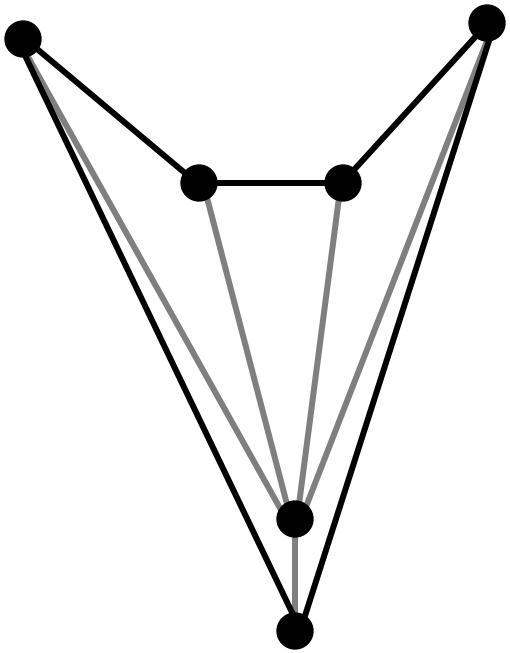
Case 2: One non-convex angle

Place v just inside the non-convex angle.



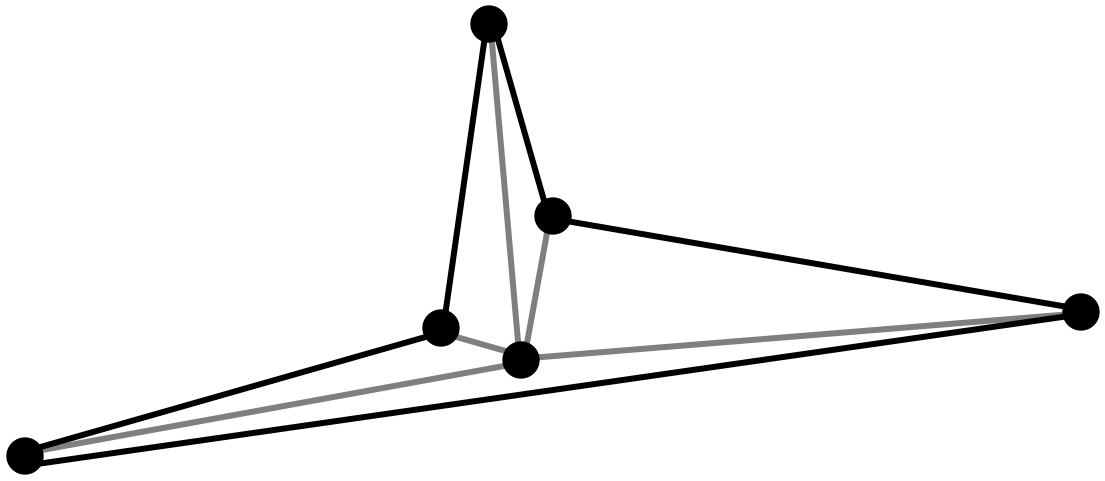
Case 3: Two adjacent non-convex angles

Place v near opposite vertex.



Case 4: Two non-adjacent non-convex angles

Place v near the far wall where it can see the hidden vertex.



Case 5: More than two non-convex angles

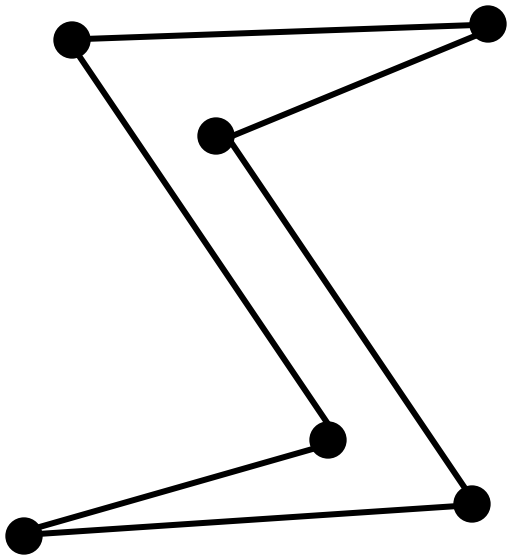
Not possible! Sum of angles is 180(n-2) degrees for an n-gon.

Question: Hexagons?

Is this lemma true for hexagons?

A)Yes

B)No



Proof of Fary's Theorem

- G connected, planar graph.
- If $v \leq 3$ can draw.
- Triangulate G.

Claim: Any such G will have an *interior* vertex of degree at most 5.

Proof of Claim

Handshake Lemma:

$$\Sigma\, d(v) = 2e = 6v - 12$$

Rearrange:

$$\Sigma\, (6\text{-}d(v)) = 12$$

The three boundary vertices each contribute at most 4 each. They cannot all contribute 4 unless they do no connect to anything else. So *some* other vertex must have 6-d(v) positive.

Low Degree v

- By Inductive Hypothesis, can draw G-v with straight lines.
- Neighborhood of v a polygon with at most 5 sides.
- By Lemma, can insert v somewhere where it has straight line paths to its neighbors.
- Using this location we can draw G with straight line edges.

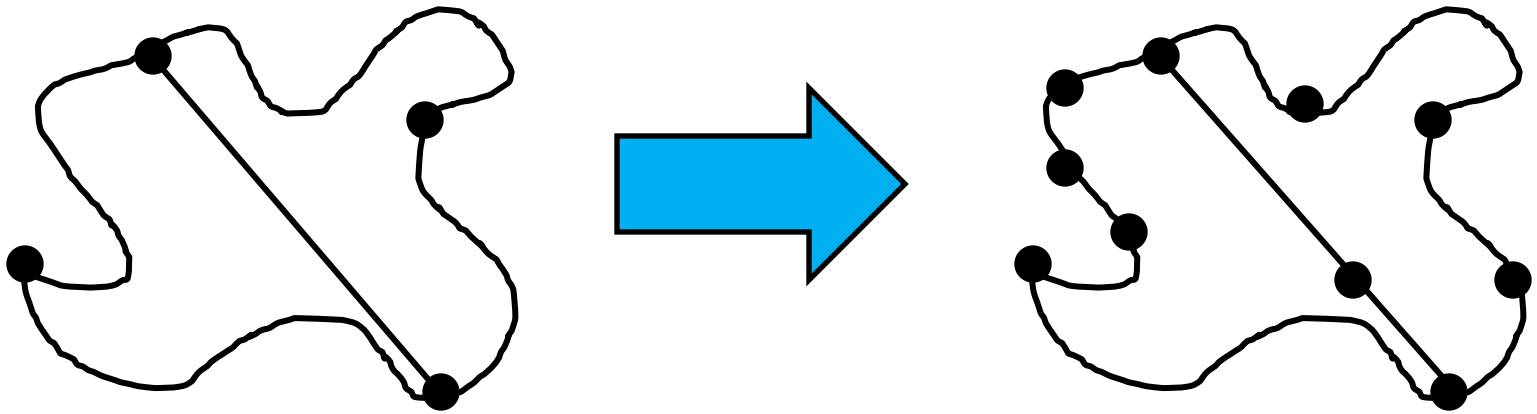
Which Graphs are Planar?

We know that K_5 and $K_{3,3}$ are *not* planar, and that polyhedral graphs and trees are, but can we come up with a complete characterization of which graphs are and are not planar?

Subdivisions

Here's one thing that doesn't much affect planarity:

Definition: A *subdivision* of a graph G is obtained by placing vertices in the middle of some of its edges.



Subdivisions II

Lemma: If G' is a subdivision of G, then G' is planar if and only if G is.

Proof:

- Given a plane embedding of G add vertices in the middle of edges to get embedding of G'.
- Given embedding of G', remove vertices and join edges, to get embedding of G.

Non-Planar Graphs

Graphs that are non-planar:

- K_5 and $K_{3,3}$.
- Any subdivision of K_5 and $K_{3,3}$.
- Any graph that contains a subdivision of K_5 and $K_{3,3}$ as a subgraph.
- Nothing else!

Kuratowski's Theorem

Theorem (V. 7.0.1): A finite graph G is planar if and only if it has no subdivision of a K_5 or $K_{3,3}$ as a subgraph.

Reduce to Single Connected Component

- G planar if and only if each component is
  - One component not planar neither is G
  - Each component planar draw them separated
- G has K_5 or $K_{3,3}$ if and only if some component does
  - Component has subgraph so does G
  - Each K_5 or $K_{3,3}$ is connected, must be in connected component

Reduce to Single Block

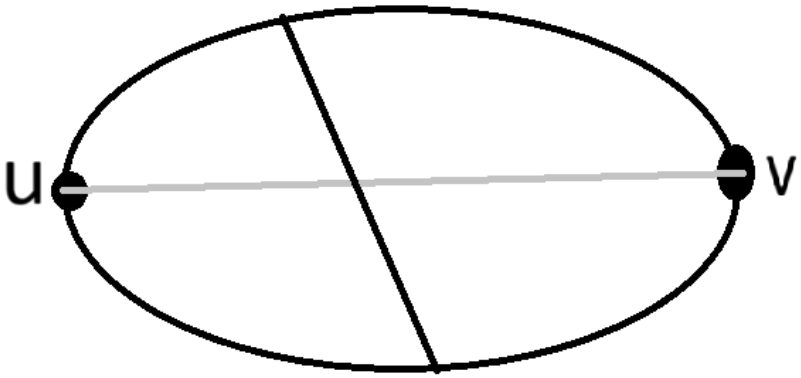
- G planar if and only if each block is
  - One block not planar neither is G
  - Each block planar draw them one at a time connecting at cut vertices
    - Block graph is a tree
- G has K_5 or $K_{3,3}$ if and only if some block does
  - Block has subgraph so does G
  - Each K_5 or $K_{3,3}$ is 1-connected, must be in block

Induct on Ears

- G has an ear decomposition
- G = G’ + ear
- Show that if Kuratowski holds for G’, it holds for G
- Suppose G’ is planar have an ear connecting u to v, is G planar?

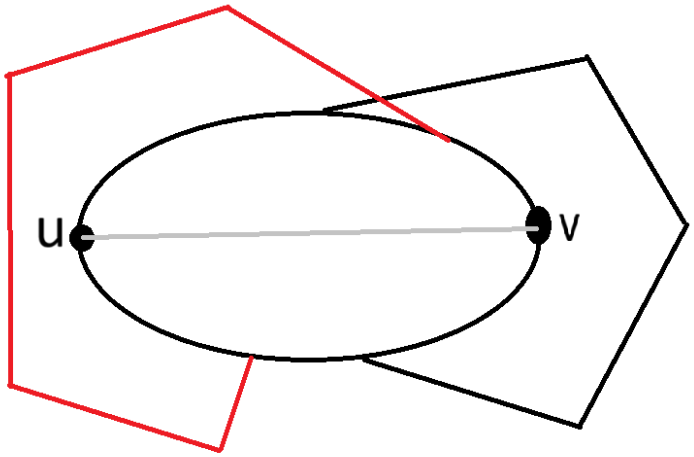
Adding an Ear

- u, v in cycle C.
- Try to add connection along inside
- Can do unless some other path separates them.



Adding an ear

- Can draw the path outside of C
- Unless two paths cross
- But that gives $K_{3,3}$
- Some extra case analysis



Coloring Problems (Ch 1.6)

- Introduction and Definitions
- Basic Results
- Brooke's Theorem
- Colorings of Planar Graphs
- Edge Colorings

Bipartite Graphs

Recall, one way to think about bipartite graphs was to color the vertices black and white so that no edge connected two vertices of the same color.

What if we allow more colors?

Colorings

Definition: A (vertex) *coloring* of a graph G is an assignment of a color to each vertex of G so that no two adjacent vertices have the same color. This is an *n-coloring* if only n different colors are used.

Definition: The *Chromatic Number*, $\chi(G)$, of a graph G is the smallest number n so that G has an n-coloring.

Basic Facts

- A graph has $\chi(G) = 1$ if and only if G has no edges.
- A graph has $\chi(G) \leq 2$ if and only if G is bipartite.
- Determining $\chi(G)$ for more complicated graphs is difficult. For 2-colorings once you color a vertex, there is only one possible choice for its neighbors. For 3-colorings, you have 2.