Ramsey Theory (Ch 1.8)

- Introduction
- Definitions
- Existence of Ramsey Numbers
- Derivation of Small Ramsey Numbers
- Lower Bounds
- Other Ramsey Problems

A Party

Suppose you are at a party with at least 6 people in it. Show that there are either 3 people who all know each other or 3 people who are all strangers to each other.

Proof

Pick someone at the party, say Alice.

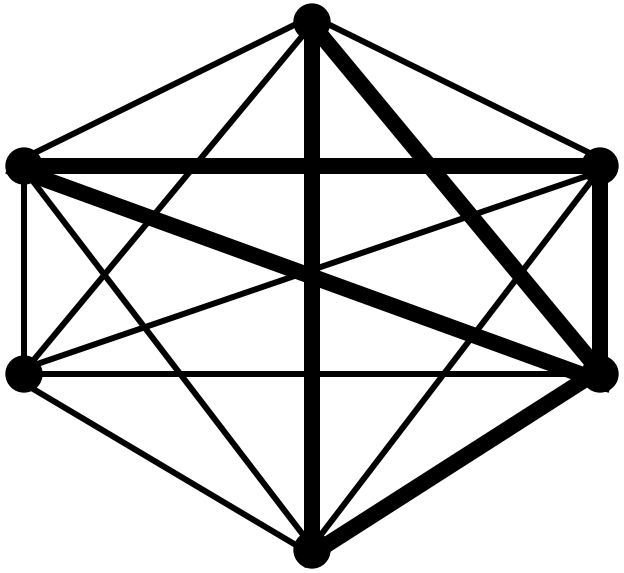
There are 5 other people so, she either knows at least 3 of them or doesn't know at least 3.

Say she knows Bob, Charles and Diane.

- If any two of Bob, Charles or Diane know each other, we have three mutual friends.
- If no two know each other, we have three mutual strangers.

Graph Theory

- 6 people
- Every pair either friends or strangers
- Three friends or three strangers.



Friends
Strangers

Theorem

Theorem: Given any coloring of the edges of a K_6 with two colors, there is always a monochromatic K_3 subgraph.

Ramsey's Theorem

Theorem: For any positive integers p and q there exists a number N so that for any $n \geq N$ and any red-blue coloring of the edges of a K_n, there is either a red K_p or a blue K_q.

Definition: The smallest such number N is called the *Ramsey Number*, R(p,q).

Proof I

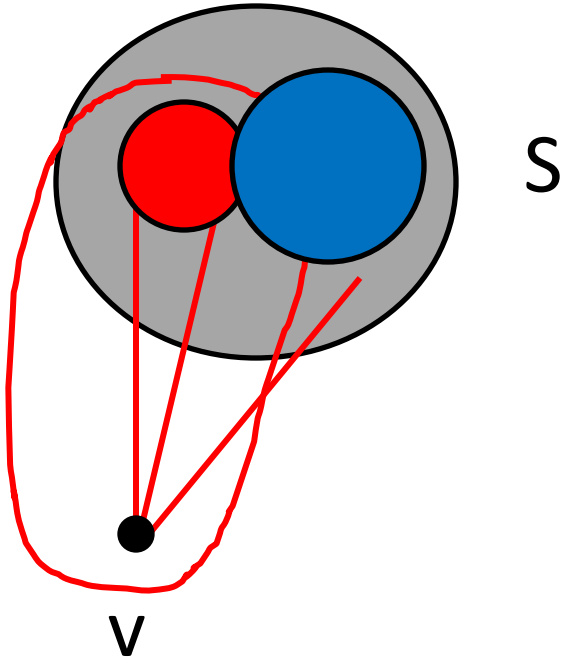
- We proceed by induction on p+q.
- Base case: If p = 1 or q = 1, then any coloring of a K_1, has a red/blue K_1. So R(1,q) = R(p,1) = 1.
- Inductive step: Assume R(p',q') is finite for all p'+q' < p+q.
- For p, q > 1, we will show not only that R(p,q) is finite, but that
  $R(p,q) \leq R(p-1,q)+R(p,q-1)$.

Proof II

- Take $n \geq R(p-1,q)+R(p,q-1)$. Color K_n.
- Consider vertex v, has at least $R(p-1,q)+R(p,q-1)-1$ edges out of it.
- Either has $R(p-1,q)$ red edges or $R(p,q-1)$ blue edges.
- WLOG v has $R(p-1,q)$ red edges to other vertices S.

Proof III

- Have v, S
  - All edge v to S red.
  - $|S| \geq R(p-1,q)$.
- IH: Color of S has either:
  - Red K_{p-1}.
    - v + clique is red K_p
  - Blue K_q
    - Gives blue K_q
- Either way we're done.



Ramsey Theory

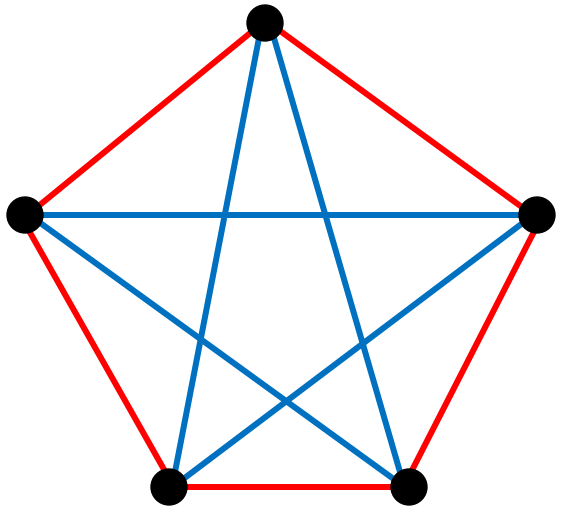
Ramsey Theory more generally studies these kinds of patterns and when certain types of structures *must* exist within sufficiently much noise.

Computing Ramsey Numbers

- R(p,q) = R(q,p) by symmetry.
- R(1,n) = 1 as discussed.
- R(2,n) = n
  - Any red edge gives red K_2.
  - All blue edges gives blue K_n.
  - But coloring a K_{n-1} all blue gives neither.

R(3,3) = 6

- $R(3,3) \leq R(2,3)+R(3,2) = 3+3 = 6$.



R(3,4) = 9

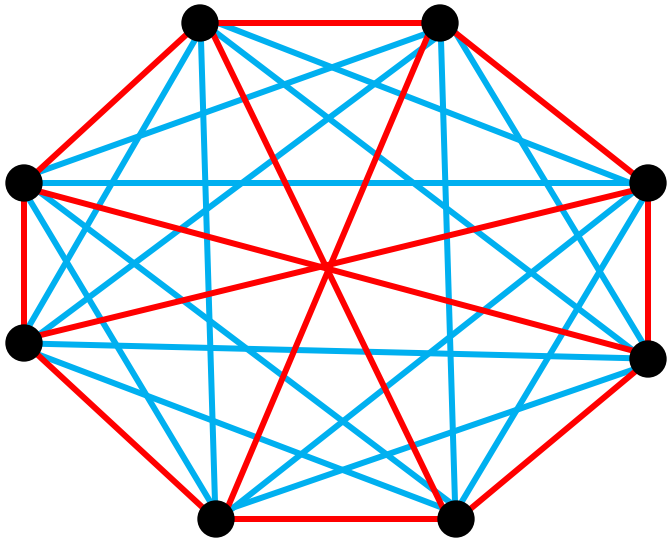
Upper bound: Show that any red-blue coloring of a K_9 has either a red K_3 or blue K_4.

Note: Bound we know R(3,3)+R(2,4) = 10.

- By previous argument, if any vertex has 6 blue edges or 4 red edges, done.
- Otherwise, each vertex must have 5 blue, 3 red edges.
- Violates Handshake Lemma. Total number of red edges = 9*3/2 = 13.5.

R(3,4) = 9

Lower bound: Need a coloring of a K_8 with no red K_3 or blue K_4.



R(4,4) = 18

- $R(4,4) \leq R(3,4) + R(4,3) = 9 + 9 = 18$
- Lower Bound:
  - Arrange 17 vertices in a circle
  - Draw a red edge between two if they are separated by 1, 2, 4, or 8 steps
  - Other edges are blue
  - Symmetry makes analysis relatively easy

Other Known Ramsey Numbers

- $R(3,5) = 14$
- $R(3,6) = 18$
- $R(3,7) = 23$
- $R(3,8) = 28$
- $R(3,9) = 36$
- $R(4,5) = 25$

Some Additional Bounds

- $35 \leq R(4,6) \leq 41$
- $43 \leq R(5,5) \leq 49$
- $58 \leq R(5,6) \leq 87$
- $102 \leq R(6,6) \leq 165$

Computing Ramsey Numbers is Hard

Computing Ramsey numbers is notoriously hard, which is why so few of them are known. The problem is that even if you have a coloring determining whether it has monochromatic subgraphs is already difficult. Finding the best colorings is even harder. And as far as anyone can tell there doesn't seem to be an easy formula for Ramsey numbers.

Bounds on Ramsey Numbers

We will focus on proving bound to at least get an idea of how big Ramsey numbers are.

Upper Bound

Theorem: $R(p,q) \leq 2^{p+q}$.

Proof: By induction on p+q.

- If p=1 or q=1, $R(p,q) = 1 < 2^{p+q}$.
- Assume the inequality holds for smaller p+q.
  - $R(p,q) \leq R(p-1,q)+R(p,q-1)$
    $\leq 2^{p+q-1}+2^{p+q-1} \leq 2^{p+q}$.

Lower Bound

Theorem (1.66): If $n \geq 3$, $R(n,n) \geq 2^{n/2}$.

Note: $R(p,q) \geq 2^{\min(p,q)/2}$.

Note 2: Combined with the upper bound, this says that symmetric Ramsey numbers are exponentially large.

Note 3: Bound is hard. We want to find a coloring with no monochromatic subgraph. But actual constructions tend to produce patters. How do we avoid them?

Random Construction

Color the edges of a K_N randomly. On average how many monochromatic K_ns?

- $\approx N^n$ many collections of n vertices.
- Each has a $\approx 2^{-n(n-1)/2}$ probability of being monochromatic
- Average number of monochromatic K_ns is roughly $N^n/(2^{n(n-1)/2}) \approx [N/2^{(n-1)/2}]^n$.
- If N much smaller than $2^{n/2}$, this is less than 1, so *some* coloring must have none.

Graph Ramsey Numbers

Traditional Ramsey Numbers look for complete subgraphs, but we can consider other kinds instead.

Definition: For graphs G and H, we define the graph Ramsey number R(G,H) to be the minimum n so that any red-blue coloring of K_n has either a red copy of G or a blue copy of H.

Finiteness

Note that G and H are contained in complete graphs, so this is finite.

Theorem (1.67):

$R(G,H) \leq R(|V_G|, |V_H|)$

Proof: Let $m = R(|V_G|, |V_H|)$. Any red-blue coloring of K_m has either a monochromatic complete red graph on $|V_G|$ or monochromatic blue complete graph on $|V_H|$. These contain a red copy of G or blue copy of H.

Example

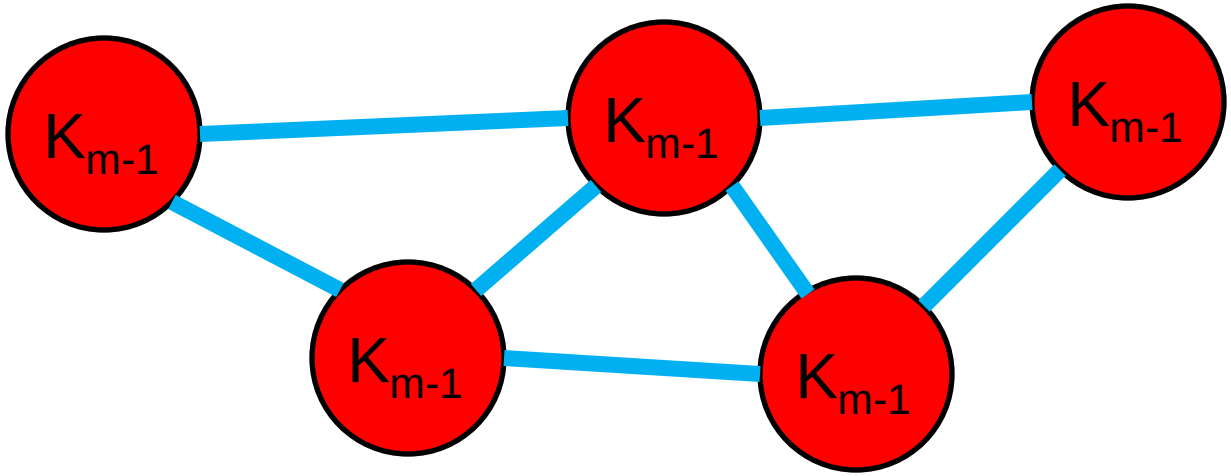
Theorem (1.70): If m and n are integers with m-1 dividing n-1 and T_m is a tree with m vertices then

$$R(T_m, K_{1,n}) = m+n-1.$$

Lower Bound

- Need a coloring of K_{m+n-2} without a red T_m or blue $K_{1,n}$.
- Note: m-1 divides m+n-2 = (m-1)+(n-1).

Coloring



- Red K_{m-1}s connected by blue edges.
- No Red T_m: All but CCs size m-1.
  - No Blue $K_{1,n}$: Each vertex has blue degree $(m+n-3) - (m-2) = n-1$.

Upper Bound

- Need to show that any red-blue coloring of a K_{n+m-1} has either a red T_m or a blue $K_{1,n}$.
- If any vertex has n or more blue edges, have blue $K_{1,n}$.
- Otherwise, consider G_r, graph of red edges.
  - Note that $\delta(G_r) \geq m-1$.

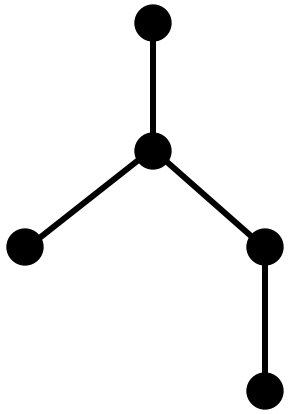
Lemma

Lemma (1.16): Let T be any tree on k vertices and G a graph with $\delta(G) \geq k-1$. Then G contains a copy of T.

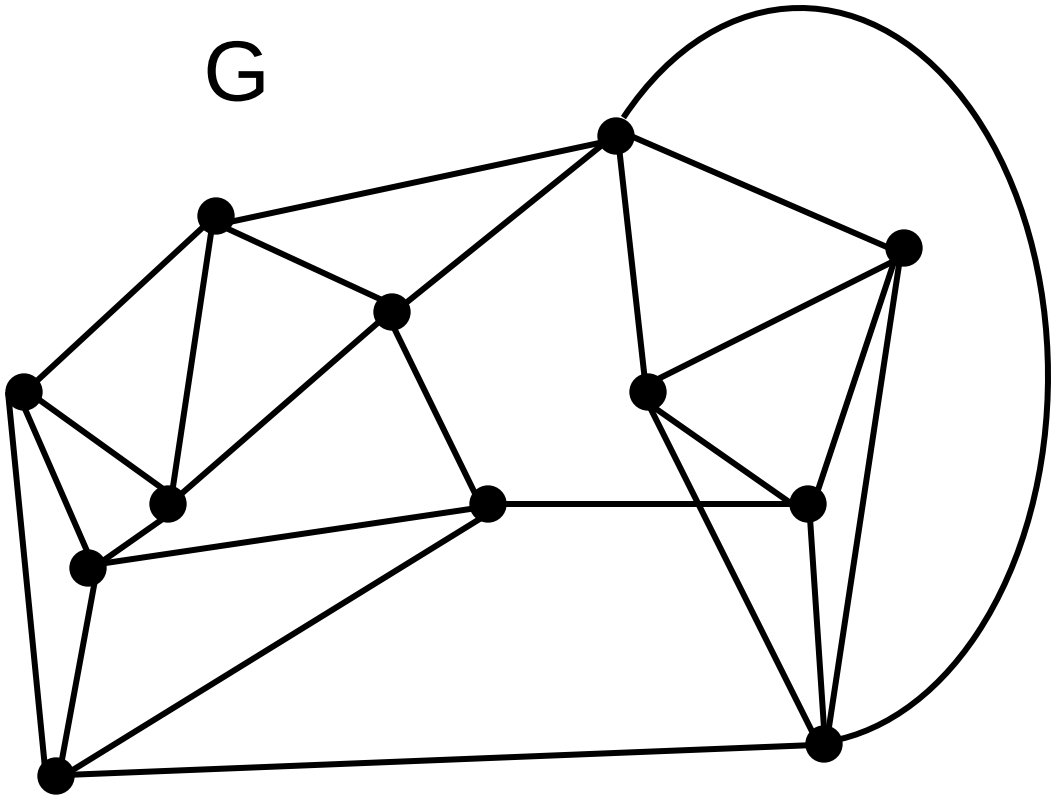
Apply to G_r and T_m to get final result.

Idea: Build T One Vertex at a Time

T



G



Proof by Induction on k

- Base case: k = 1
  - Can embed single point
- Assume can embed any tree on k-1 vertices.
- Let v be a leaf of T. Removing v and edge (u,v) gives T’.
- By IH, embed T’ in G.
- Need new neighbor of u to be v.
- u has k-1 neighbors, only k-2 are used.