CSE 203A: Randomized Algorithms

Spring 2026

Lecture (Lecture Number 13): KSAT and Random Walks
Date: 27th April
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Lecture Overview

These are the two topics that shall be covered this lecture:

- Generalization of k-SAT problem
- Basics of random walks

1 Introduction

In this lecture, we continued our study of randomized algorithms by extending the randomized local-search algorithm for 3-SAT to the more general k-SAT setting. The main idea was to analyze how likely a random assignment is to make progress toward a satisfying assignment when a variable from a violated clause is flipped. After this, the lecture shifted toward the basics of random walks and Markov chains, introducing transition probabilities, probability distributions over states, transition matrices, eigenvalue intuition, and the idea of a steady-state distribution. This connects naturally to the broader course theme of understanding randomized processes through probability, convergence, and concentration-style reasoning.

2 Brief Reminders about Previous Topics

Algorithm 1: 3 SAT algorithm

Input: Input x
Output: Approximate solution
Start at random assignment X
repeat
  if X *satisfies the assignment* **then**
    break
  else
    find a violated clause and flip a random variable in that clause
until *3n times*
return candidate

3 Main Lecture Content

3.1 K-SAT applicability

The above algorithm still works for K-SAT except that flipping random variable, there is at least a 1/k chance in correcting a bit.

How likely is it that x is αn steps away from solution y?

$\frac{(\binom{n}{\alpha n})}{2^n}(\frac{1}{k^{\alpha n}})$

$\approx \frac{(\frac{1}{\alpha})^{\alpha}(\frac{1}{1-\alpha})^{1-\alpha}\frac{1}{k^{\alpha n}}}{2^n}$

$> 2^{-n(1-\frac{1}{2k})}$

Therefore, overall big O is $O(2^{n(1-\frac{1}{O(k)})})$

3.2 Markov Chains

Sequence of R.V: $X_1, X_2, ...$

$$\Pr(X_{n+1} \mid X_n, X_{n-1}, \ldots, X_0) = \Pr(X_{n+1} \mid X_n)$$

Furthermore, it is time independent.

$Pr(X_{t+1} = i \mid X_t = j) = Pr(X_{s+1} = i \mid X_s = j)$

3.2.1 How do we describe a random walk?

- Transition Rules
- Current state - Its a probability distribution for discrete world.

  We will define $q_i^t = Pr(X_t = i)$

Formula to know: $q^t = q^0 P^t$ where q^t is a probability vector for time step t for discrete set of states and P is a transition probability matrix.

How do we process this? We use eigenvalues.

Let's say that P has eigenvalues v_i and λ_i. If $q_0 = \sum a_i v_i$, $q_t = \sum a_i \lambda^t v_i$

For large t, the sum is dominated with eigenvalues with highest absolute value. The largest possible absolute value is 1.

Through proofs that show that P has both a left and right eigenvalue of 1, this means that there exists at least a steady state distribution, $\pi = \pi P$. This means that once you arrive at this vector, you will always stay in this state for future time steps when you multiply by P.

3.2.2 Conjecture: $q^t \to \pi$ as t $\to \infty$

The above does not hold in the following condition(s):

- Multiple steady states from disconnected subgraphs

- Periodic behavior
- Technical issues for infinite Markov Chains

The conjecture only hold finite, connected, aperiodic Markov Chains.

Proof:

There exists some steady state π and two copies of random walk, X and Y, where X_0 is distributed according to q_0, while y_0 is distributed according to steady state.

X and Y evolve independently until $X_t = Y_t$. When they hit the same value, they evolve in sync.

WTS:

$$\Pr(X_t \mid Y_t) = 1$$

as t $\to \infty$

All we need to show is $\exists$ N, $\epsilon > 0$ such that for any X_t and Y_t:

$$\Pr(X_{N+t} = Y_{N+t} \mid X_t = a, Y_t = b) \geq \epsilon$$

How do we show this?

Lets say x starts at state a and y starts at state b. We know there exists some probability δ that x will get to b in l steps. While y may no longer be at b, x will start some loops and eventually come back to b so will y. We know set of possible return times has gcd 1 meaning there is a small infinitesimal chance that they will end up meeting again.

4 Further Remarks

- The k-SAT analysis generalizes the randomized local-search algorithm for 3-SAT.
- For a violated clause of size k, flipping a random variable has probability at least $1/k$ of making progress toward a satisfying assignment.
- As k increases, progress becomes less likely, so the running-time bound becomes worse.
- Markov chains provide a way to model random processes where the next state depends only on the current state.
- Steady-state distributions describe the long-term behavior of a random walk when convergence occurs.
- Convergence can fail because of disconnected components, periodic behavior, or infinite state spaces.

5 Summary

The lecture covered two main ideas. First, the randomized algorithm for 3-SAT can be generalized to k-SAT, although the probability of making progress in one step becomes at least $1/k$ instead of $1/3$. This leads to a running-time analysis of roughly

$$O\left(2^{n\left(1-\frac{1}{O(k)}\right)}\right).$$

Second, we introduced Markov chains and random walks. A Markov chain is a sequence of random variables where the next state depends only on the current state, not the full history. For a finite discrete state space, the distribution after t steps can be written as

$$q^t = q^0 P^t,$$

where P is the transition matrix. The lecture also introduced steady-state distributions π satisfying $\pi = \pi P$, and discussed when a random walk converges to such a distribution. In particular, convergence is expected for finite, connected, aperiodic Markov chains, while disconnected components, periodicity, or infinite state spaces can prevent convergence.

References

[1] R. Motwani and P. Raghavan, *Randomized Algorithms*, Cambridge University Press, 1995.