

CSE 203A Exam 1 (Spring 2026)

Question 1 (Expected Fixed Points, 30 points). *Given a random permutation π of $[n]$, what is the expected number of fixed points (i.e. values $i \in [n]$ so that $\pi(i) = i$) of π ?*

By linearity of expectation, we have

$$\mathbb{E}[\# \text{ of fixed points}] = \mathbb{E}\left[\sum_{i=1}^n \mathbb{1}_{i \text{ is a fixed point}}\right] = \sum_{i=1}^n \mathbb{E}[\mathbb{1}_{i \text{ is a fixed point}}] = \sum_{i=1}^n \Pr[i \text{ is a fixed point}].$$

A random permutation of $[n]$ maps an element $i \in [n]$ to every element of $[n]$ with equal probability (i.e., $1/n$), so the expected number of fixed points is $n(1/n) = 1$.

Question 2 (Same Row Matrix, 35 points). Given an $n \times n$ matrix A with entries in \mathbb{F}_p for $p > n^2$ [*in-class announcement changed this to $p > n^3$*], give an $O(n^2)$ time **coRP** algorithm for determining whether there exist two rows of A that are permutations of each other. You may assume that \mathbb{F}_p arithmetic can be done in constant time.

Algorithm. For each row r of A , define the polynomial $f_r(x) = \prod_i (x - A_{r,i})$ over \mathbb{F}_p . Randomly sample a field element $x \in \mathbb{F}_p$ and evaluate $f_r(x)$ for every row r . Accept if $f_{r_1}(x) = f_{r_2}(x)$ for any pair r_1, r_2 of distinct rows, and reject otherwise.

Runtime. Assuming \mathbb{F}_p arithmetic can be done in constant time, each $f_r(x)$ can be evaluated in $O(n)$ time, so all n polynomials can be evaluated in $O(n^2)$ time. Comparing two (already computed) polynomial values takes constant time, so checking all $\binom{n}{2}$ pairs can be done in $O(n^2)$ time. Thus, the algorithm takes $O(n^2)$ time overall.

Correctness. If r_1, r_2 are permutations of each other, then the polynomials f_{r_1}, f_{r_2} have the same factors, so the commutativity of multiplication implies f_{r_1}, f_{r_2} are the same polynomial. In particular, $f_{r_1}(x) = f_{r_2}(x)$ for all $x \in \mathbb{F}_p$, so the algorithm always succeeds.

If r_1, r_2 are not permutations of each other, then f_{r_1}, f_{r_2} are distinct polynomials, so their difference $g_{r_1, r_2} := f_{r_1} - f_{r_2}$ is a nonzero polynomial of degree at most n . Thus,

$$\Pr_x[f_{r_1}(x) = f_{r_2}(x)] = \Pr_x[g_{r_1, r_2}(x) = 0] \leq \frac{\deg(g_{r_1, r_2})}{p} < \frac{n}{n^3} = \frac{1}{n^2}.$$

Applying a union bound, we find that the probability any two polynomials are equal on a random $x \in \mathbb{F}_p$ (i.e., that the algorithm fails) is

$$\Pr_x \left[\bigvee_{r_1 \neq r_2} (f_{r_1}(x) = f_{r_2}(x)) \right] \leq \sum_{r_1 \neq r_2} \Pr_x[f_{r_1}(x) = f_{r_2}(x)] \leq \frac{\binom{n}{2}}{n^2} \leq \frac{1}{2},$$

as desired.

Question 3 (Graph Products, 35 points). Consider two finite, undirected graphs G and H . Define a new graph $G \otimes H$ whose vertex set consists of pairs (u, v) with $u \in G$ and $v \in H$ and has edges (u, v) to (u', v') where u' is a neighbor of u and v' is a neighbor of v .

If the transition matrix of the random walk on G has eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ and the transition matrix of H has eigenvalues $\mu_1, \mu_2, \dots, \mu_m$, what are the eigenvalues of the transition matrix for $G \otimes H$? Justify your answer.

The eigenvalues are $\lambda_i \mu_j$ for all $i \in [n]$ and $j \in [m]$.

Let P_G, P_H be the transition matrices of the random walks on G, H , respectively. By definition, taking a random step in the graph $G \otimes H$ is equivalent to independently taking a random step in the graph corresponding to G and the graph corresponding to H . More formally, we can write the transition matrix for $G \otimes H$ as

$$P_{G \otimes H}((u, v), (u', v')) = P_G(u, u') \cdot P_H(v, v').$$

Now let a be an eigenvector of P_G with eigenvalue λ , and let b be an eigenvector of P_H with eigenvalue μ . Define the nm -dimensional vector c by $c_{(u,v)} = a_u b_v$. Then by the definition of a random walk, we have

$$\begin{aligned} (P_{G \otimes H} c)_{(u,v)} &= \frac{1}{\deg_G(u) \cdot \deg_H(v)} \sum_{u':(u',u) \in E(G)} \sum_{v':(v',v) \in E(H)} c_{(u',v')} \\ &= \frac{1}{\deg_G(u) \cdot \deg_H(v)} \sum_{u':(u',u) \in E(G)} \sum_{v':(v',v) \in E(H)} a_{u'} b_{v'} \\ &= \frac{1}{\deg_G(u) \cdot \deg_H(v)} \left(\sum_{u':(u',u) \in E(G)} a_{u'} \right) \left(\sum_{v':(v',v) \in E(H)} b_{v'} \right) \\ &= (P_G a)_u \cdot (P_H b)_v \\ &= \lambda a_u \cdot \mu b_v = \lambda \mu c_{(u,v)}. \end{aligned}$$

That is, c is an eigenvector with eigenvalue $\lambda \mu$.

It remains to verify that the eigenvectors constructed in this fashion span the entire space. Since G and H are undirected, the transition matrices P_G and P_H have eigenvector bases a_1, \dots, a_n and b_1, \dots, b_m , respectively. As above, define $c_{i,j}$ to be $(c_{i,j})_{(u,v)} = (a_i)_u (b_j)_v$. Since the vector space has dimension nm , it suffices to show that the vectors $\{c_{i,j}\}$ are linearly independent. Suppose by contradiction there exist scalars $\{\alpha_{i,j}\}$, not all zero, such that $\sum_{i,j} \alpha_{i,j} c_{i,j} = \vec{0}$. Then for every (u, v) , we have $\sum_{i,j} \alpha_{i,j} (a_i)_u (b_j)_v = 0$. Fixing v and considering the vector in the u -coordinate, we have

$$\sum_{i=1}^n \left(\sum_{j=1}^m \alpha_{i,j} (b_j)_v \right) a_i = \vec{0}.$$

Recall that a_1, \dots, a_n are linearly independent, so $\sum_{j=1}^m \alpha_{i,j} (b_j)_v = 0$ for every i and v . This implies that $\sum_{j=1}^m \alpha_{i,j} b_j = \vec{0}$, but since the b_j 's are linearly independent, we must have all the $\alpha_{i,j}$'s equal 0, a contradiction. That is, the eigenvectors $\{c_{i,j}\}$ span the entire space, so the eigenvalues are exactly of the form $\lambda_i \mu_j$ (counted with multiplicity).