Announcements

- No HW this week
- 3-4pm discussion section today over zoom
- Final exam next week
  - Wednesday at 11:30
  - In this classroom
  - Randomized seating
  - Comprehensive (with slight emphasis on new material in Chapters 8 and 9)
  - 6 Questions in 3 hours
  - 12 one-sided pages of notes

Last Time

- NP Problems and NP completeness

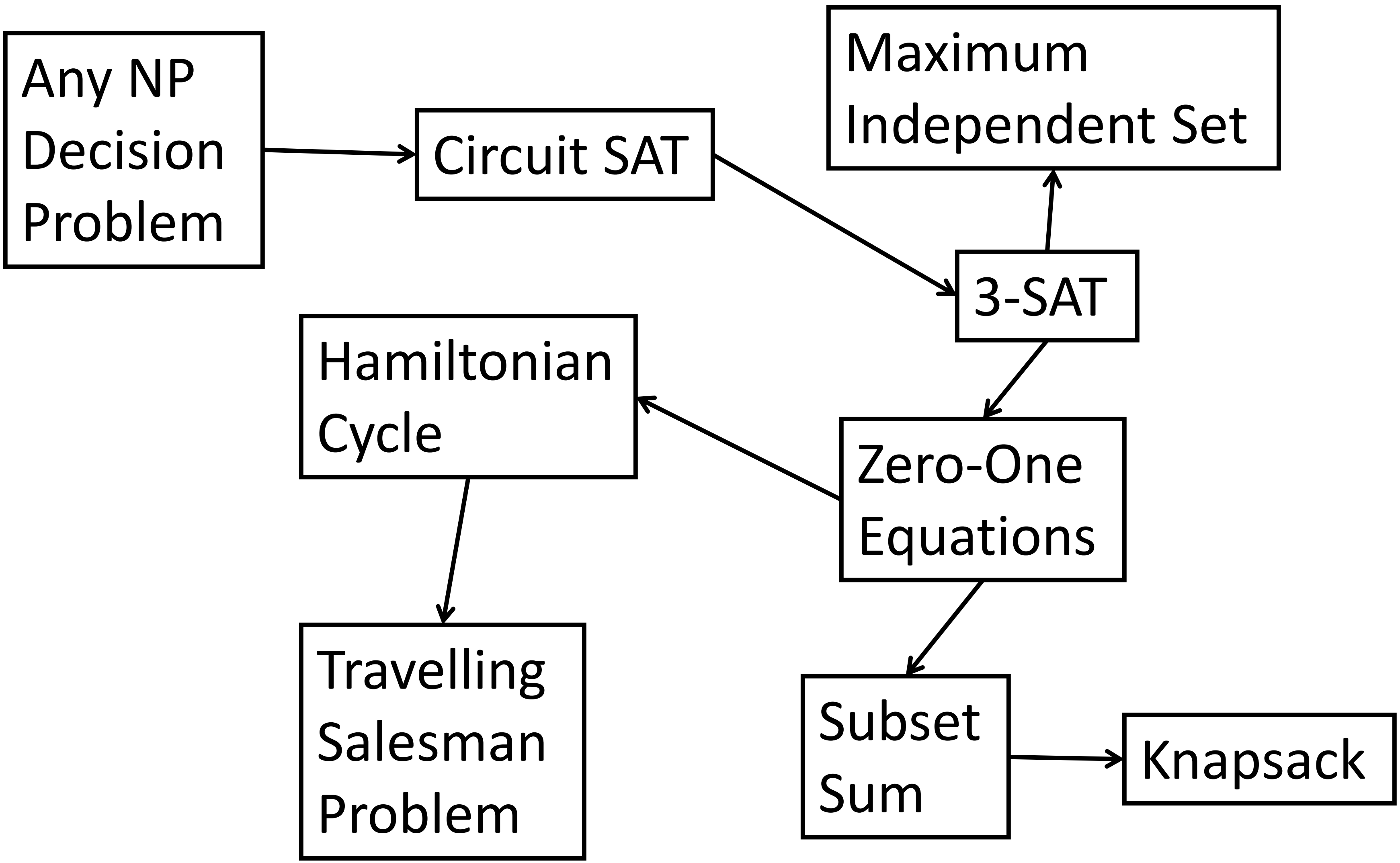
NP

Such problems are said to be in <u>Nondeterministic Polynomial</u> time (NP).

<u>NP-Decision</u> problems ask if there is some object that satisfies a polynomial time-checkable property.

<u>NP-Optimization</u> problems ask for the object that maximizes (or minimizes) some polynomial time-computable objective.

Reduction Summary



NP Complete

The problems on the last slide are all NP-Complete/Hard. This means that if *any* problem in NP is fundamentally hard (a widely held belief), then these particular problems are.

Today

- Dealing with NP Completeness
- Basic Methods
- Backtracking / Branch and Bound

Dealing With NP-Completeness (Ch 9)

- Backtracking/Branch and Bound
- Heuristic Search
- Approximation Algorithms

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Unfortunately, this doesn't solve your original problem. Even if it's NP-Complete you still need to solve it anyway.

Bad News

If your problem is NP-Hard/NP-Complete…

Then unless P=NP, there is no algorithm that gives the exact answer to your problem on all instances in polynomial time.

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See if you can modify problem.

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Fixed Parameter Tractability

See if you can make further assumptions.

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Efficient Search

Sudoku

Consider the logic puzzle Sudoku (or any similar logic puzzle).

Fill a 9x9 grid of numbers with 1-9 so that:

- Each row has all numbers
- Each column has all numbers
- Each of the main 3x3 sub squares has all numbers
- Some entries are pre-filled

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- So in general, you cannot do much better than brute force search.
- True brute force search would consider $9^{81} \approx 2 \cdot 10^{77}$ possibilities.
- In practice, people can solve them while waiting for the dentist.
  - How?

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- Use that information to help fill out more squares.
- Hopefully, you can keep going until the entire problem is solved.

Example

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So we must have x = y = z = True, which is a solution.

Getting Stuck

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How do you get out?

Option 1: Stronger deductive rules.

Sudoku Inference Rules

More complicated deduction rules allow you to go further without getting stuck. Common Sudoku rules include:

1) Find a square that only one number can fill.
2) Find a region with only one place for a given number.
3) Find a pair of squares in the same row that must contain two numbers (which then cannot appear elsewhere in that row).
4) Find a rectangle whose corners must contain 2 copies of a number. That number cannot appear elsewhere in those rows/columns.
5) Find 3 rows & 3 columns whose intersections must contain 3 copies of a number. That number cannot appear elsewhere in those rows and columns.

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Guess and check.

Guess and Check

- Make a guess for some entry.
- Try to solve the resulting puzzle (perhaps doing more guessing).
- If you find a solution, great!
- If not, you have deduced that your original guess was wrong.

Example

$$(x \vee y) \wedge (y \vee z) \wedge (z \vee x) \wedge (\bar{x} \vee \bar{y}) \wedge (\bar{y} \vee \bar{z}) \wedge (\bar{z} \vee \bar{x})$$

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2nd clause: y = False
4th clause: z = False
Contradicts 1st clause.

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1^{st} clause: y = True
3^{rd} clause: z = True
Contradicts 4^{th} clause.

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$$(x \vee y) \wedge (y \vee z) \wedge (z \vee x) \wedge (\bar{x} \vee \bar{y}) \wedge (\bar{y} \vee \bar{z}) \wedge (\bar{z} \vee \bar{x})$$

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1st clause: y = True
3rd clause: z = True
Contradicts 4th clause.

No Solutions!

Backtracking

You can combine guess and check nicely with deductions. In fact, a deduction can be thought of as just guessing the wrong way to fill things in and then concluding that it doesn’t work.

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This brings us to the general algorithm of Backtracking. This takes some search problem P with some space S that needs to be searched.

Backtracking

```
Backtracking(P,S)
  If you can deduce unsolveable
    Return 'no solutions'
  Split S into parts S_1,S_2,…
  For each i,
    Run Backtracking(P,S_i)
  Return any solutions found
```

Splitting

How do you split S into parts?

- Pick variable x_i and set x_i = True, or x_i = False
- Try all possible numbers in a square in Sudoku
- Try all possible edges in Hamiltonian Cycle

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Which variable do we guess?

- Often helps to pick a variable that shows up a lot. Then guessing it's value will make later deductions easier.

Runtime

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SAT Solvers can use these ideas to solve problems with hundreds of variables, many many more than would be practical by brute force.

Optimization Version

Backtracking works well for decision/search problems (where a potential solution works or doesn't work), but not so well for optimization problems (where many solutions work, but you need to find the <u>best</u> one).

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If most solutions work, how do you weed out bad paths?

Branch & Bound

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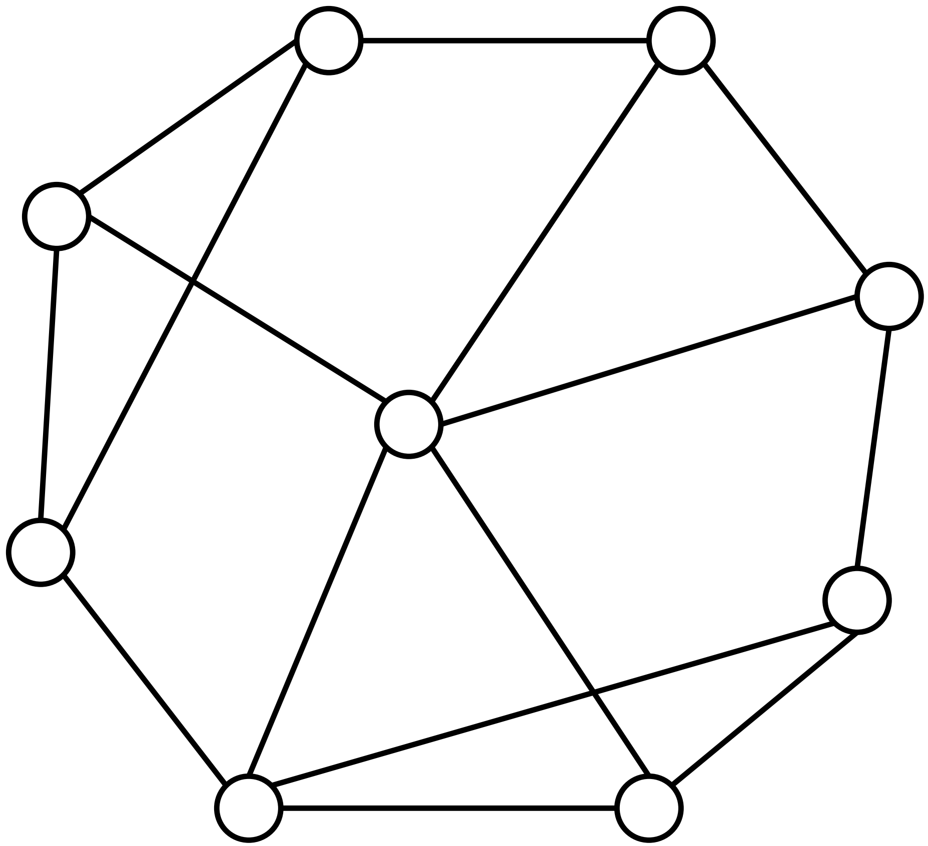
If an upper bound is smaller than your best solution so far, it cannot contain the optimum.

Example: Maximum Independent Set



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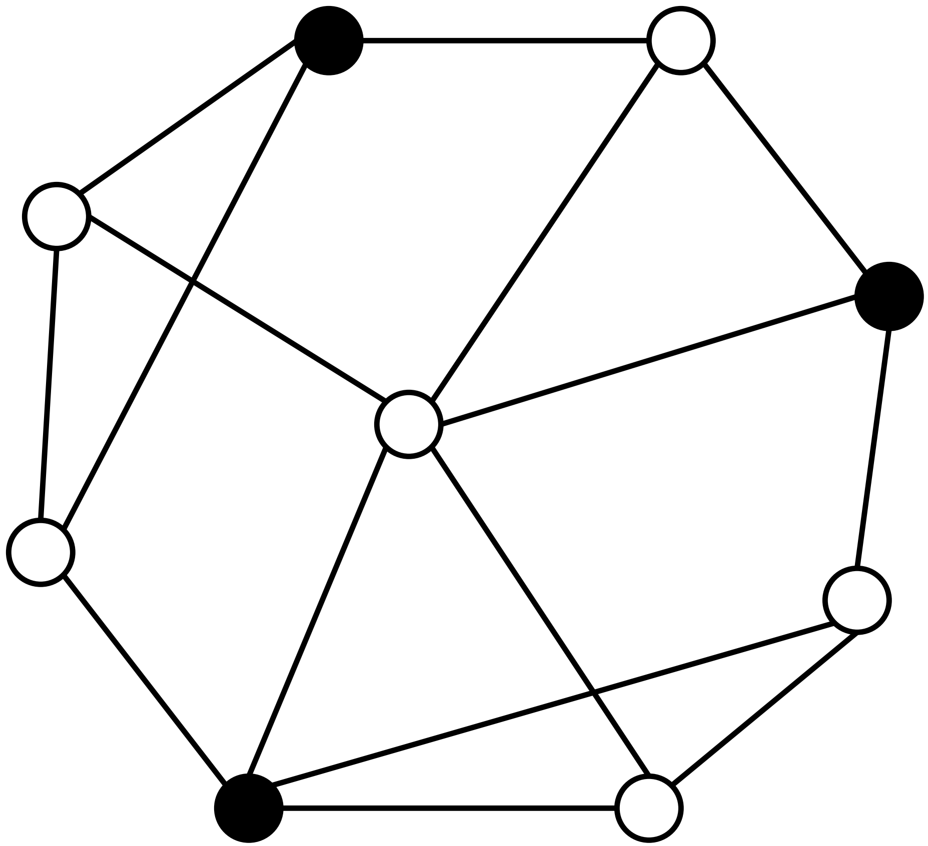

Set of size 3.

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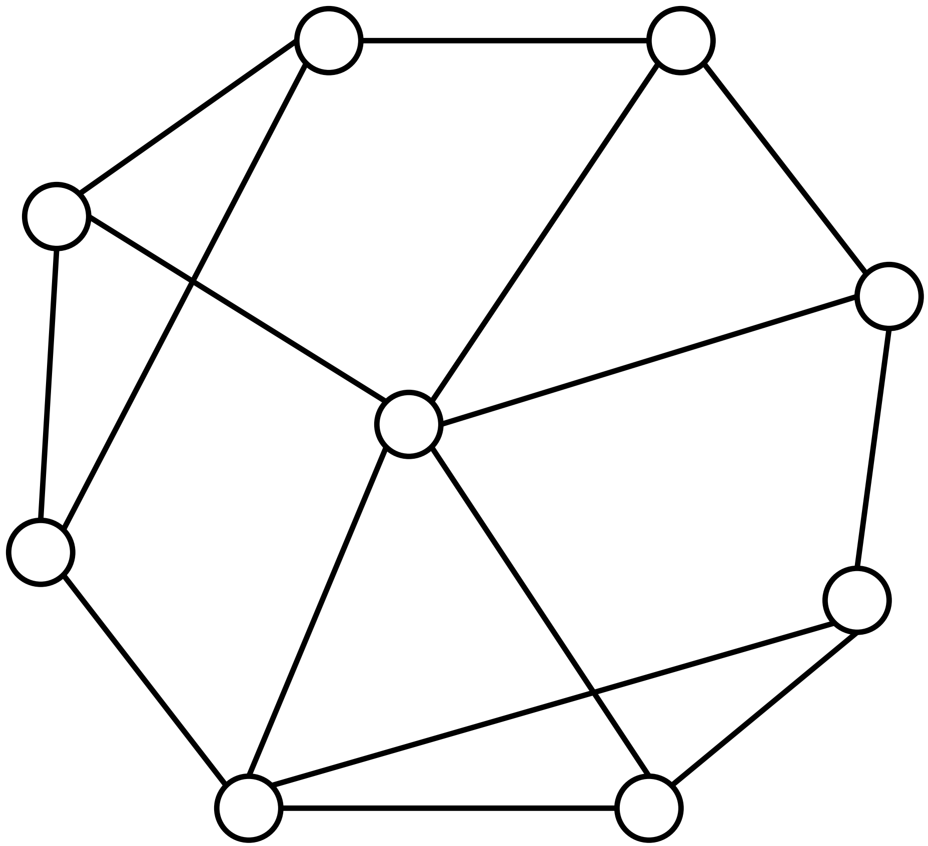

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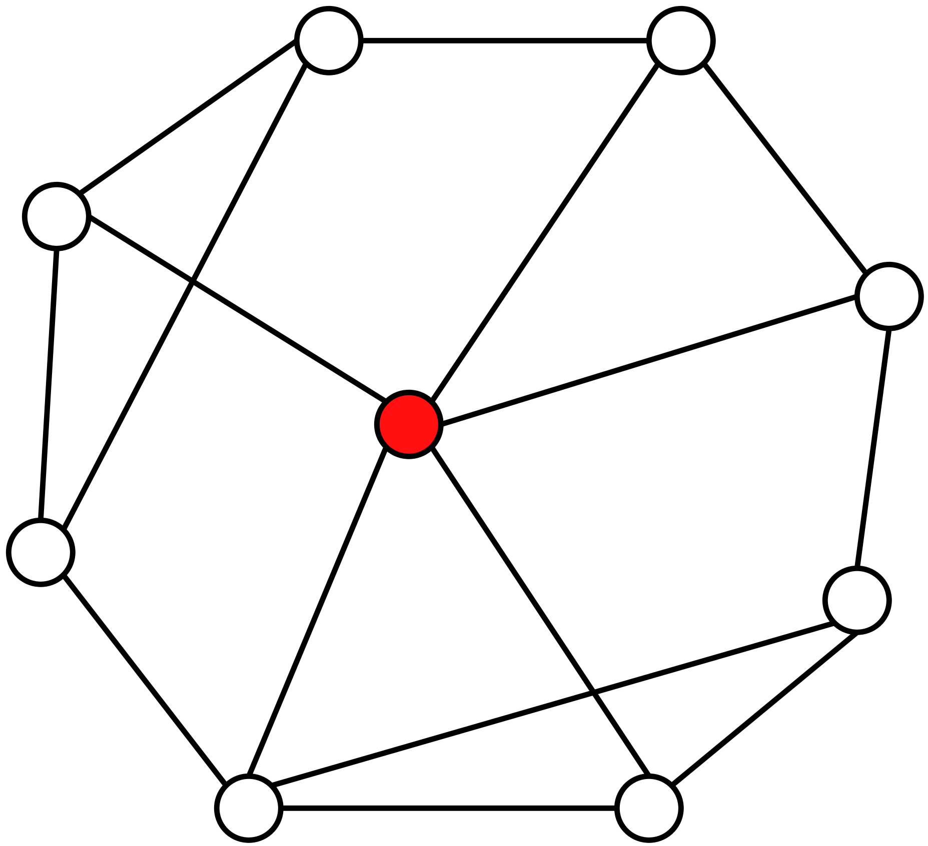

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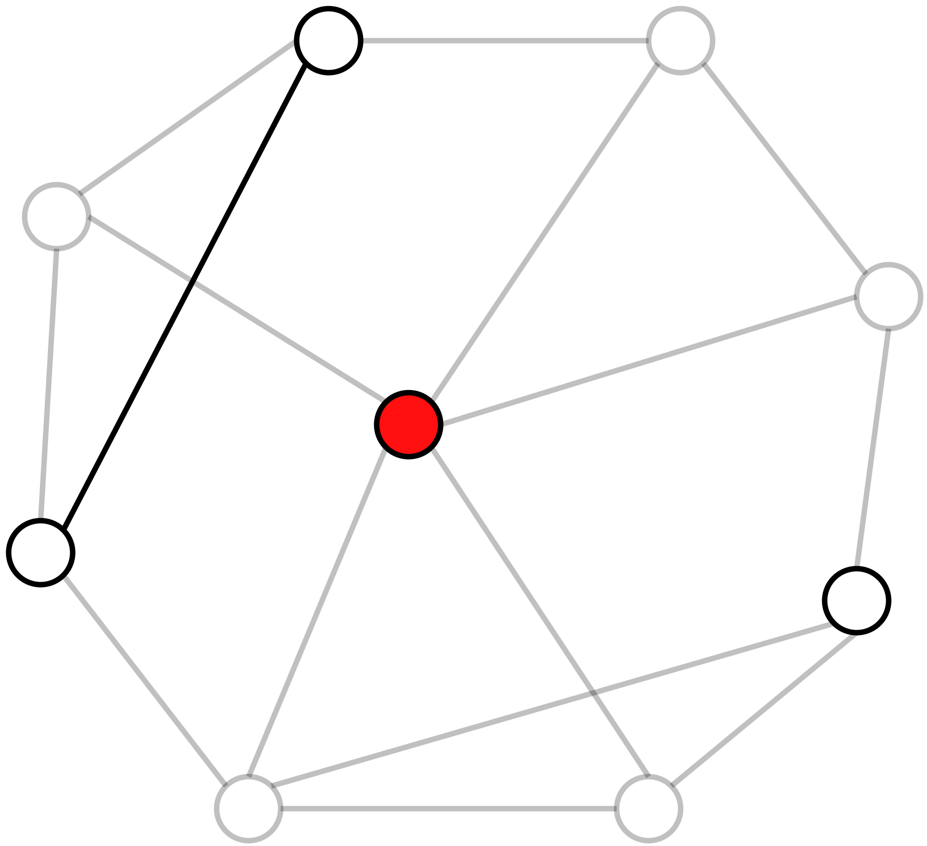

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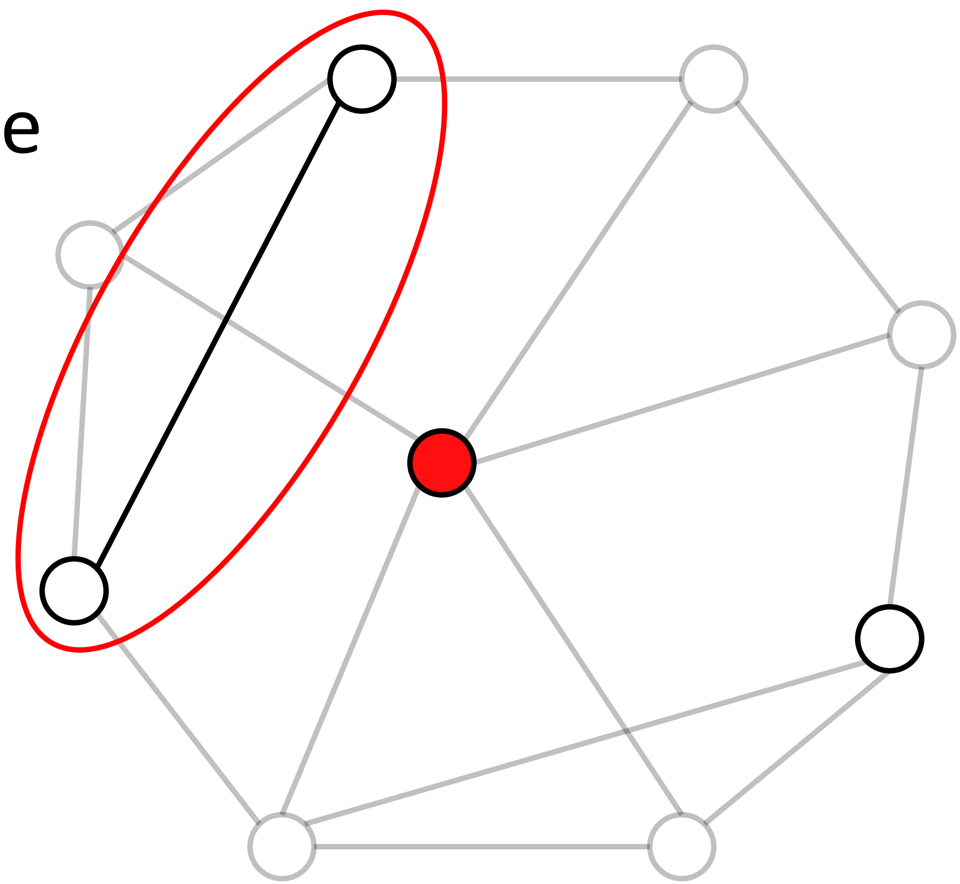
Can't have any of these.

Example: Maximum Independent Set

Can't have both of these



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Example: Maximum Independent Set

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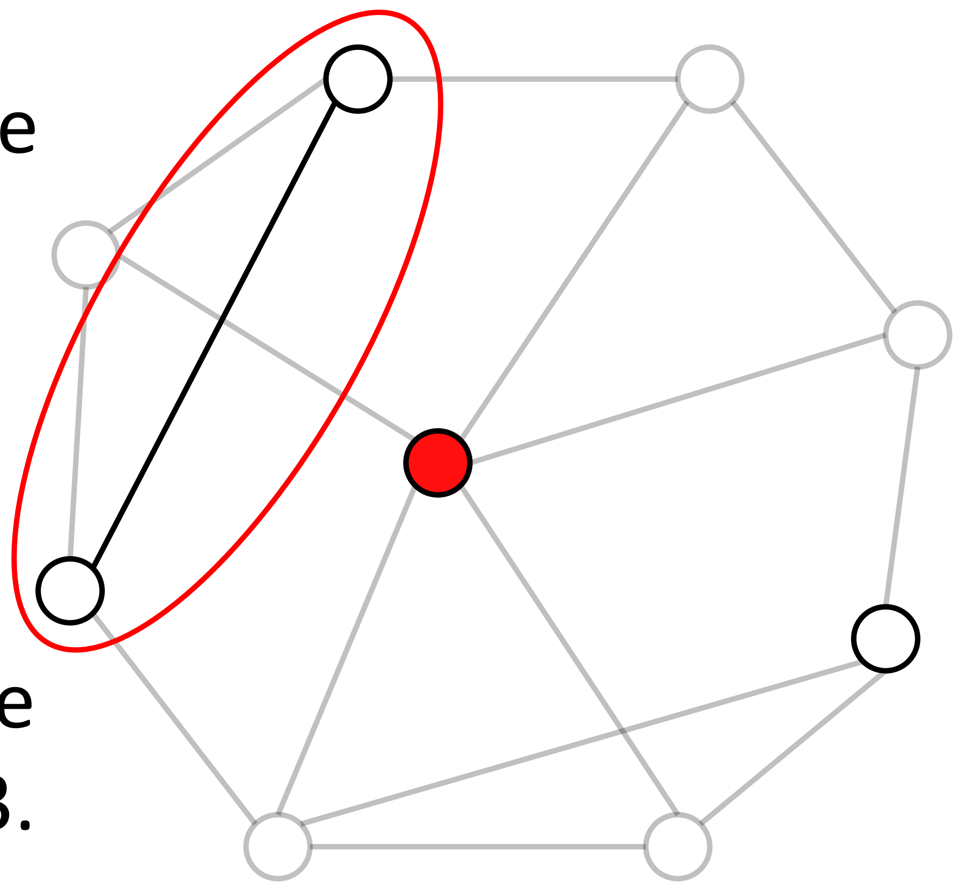
Set of size at most 3.



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If we have this point…

Can’t have any of these.

Branch and Bound

```
BranchAndBound(Best,S)
  If UpperBound(S) ≤ Best
    Return 'no improvement'
  If S a full solution
    Return value of S
  Split S into S₁,S₂,…
  For each Sᵢ
    New ← BranchAndBound(Best,Sᵢ)
    Best = Max(New,Best)
  Return Best
```