Announcements

- Homework 5 due today
  - Q3: You should not assume that the T_i are increasing
- Exam 3 next week
  - Same format
  - Topics:
    - Huffman codes
    - MSTs
    - Dynamic Programming
    - LCSS/Knapsack/CMM/All-pairs shortest path/MIS in trees/Travelling salesman

Today

- NP Problems and NP completeness

NP-Completeness (Ch 8)

- NP-Problems
- Reductions
- NP-Completeness & NP-Hardness
- SAT
- Hamiltonian Cycle
- Zero-One Equations
- Knapsack

Brute Force Algorithms

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- Is there a path from s to t in G?
- What is the longest common subsequence?
- What is the closest pair of points?
- Does G have a topological ordering?

NP

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NP-Decision problems ask if there is some object that satisfies a polynomial time-checkable property.

NP-Optimization problems ask for the object that maximizes (or minimizes) some polynomial time-computable objective.

Optimization vs. Decision

Note that these are not too different.

- Every decision problem can be phrased as an optimization problem (objective has value 1 if the object satisfies the condition and 0 otherwise).
- Every optimization problem has a decision form (can we find an example whose objective is more than x).

Examples of NP Problems

- SAT
- TSP
- Hamiltonian Cycle
- Knapsack
- Maximum Independent Set

SAT

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$$\text{No satisfying assignment.}$$

Applications of SAT

- Circuit Design
- Logic Puzzles
- Cryptanalysis

Hamiltonian Cycle (in text as Rudruta Path)

Given an undirected graph G is there a cycle that visits every vertex exactly once?

General Knapsack

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If weights are allowed to be large (written in binary), don't have a good algorithm.

Question: Decision vs. Optimization

Which of the following are NP-Decision problems? (Multiple correct answers)

A) SAT

B) Hamiltonian Cycle

C) General Knapsack

D) Maximum Independent Set

E) Travelling Salesman

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Brute Force Search

- Every NP problem has a brute force search algorithm.
- Throughout this class we have looked at problems with algorithms that substantially improve on brute force search.
- Does every NP problem have a better-than-brute-force algorithm?

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Most computer scientists believe P ≠ NP. (But proving anything is very very hard)

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- Can search for algorithms (to show it's easy).
- Can try to prove that it is hard, but this is extremely difficult.
- Can try to relate its difficulty to that of other problems.

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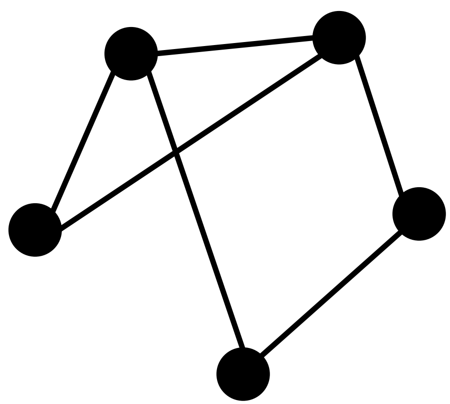
We show that <u>if</u> there is an algorithm for solving A, then we can use this algorithm to solve B. Therefore, B is no harder than A.

Hamiltonian Cycle → TSP

Hamiltonian
Cycle Instance



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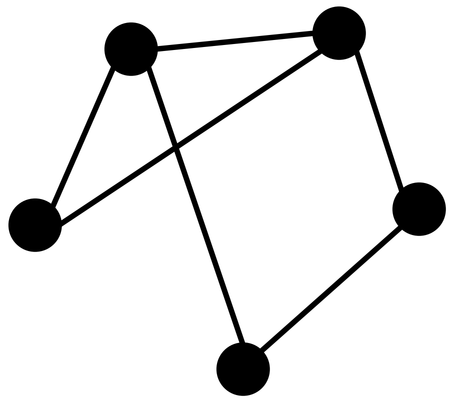
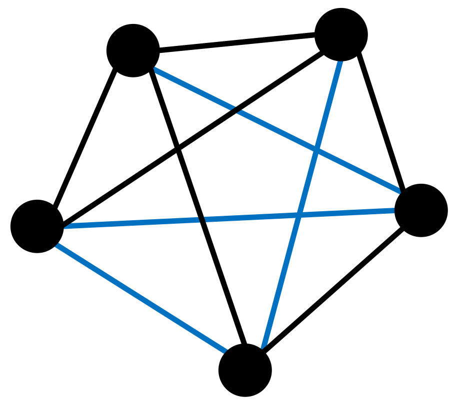


TSP Instance



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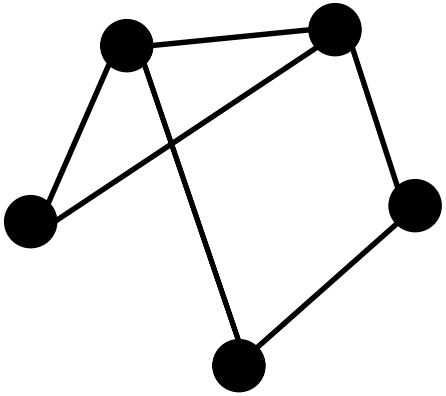

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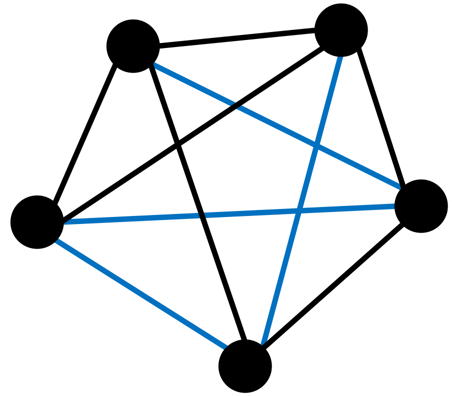

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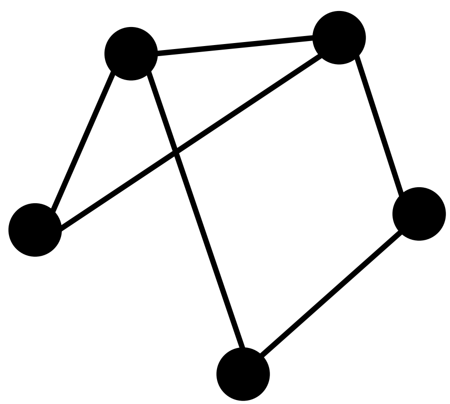

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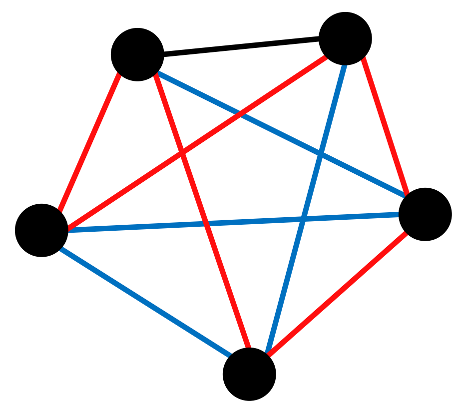

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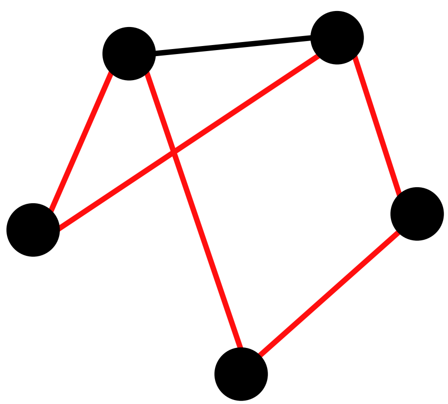

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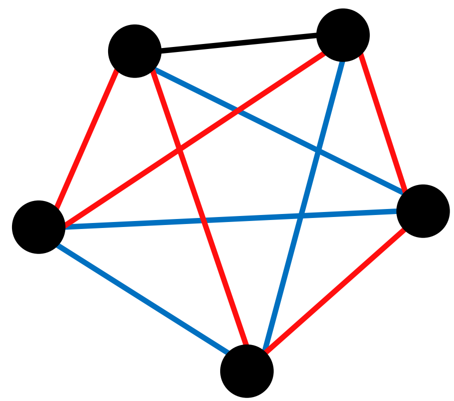

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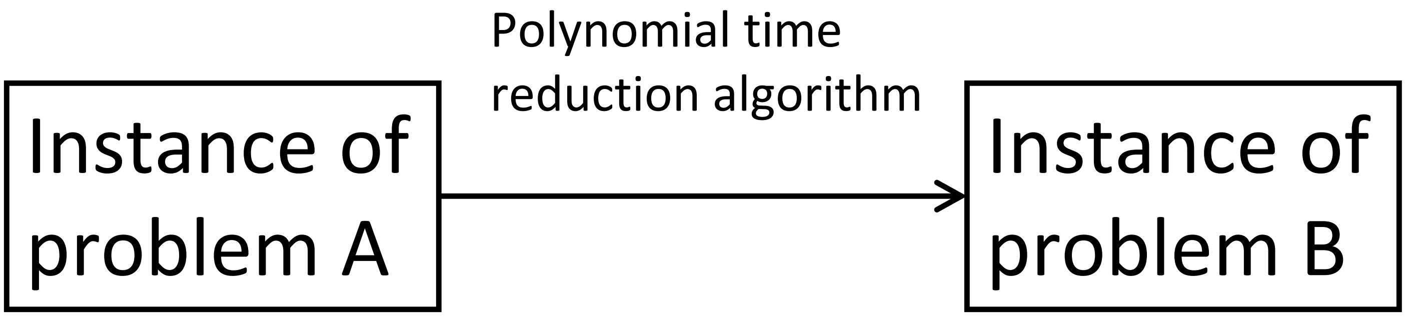
If, we have an algorithm that solves TSP, we can use it to solve Ham. Cycle.

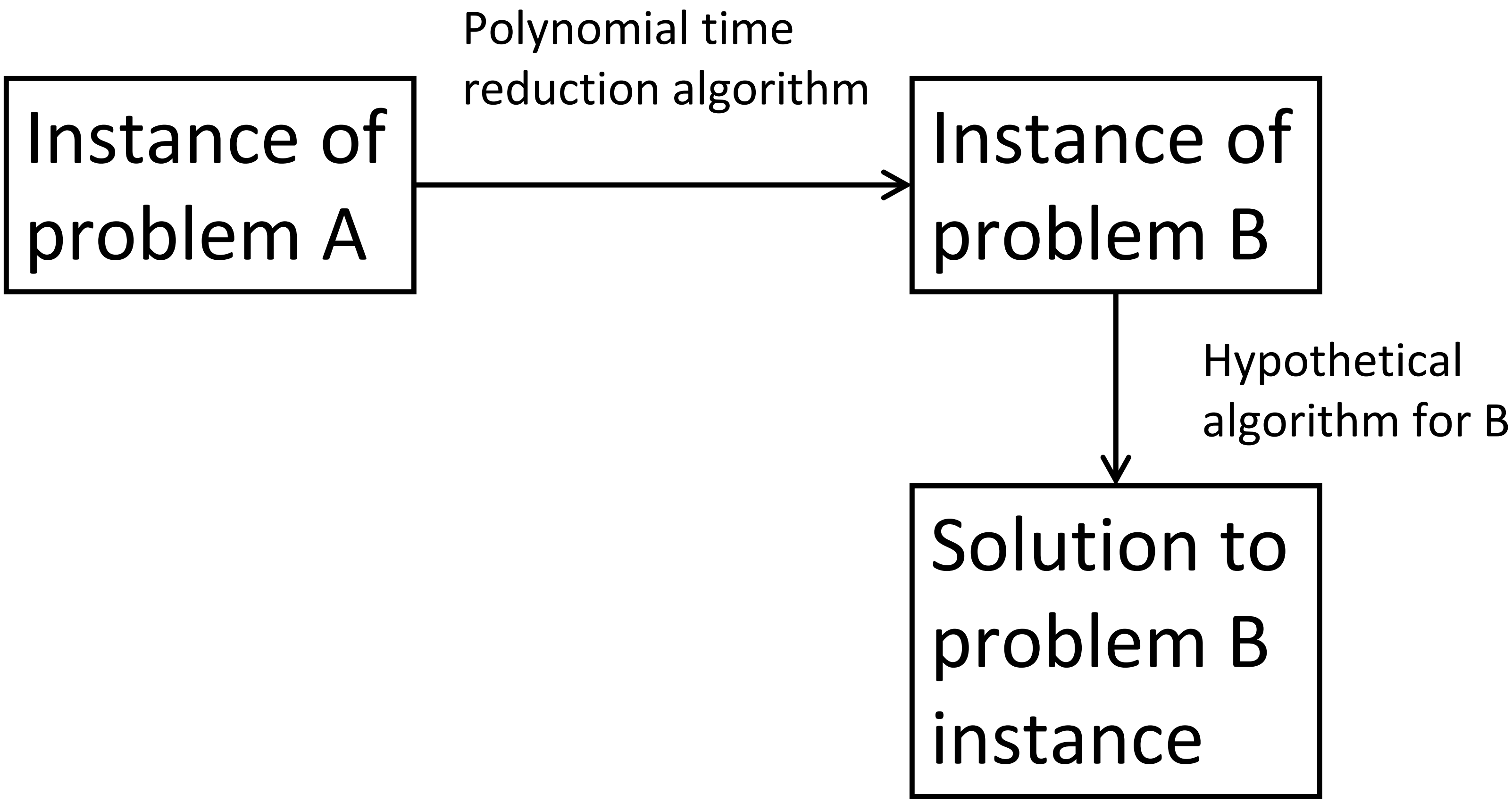
Reduction A → B

Instance of
problem A

Reduction A → B



Reduction A → B



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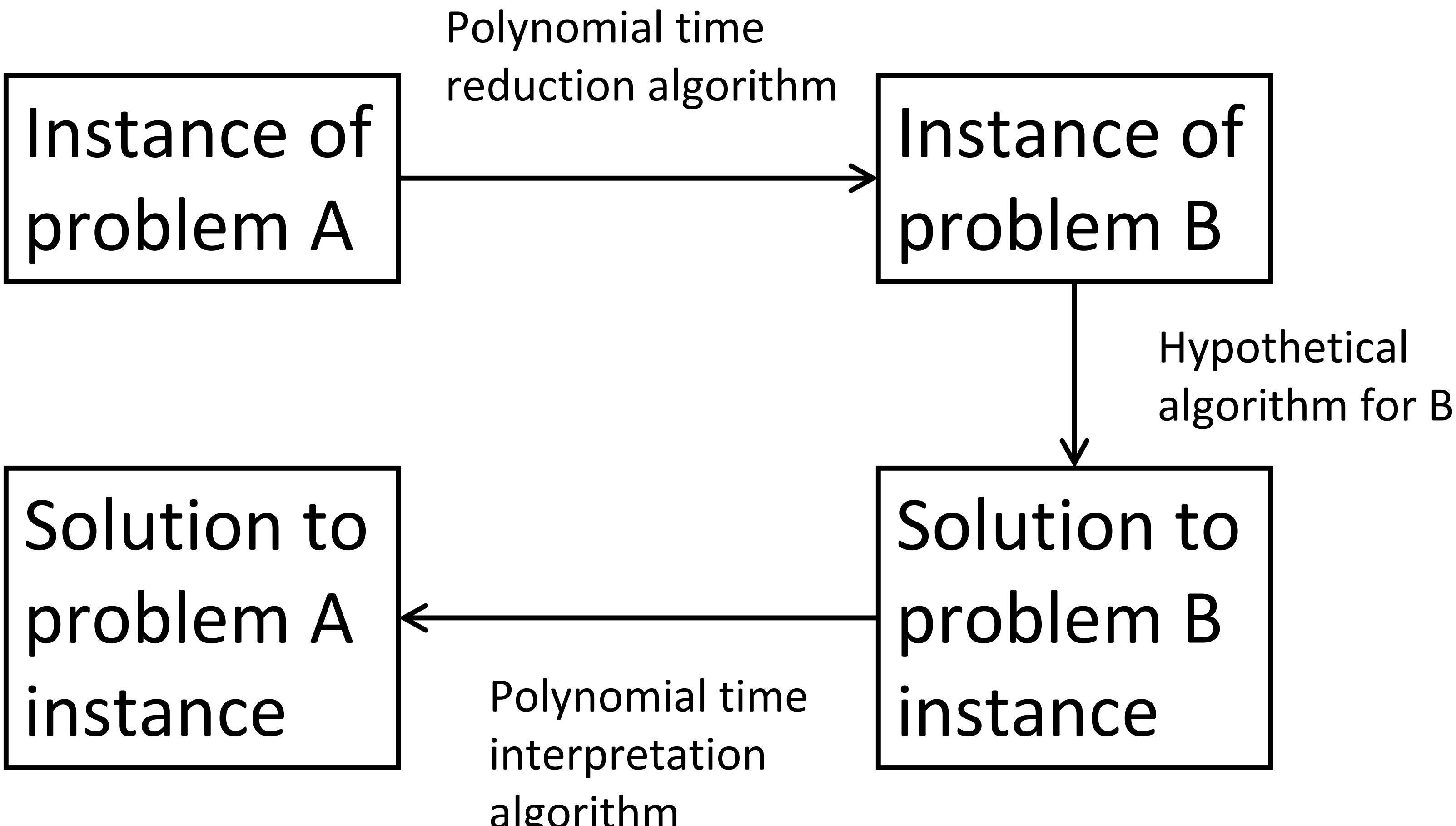

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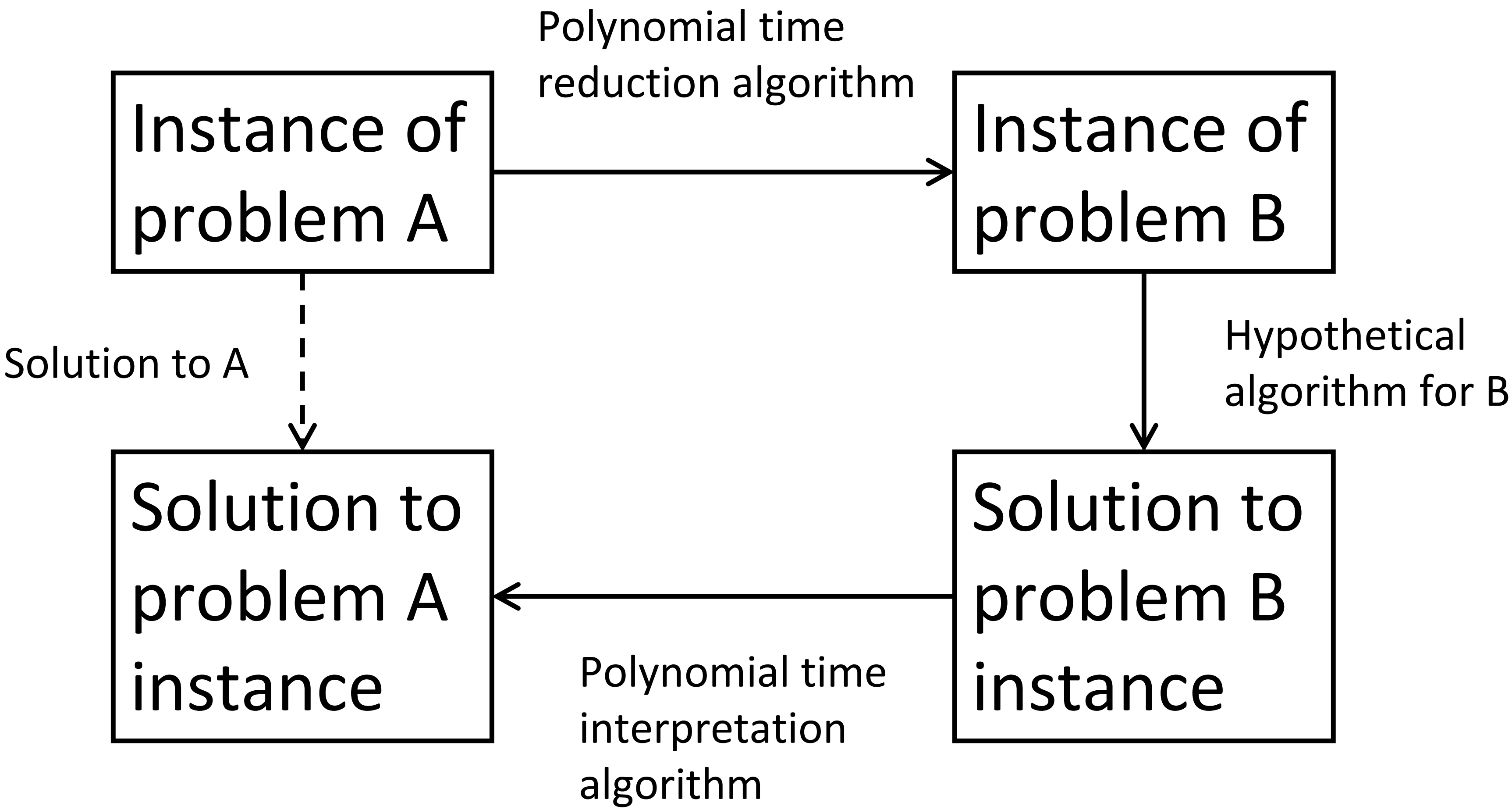

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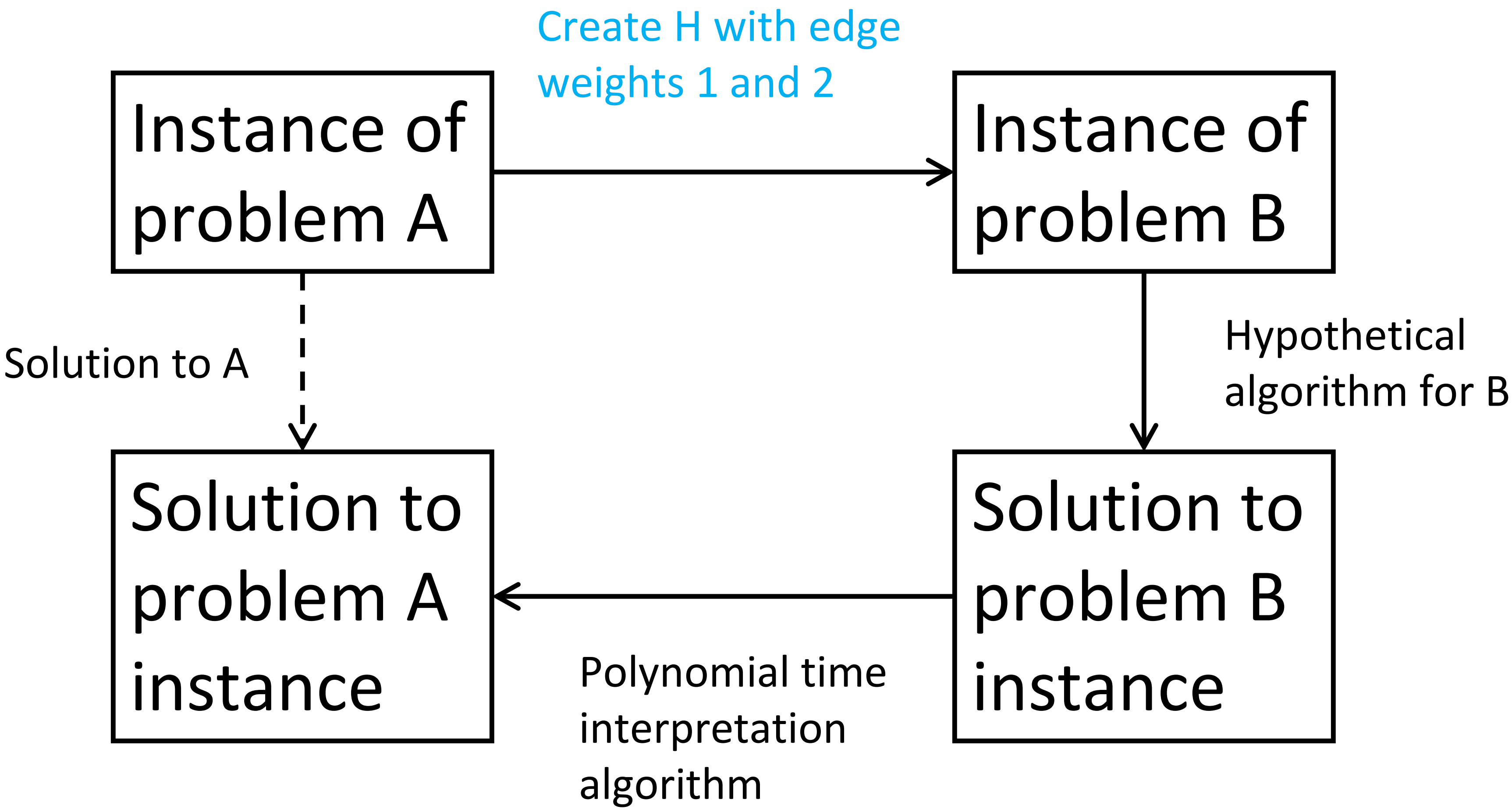

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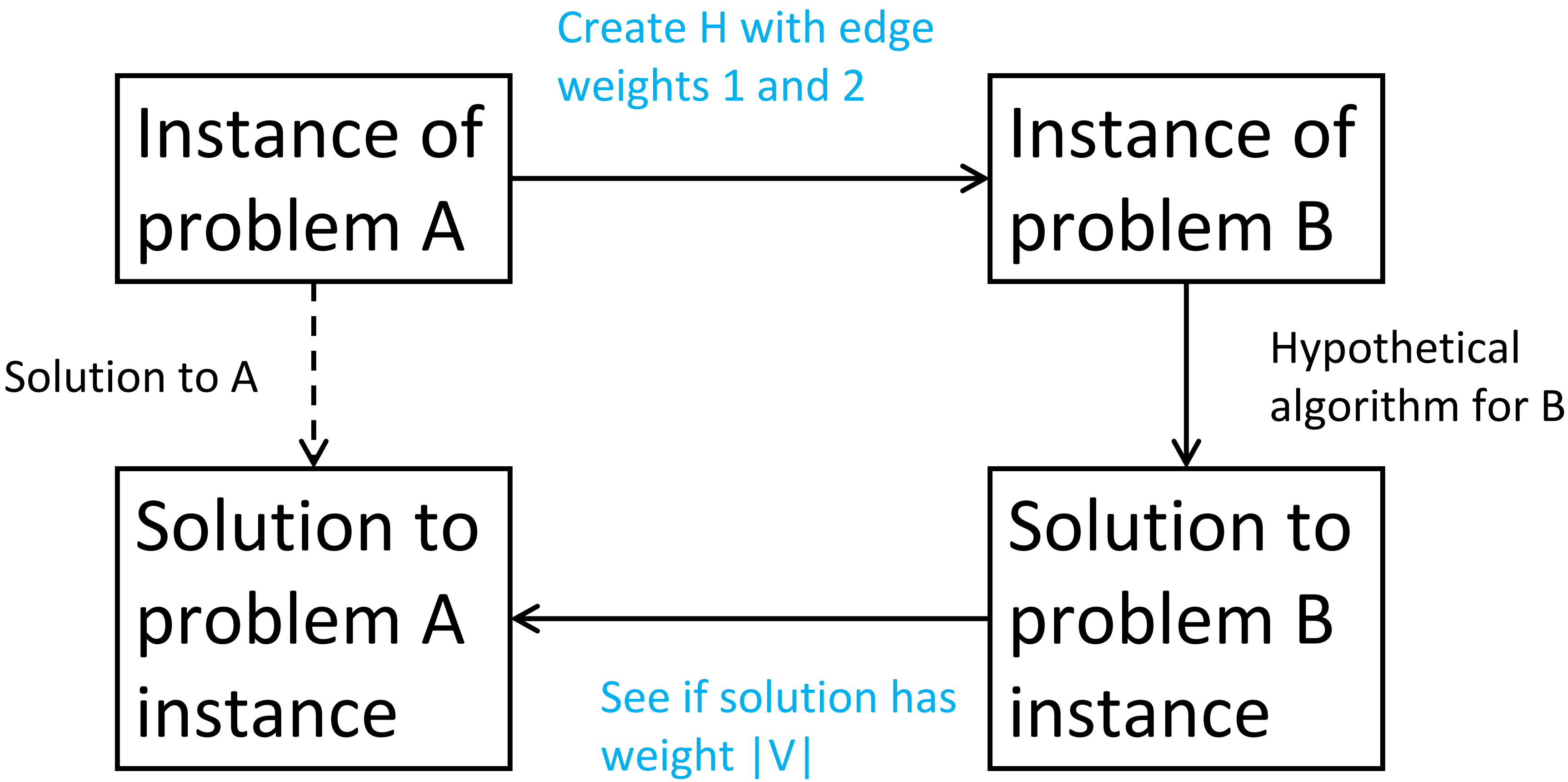

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Reduction A $\rightarrow$ B

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- Given an algorithm to solve B, we can turn it into an algorithm to solve A.
- This means that A might be <u>easier</u> to solve than B, but cannot be <u>harder</u>.

Circuit SAT

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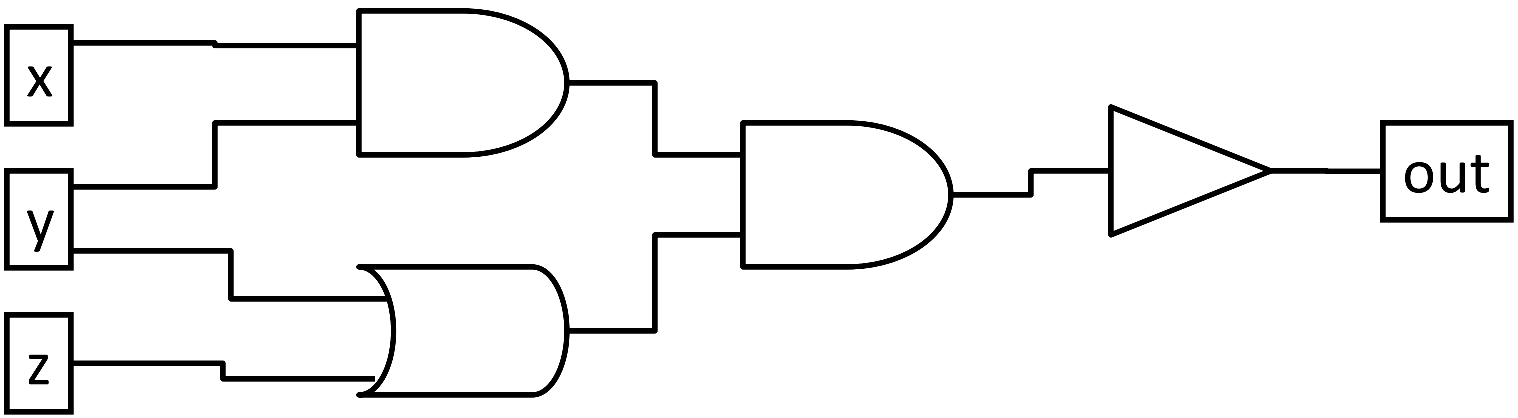

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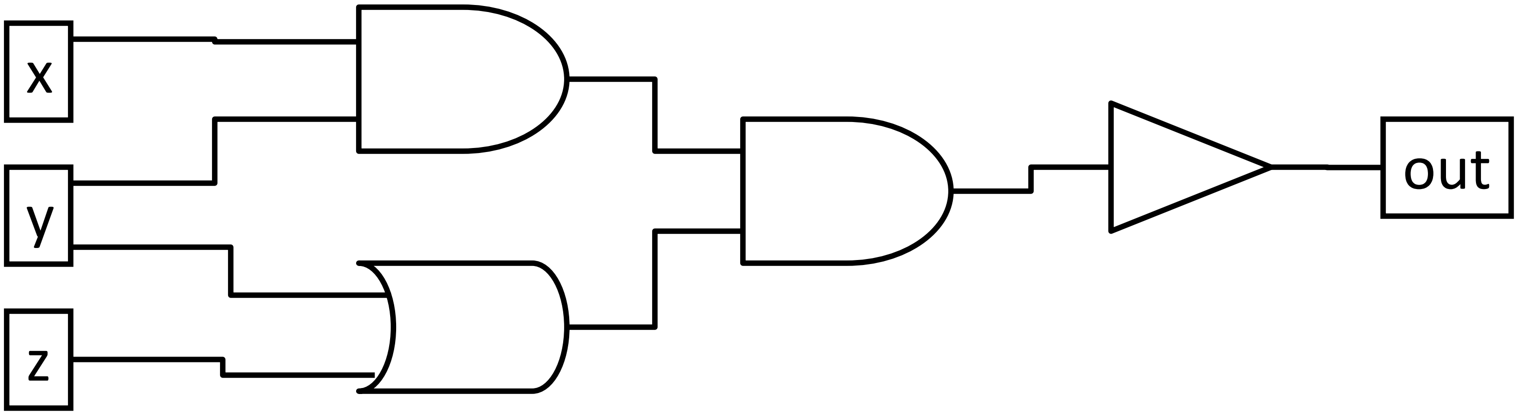

Important Reduction:

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- In other words, for some polynomial-time computable function F, it asks if there is an X so that F(X) = 1.
- Create a circuit C that computes F. The problem is equivalent to asking if there is an input for which C outputs 1.

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Note: Decision problems can be NP-Complete. For optimization problems, it is called NP-Hard.

Other NP-Complete/Hard Problems

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- Maximum Independent Set
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How do we show this? By finding reductions from other NP-Hard/Complete Problems.