Announcements

- Homework 4 Solutions online
- Homework 5 online due Friday

Last Time

- Dynamic Programming

Dynamic Programming

Our final general algorithmic technique:

1. Break problem into smaller subproblems.
2. Find recursive formula solving one subproblem in terms of simpler ones.
3. Tabulate answers and solve all subproblems.

Notes about DP

- General Correct Proof Outline:
  - Prove by induction that each table entry is filled out correctly
  - Use base-case and recursion
- Runtime of DP:
  - Usually
    [Number of subproblems]x[Time per subproblem]

More Notes about DP

- Finding Recursion
  - Often look at first or last choice and see what things look like without that choice
- Key point: Picking right subproblem
  - Enough information stored to allow recursion
  - Not too many

Today

- Chain Matrix Multiplication
- All Pairs Shortest Path

Chain Matrix Multiplication

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Runtime O(nmk)*

*Can do slightly better with Strassen, but we'll ignore this for now.

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How long does it take?

Example

A is 2x3,

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$2 \cdot 3 \cdot 3 = 18$

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(AB)C

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Runtime: 18 + 6 = 24

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$A(BC)$

$3 \cdot 3 \cdot 1 = 9$

$2 \cdot 3 \cdot 1 = 6$

Runtime: $9 + 6 = 15$

Multiplication order matters!

$(AB)C$

$2 \cdot 3 \cdot 3 = 18$

$2 \cdot 3 \cdot 1 = 6$

Runtime: $18 + 6 = 24$

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In particular, assume A_1 is an n_0 x n_1 matrix, A_2 is n_1 x n_2, generally A_k is an n_{k-1} x n_k matrix.

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  - $n_0n_kn_m$ to do final multiply
- Recursion $CMM(A_1,\ldots,A_m) = \min_k[CMM(A_1,\ldots,A_k)+CMM(A_{k+1},\ldots,A_m)+n_0n_kn_m]$

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- These require $CMM(A_i,A_{i+1},\ldots,A_j)$, but nothing else.
- Only need subproblems $C(i,j) = CMM(A_i,A_{i+1},\ldots,A_j)$ for $1 \leq i \leq j \leq m$.
  - Fewer than m^2 total subproblems.
  - Critical: Subproblem reuse.

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Solution order: Solve subproblems with smaller j-i first. This ensures that the recursive calls will already be in your table.

Example

Compute ABCD where

A is 2x5, B is 5x4, C is 4x3, D is 3x5

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Finish

Start

	A	B	C	D
A				
B				
C				
D				

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Finish

Start

	A	B	C	D
A				
B	X			
C	X	X		
D	X	X	X	

Illegal calls

Example

Compute ABCD where

A is 2x5, B is 5x4, C is 4x3, D is 3x5

Finish

Start

	A	B	C	D
A	0			
B	X	0		
C	X	X	0	
D	X	X	X	0

Base

Case

Example

Compute ABCD where

A is 2x5, B is 5x4, C is 4x3, D is 3x5

Finish

Start

	A	B	C	D
A	0	40		
B	X	0		
C	X	X	0	
D	X	X	X	0

$2 \cdot 5 \cdot 4 = 40$

Example

Compute ABCD where

A is 2x5, B is 5x4, C is 4x3, D is 3x5

Finish

Start

	A	B	C	D
A	0	40		
B	X	0	60	
C	X	X	0	
D	X	X	X	0

$5 \cdot 4 \cdot 3 = 60$

Example

Compute ABCD where

A is 2x5, B is 5x4, C is 4x3, D is 3x5

Finish

Start

	A	B	C	D
A	0	40		
B	X	0	60	
C	X	X	0	60
D	X	X	X	0

$4 \cdot 3 \cdot 6 = 60$

Example

Compute ABCD where

A is 2x5, B is 5x4, C is 4x3, D is 3x5

Finish

Start

	A	B	C	D
A	0	40	64	
B	X	0	60	
C	X	X	0	60
D	X	X	X	0

$2 \cdot 5 \cdot 3 + 60 = 90$

$2 \cdot 4 \cdot 3 + 40 = 64$

Example

Compute ABCD where

A is 2x5, B is 5x4, C is 4x3, D is 3x5

Finish

Start

	A	B	C	D
A	0	40	64	
B	X	0	60	135
C	X	X	0	60
D	X	X	X	0

$5 \cdot 4 \cdot 5 + 60 = 160$

$5 \cdot 3 \cdot 5 + 60 = 135$

Example

Compute ABCD where

A is 2x5, B is 5x4, C is 4x3, D is 3x5

Finish

Start

	A	B	C	D
A	0	40	64	94
B	X	0	60	135
C	X	X	0	60
D	X	X	X	0

$2 \cdot 5 \cdot 5 + 135 = 185$

$2 \cdot 4 \cdot 5 + 40 + 60$

$= 140$

$2 \cdot 3 \cdot 5 + 64 = 94$

Example

Compute ABCD where

A is 2x5, B is 5x4, C is 4x3, D is 3x5

Finish

Start

	A	B	C	D
A	0	40	64	94
B	X	0	60	135
C	X	X	0	60
D	X	X	X	0

(((AB)C)D)

Runtime

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Final Runtime: $O(m^3)$

DP Setup

Sometimes there are many ways to create a DP for a given problem, and how exactly you set it up will have a large effect on runtime.

All Pairs Shortest Paths

Problem: Given a graph G with (possibly negative) edge weights, compute the length of the shortest path between every pair of vertices.

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Problem: Given a graph G with (possibly negative) edge weights, compute the length of the shortest path between every pair of vertices.

Note: `Bellman-Ford` computes single-source shortest paths. Namely, for some fixed vertex s it computes all of the shortest paths lengths d(s,v) for every v.

Repeated `Bellman-Ford`

Easy Algorithm: Run `Bellman-Ford` with source s for each vertex s.

Runtime: $O(|V|^2|E|)$

Dynamic Program

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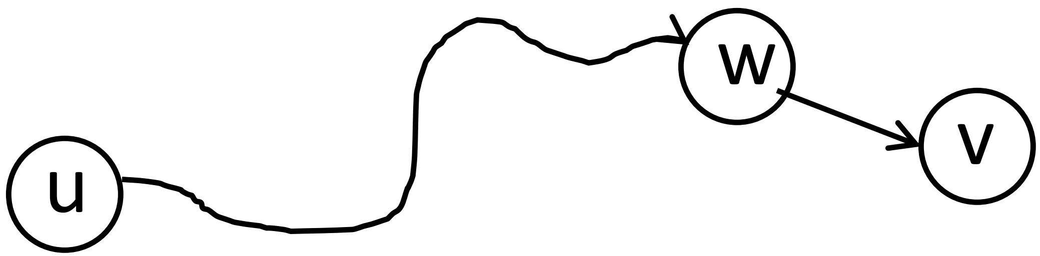

- Consider last edge.

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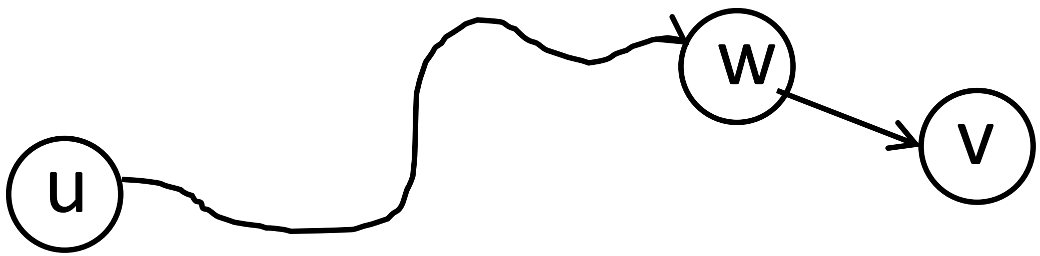

- Consider last edge.
- Length k-1 path from u to w, edge from w to v.
- $d_k(u,v) = \min_w[d_{k-1}(u,w) + \ell(w,v)]$

Matrix Multiplication Method

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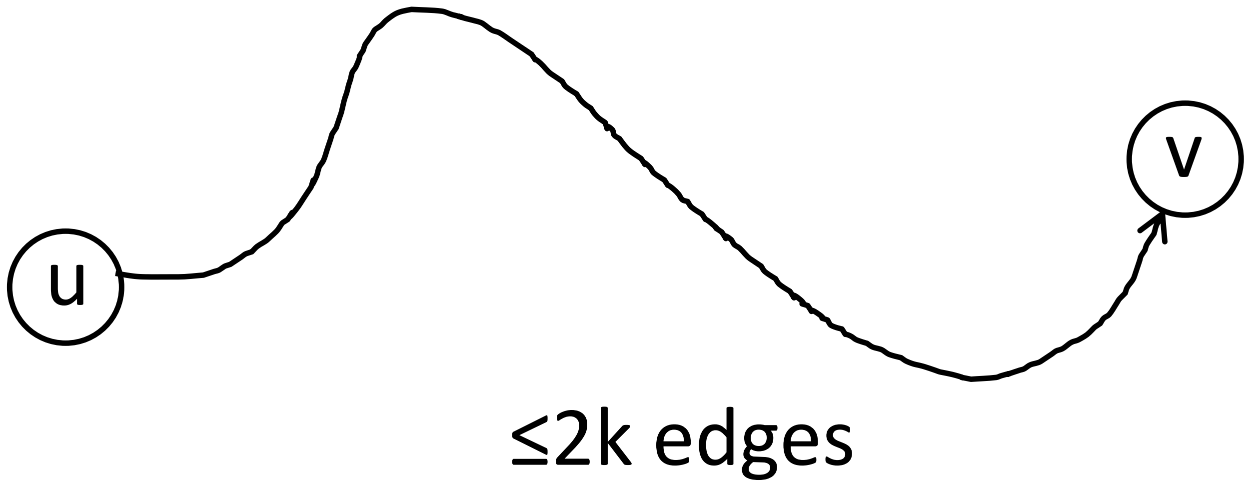
- `Bellman-Ford` is slow in part because we can only increase k by one step at a time.
- This happens because we cut off only the last edge of the optimal path.
- What if instead we cut it in the middle?

Recursion



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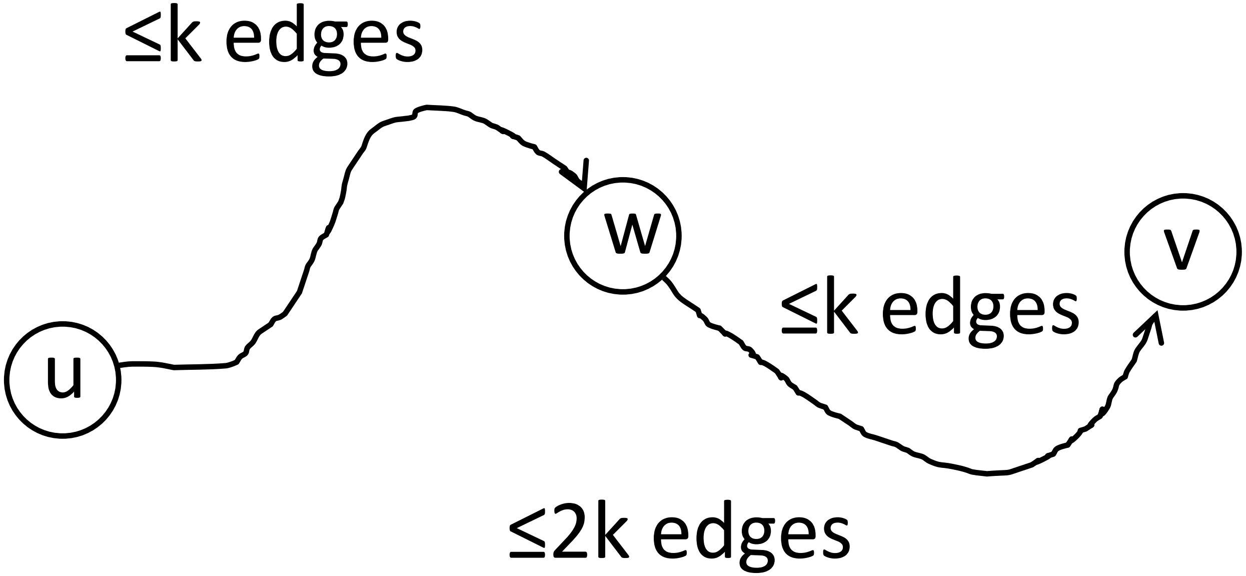

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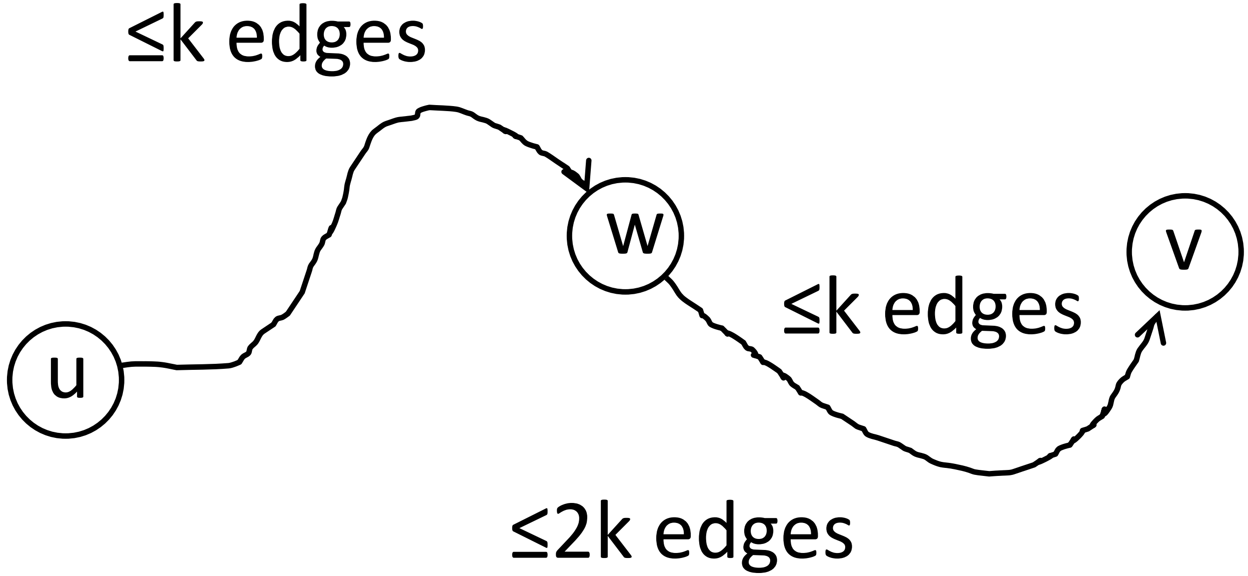

$$d_{2k}(u,v) = \min_{w \in V}(d_k(u,w) + d_k(w,v)).$$

Algorithm

Base Case:

$$d_1(u,v) = \begin{cases} 0 & \text{if } u = v \\ \ell(u,v) & \text{if } (u,v) \in E \\ \infty & \text{otherwise} \end{cases}$$

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End Condition: Compute $d_1, d_2, d_4, \ldots d_m$ with $m > |V|$.

Algorithm

$O(|V|^2)$

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$O(\log|V|)$ iterations

Algorithm

Runtime: $O(|V|^3 \log|V|)$

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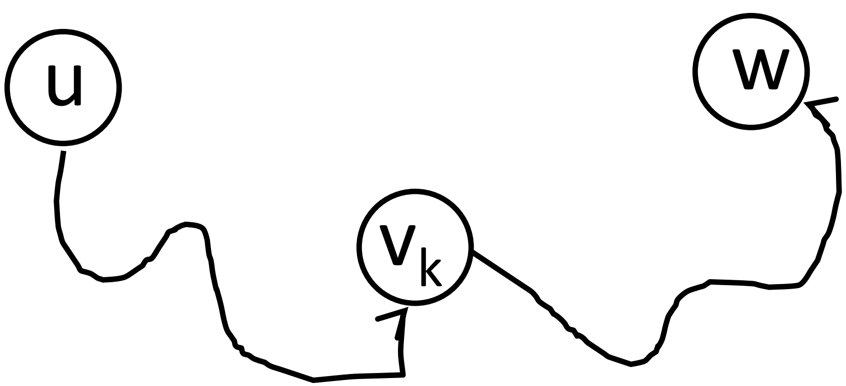
- The shortest path using v_k has length $d_{k-1}(u,v_k)+d_{k-1}(v_k,w)$.



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$O(|V|^2)$

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Algorithm

O(|V|²)

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O(|V|) Iterations

Algorithm

Runtime: $O(|V|^3)$

$O(|V|^2)$

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$O(|V|)$ Iterations

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1. Run `Bellman-Ford` once to compute d(v).
2. Problem equivalent to using
   $\ell'(u,w) = \ell(u,w)+d(u)-d(w) \geq 0$.
3. Run `Dijkstra` from every source.

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1. Run `Bellman-Ford` once to compute d(v).
2. Problem equivalent to using
   $\ell'(u,w) = \ell(u,w)+d(u)-d(w) \geq 0$.
3. Run `Dijkstra` from every source.

Runtime: $O(|V||E|+|V|^2\log|V|)$