Announcements

- Homework 3 Solutions online
- No homework this week
- Exam 2 on Friday
- W 10:30-12:00 office hours (this week) will be held online at: https://ucsd.zoom.us/j/9296249412

Last Time

- Greedy Algorithms
- Exchange Arguments

Greedy Algorithms

General Algorithmic Technique:

1. Find decision criterion
2. Make best choice according to criterion
3. Repeat until done

Surprisingly, this sometimes works.

Exchange Argument

- Greedy algorithm makes a sequence of decisions D_1, D_2, D_3,…,D_n eventually reaching solution G.
- Need to show that for arbitrary solutions A that G ≥ A.
- Find sequence of solutions
  $A=A_0$, A_1, A_2,…,A_n = G
  so that:
  - $A_i \leq A_{i+1}$
  - A_i agrees with D_1,D_2,…,D_i

Exchange Argument

In particular, we need to show that given any A_i consistent with $D_1,\ldots,D_i$ we can find an A_{i+1} so that:

- A_{i+1} is consistent with $D_1,\ldots,D_{i+1}$
- $A_{i+1} \geq A_i$

Then we inductively construct sequence

$A=A_0 \leq A_1 \leq A_2 \leq \ldots \leq A_n = G$

Thus, $G \geq A$ for any A. So G is optimal.

Today

- Huffman Codes
- Minimum Spanning Trees

Huffman Codes

- Want to encode string of letters in binary.

  Ex: `ABCDACBDAD`

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- `A = 00, B = 01, C = 10, D = 11`

A	B	C	D	A	C	B	D	A	D
00	01	10	11	00	10	01	11	00	11

Huffman Codes

- Want to encode string of letters in binary.

Ex: ABCDACBDAD

- A = 00, B = 01, C = 10, D = 11

A	B	C	D	A	C	B	D	A	D
00	01	10	11	00	10	01	11	00	11

- Use two bits to encode each letter.

Question: Encoding Length

Using the coding scheme from the last slide, how many bits are needed to encode a string of n `A`s, `B`s, `C`s and `D`s?

1) 2
2) n
3) 2n
4) 4n
5) n^2

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You need two bits per letter.

Non-Fix Length Encodings

- Suppose instead we had to decode:

`AAABAACBAABADAAA`

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- 16 Letters requires 32 bits.

Non-Fix Length Encodings

- Suppose instead we had to decode:

  `AAABAACBAABADAAA`

- 16 Letters requires 32 bits.
- Note that there are a lot of `A`s here. If we could find a way to encode them with fewer bits, we could save a lot.

Unique Decoding

Cannot do any encoding we like.

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Suppose we tried:

`A = 0, B = 1, C = 10, D = 01`

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How do you decode `01`? Either `AB` or `D`.

Unique Decoding

Cannot do any encoding we like.

Suppose we tried:

`A = 0, B = 1, C = 10, D = 01`

How do you decode `01`? Either `AB` or `D`.

Problem: The encoding for `A` is a prefix of the encoding for `D`. When you see it, you don't know if it's an `A`, or the start of a `D`.

Prefix Free Encodings

Definition: An encoding is prefix-free if the encoding of no letter is a prefix of the encoding of any other.

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Example:

`A = 0, B = 10, C = 110, D = 111`

Lemma: Any prefix-free encoding can be uniquely decoded.

Example

`A = 0, B = 10, C = 110, D = 111`

Decode:

`00010001101000100111000`

Example

A = 0, B = 10, C = 110, D = 111

Decode:

00010001101000100111000

A

Example

A = 0, B = 10, C = 110, D = 111

Decode:

00010001101000100111000

A A

Example

A = 0, B = 10, C = 110, D = 111

Decode:

00010001101000100111000

AAA

Example

A = 0, B = 10, C = 110, D = 111

Decode:

00010001101000100111000

A A A B

Example

A = 0, B = 10, C = 110, D = 111

Decode:

0001000110100010011100 0

A A A B A

Example

A = 0, B = 10, C = 110, D = 111

Decode:

00010001101000100111000

AAA B AA

Example

A = 0, B = 10, C = 110, D = 111

Decode:

00010001101000100111000

AAA B AA C

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A = 0, B = 10, C = 110, D = 111

Decode:

0001000110100010011000

A A A B A A C B

Example

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Decode:

`00010001101000100111000`

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A A A B A A C B A A B A D A A A

Only 23 bits instead of 32!

Optimal Encoding

<u>Problem:</u> Given a string, S, find a prefix-free encoding that encodes S using the fewest number of bits.

How Long is the Encoding?

If for each letter x in our string, x appears f(x) times and if we encode x as a string of length ℓ(x), the total encoding length is:

$$\Sigma f(x) \cdot \ell(x).$$

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Our example has:

11 `A`s, 3 `B`s, 1 `C`, 1 `D`.

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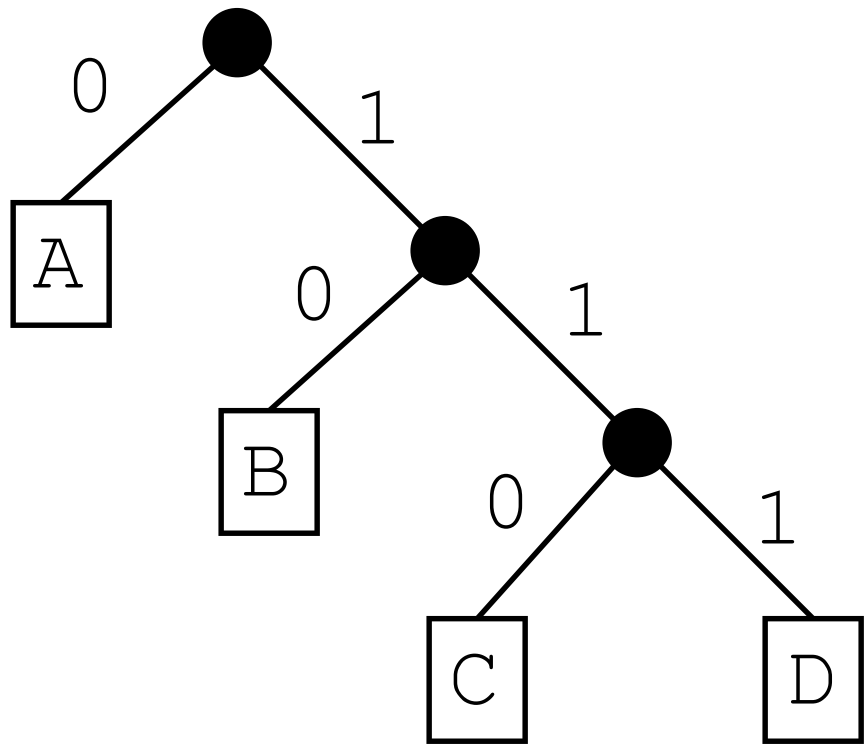
Our example has:

11 `A`s, 3 `B`s, 1 `C`, 1 `D`.

These are the frequencies. We need to find the best encoding.

Tree Representation

Can represent prefix-free encoding as a tree.



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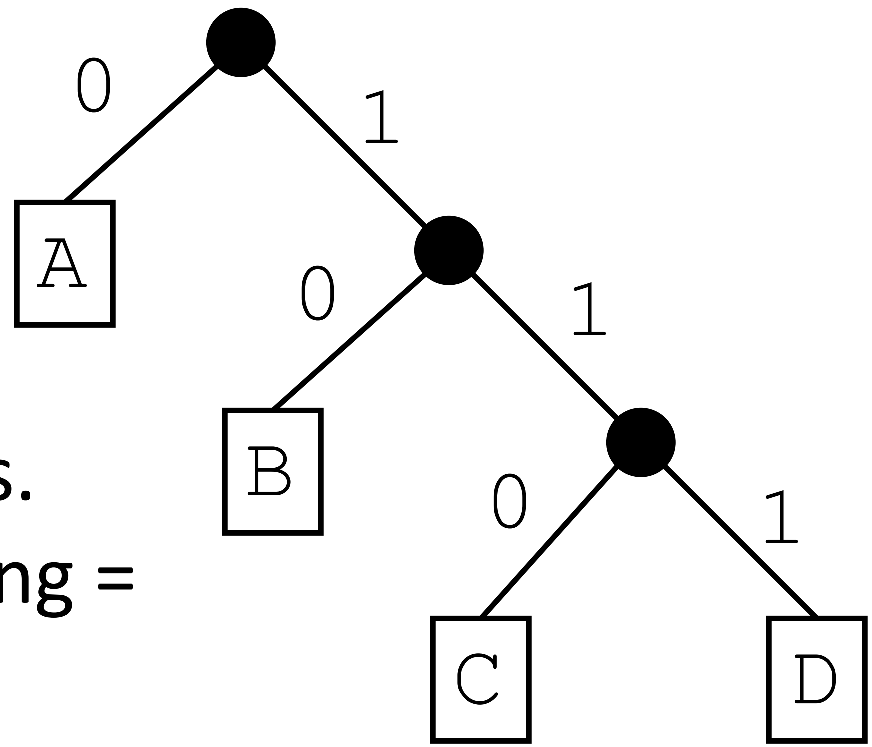
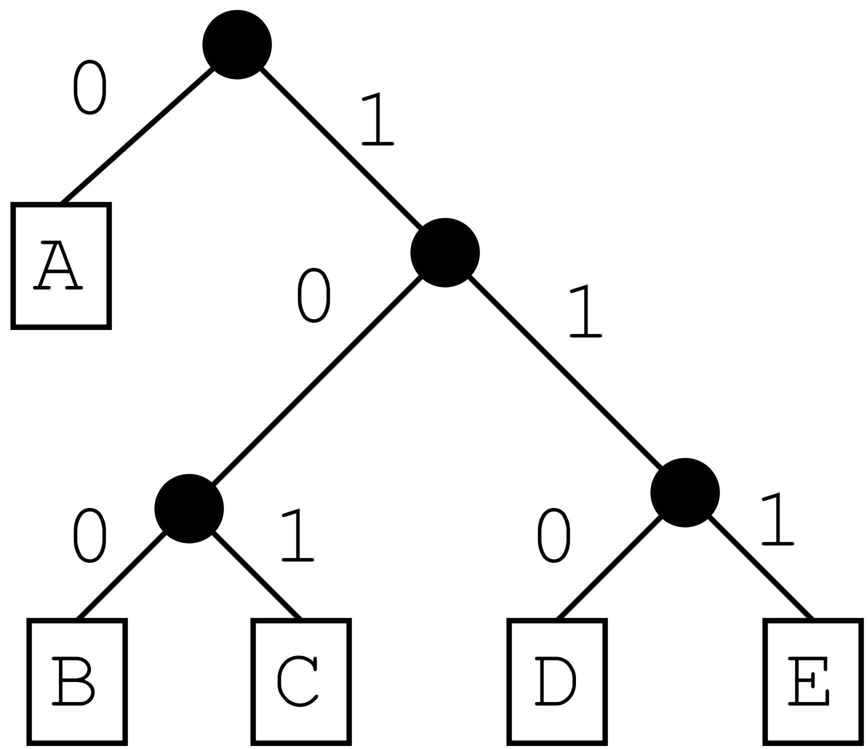


Letters are leaves.
Length of encoding =
Depth of leaf.

Question: Tree Decoding

What letter does the string `111` correspond to in this tree?



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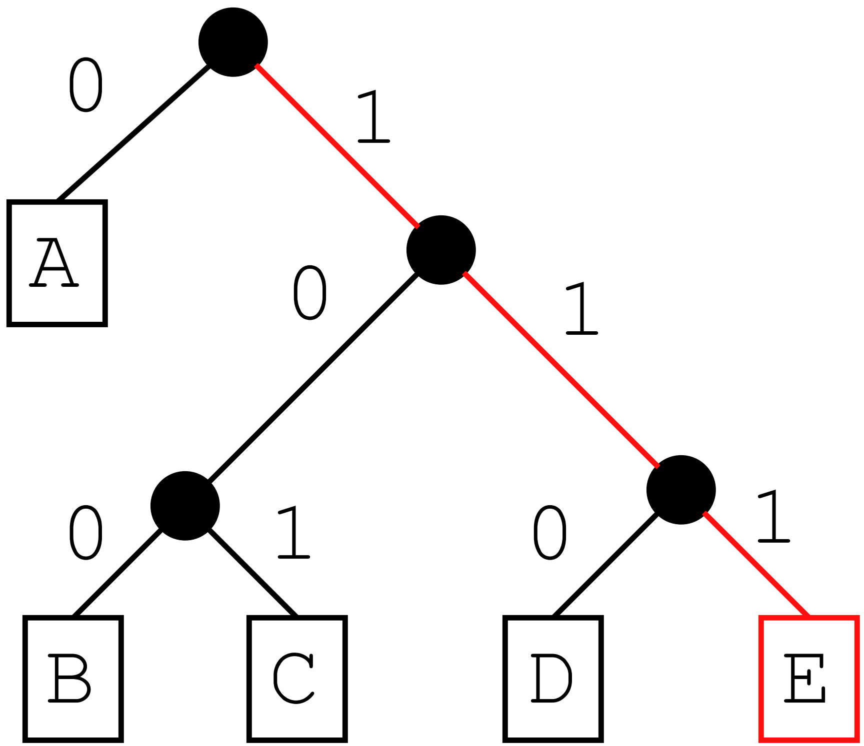
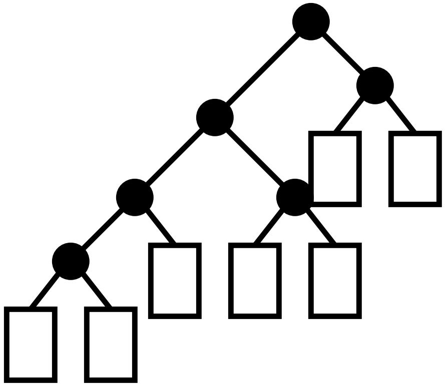


Placement of Leaves

Suppose we know the tree structure. Where do we put the letters?



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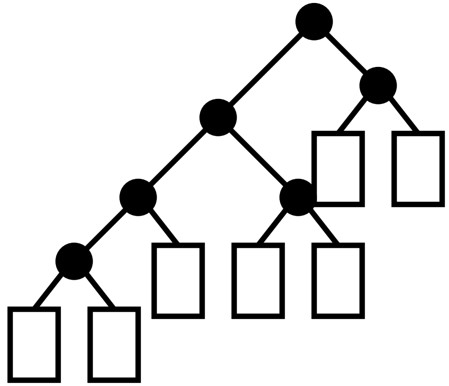

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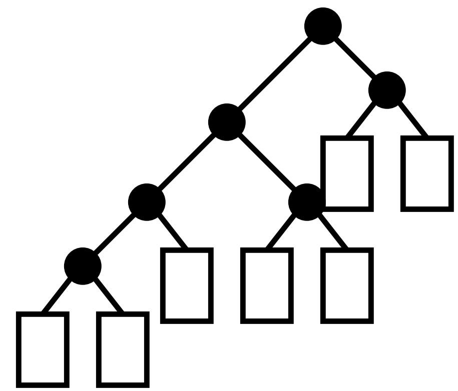
Objective = Σ freq(x)depth(x)

Placement of Leaves



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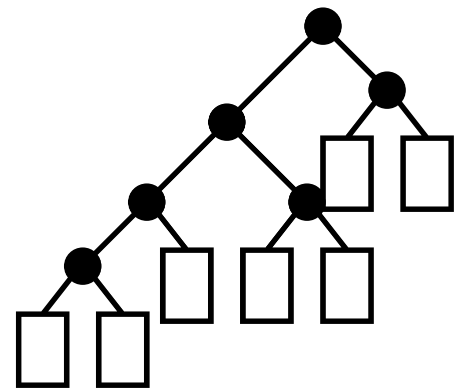

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<u>Letter frequencies</u>

`Ax10, Bx15, Cx4, Dx22, Ex31, Fx5, Gx19`

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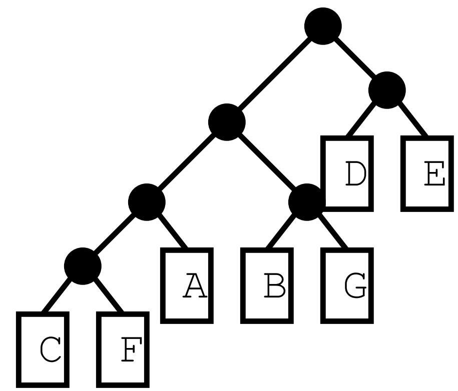

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Example

Frequencies:

`Ax30, Bx15, Cx25, Dx50, Ex65`

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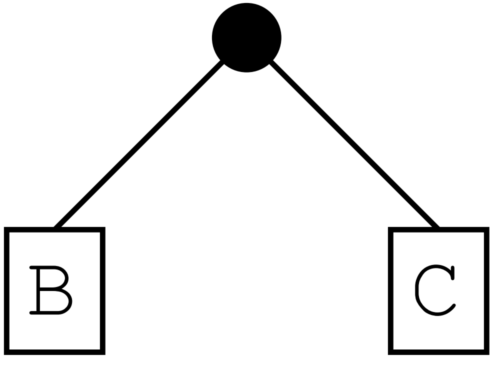

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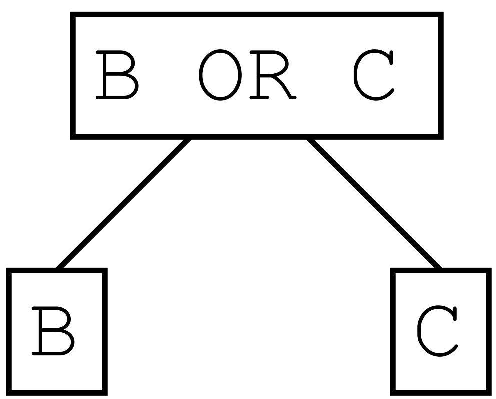

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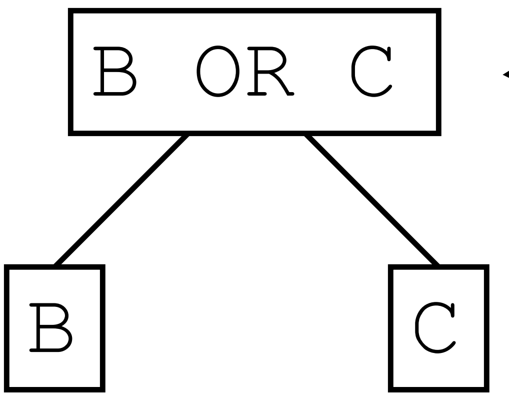
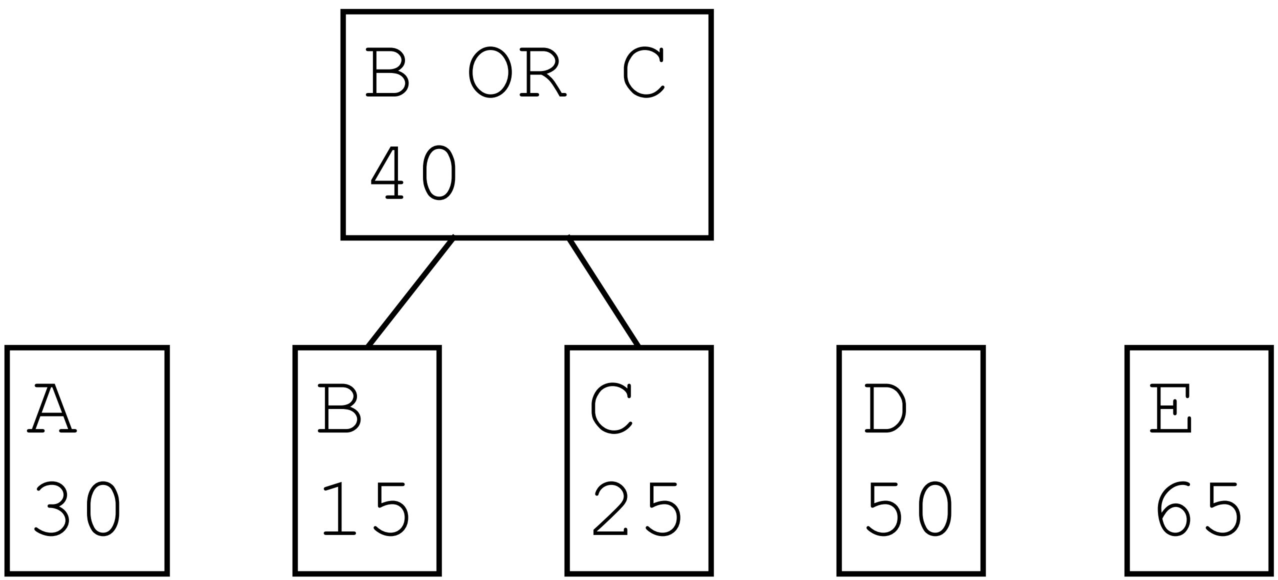


Think of as a new node of weight 15+25=40

Example

A	B	C	D	E
30	15	25	50	65

Example



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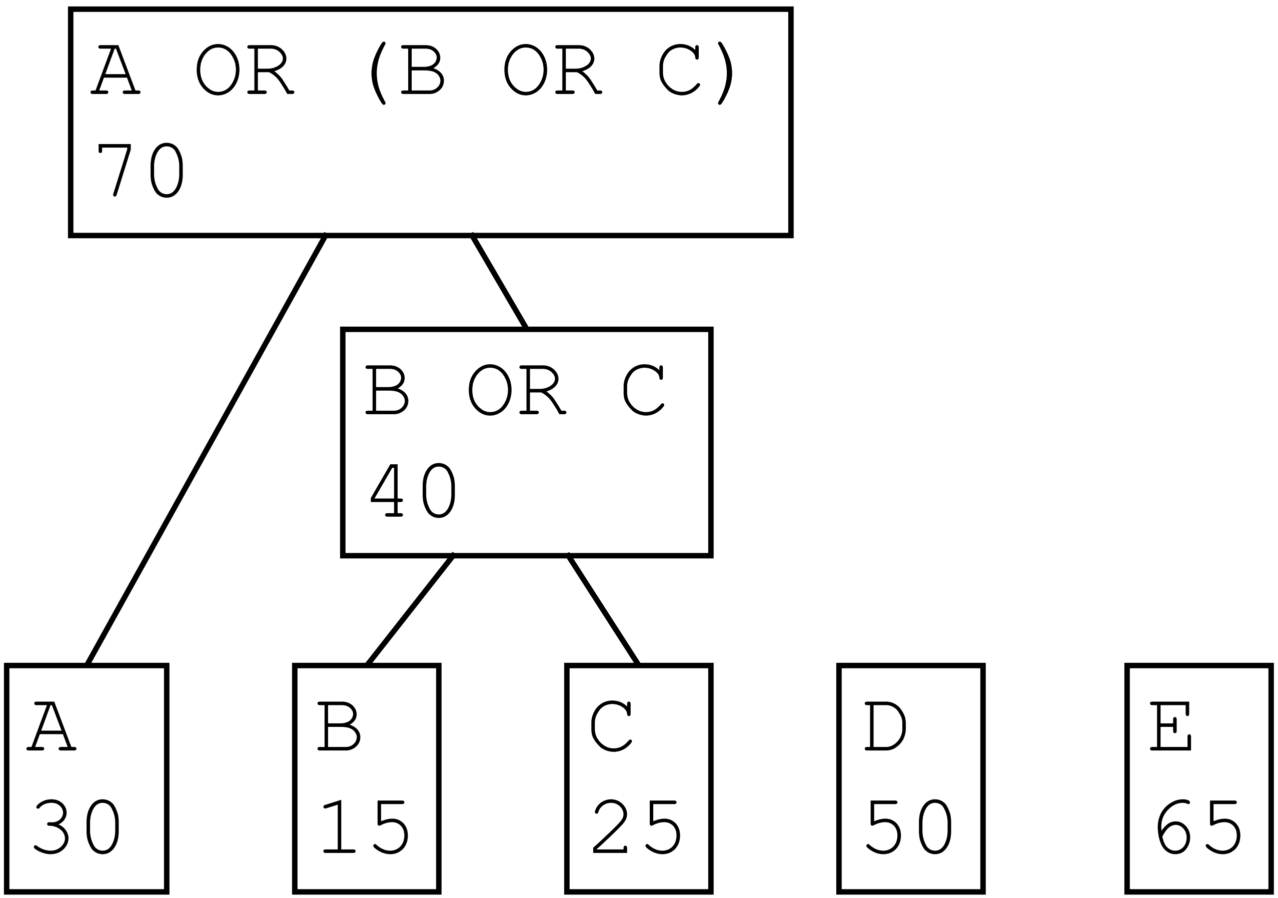

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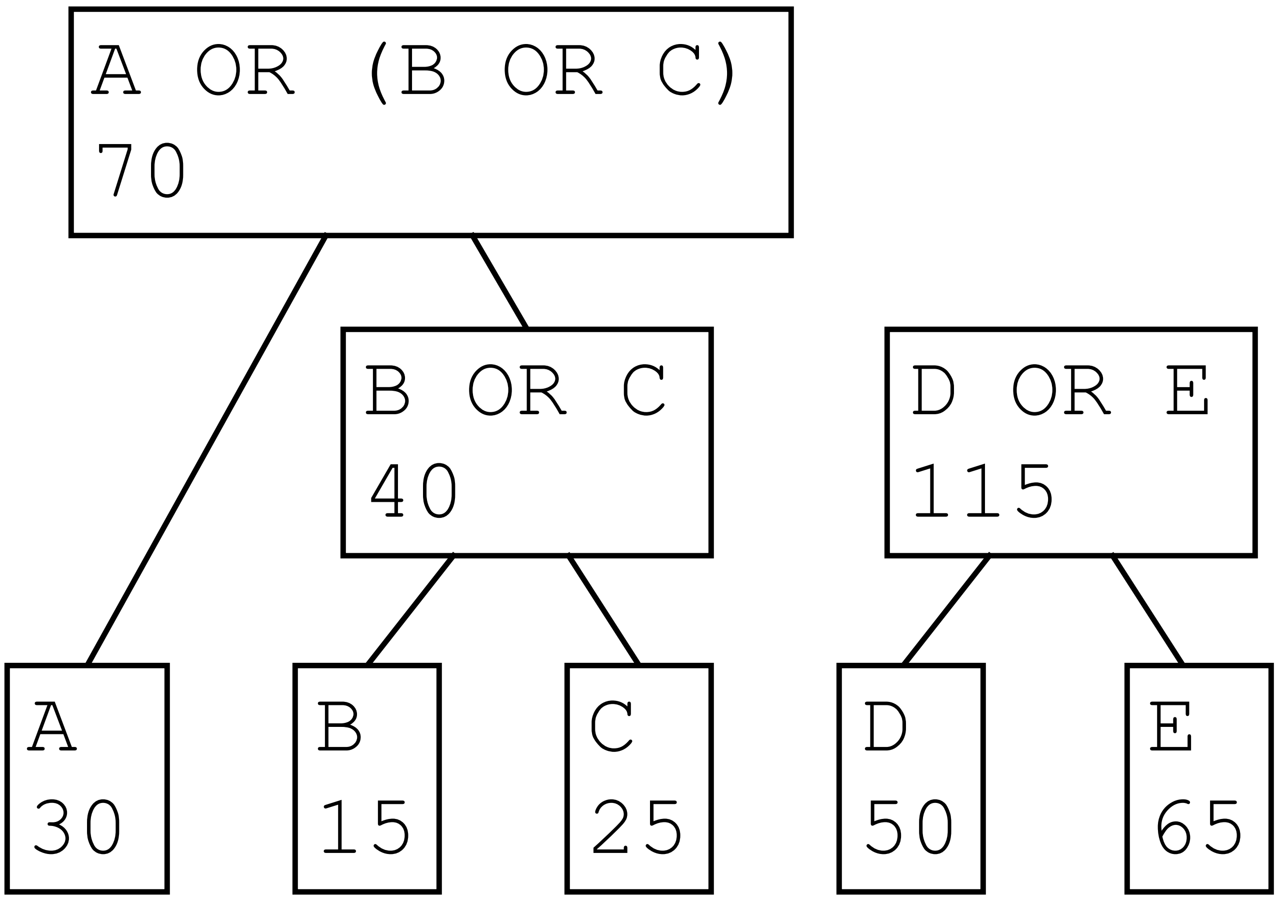

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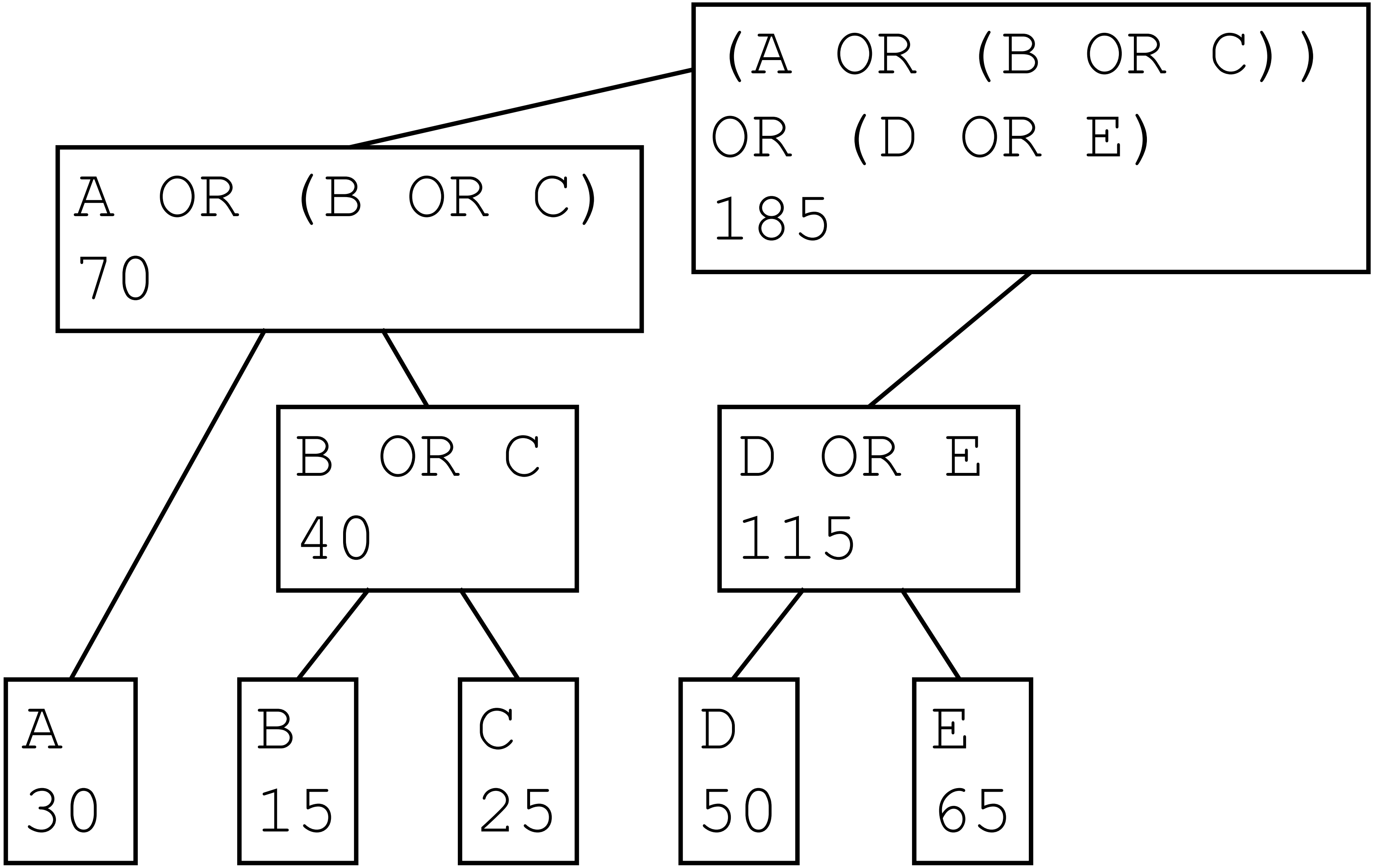
(A OR (B OR C)) OR (D OR E)
185
A OR (B OR C)
70
B OR C
40
D OR E
115
A
30
B
15
C
25
D
50
E
65

Algorithm

```
HuffmanTree(L)
  While(at least two left)
    x, y ← Two least frequent
    z new node f(z) ← f(x)+f(y)
    x and y children of z
    Replace x and y with z in L
  Return remaining elt of L
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Better with priority queue.

Optimized Algorithm

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HuffmanTree(L)
  Priority queue Q
  Insert all elements of L to Q
  While(|Q| ≥ 2)
    x ← Q.DeleteMin()
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O(n log(n)) (Priority queue Q / Insert all elements of L to Q)

O(n) Iterations (While(|Q| ≥ 2))

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O(n log(n)) — Priority queue Q / Insert all elements of L to Q

O(n) Iterations — While(|Q| ≥ 2)

O(log(n)) — loop body

Runtime: O(n log(n))

Proof of Correctness

- Know that there is a correct solution with lightest elements as siblings
- *If* we require that lightest elements are siblings, problem is *equivalent* to smaller Huffman tree problem
- By induction, smaller problem is solved correctly

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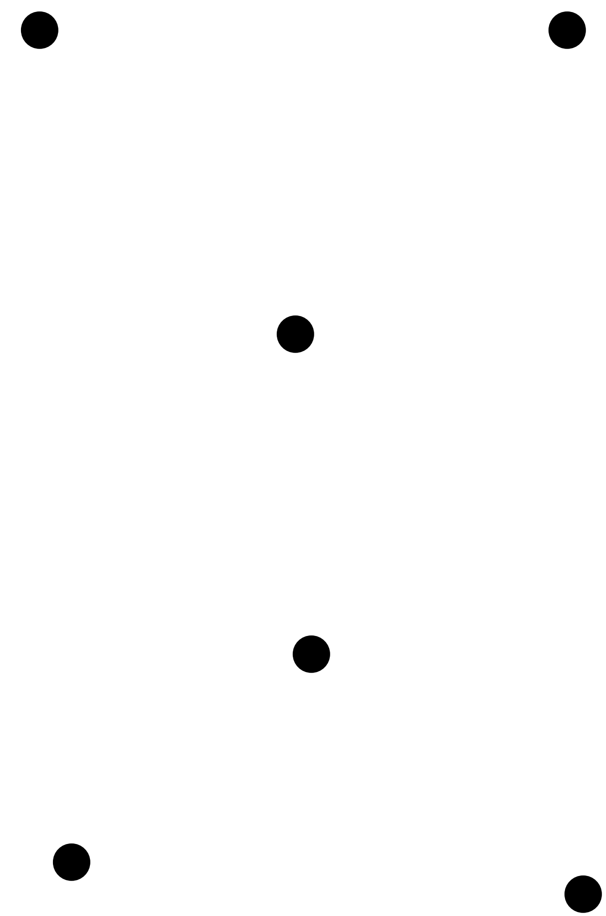

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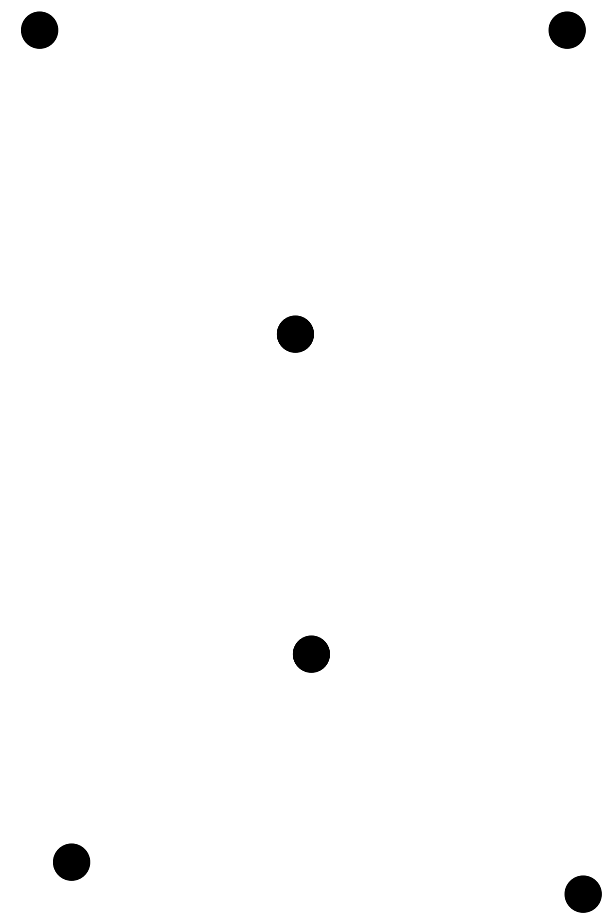
- Suppose that you have a collection of cities that you would like to connect by roads.
- There are several potential roads you could build.



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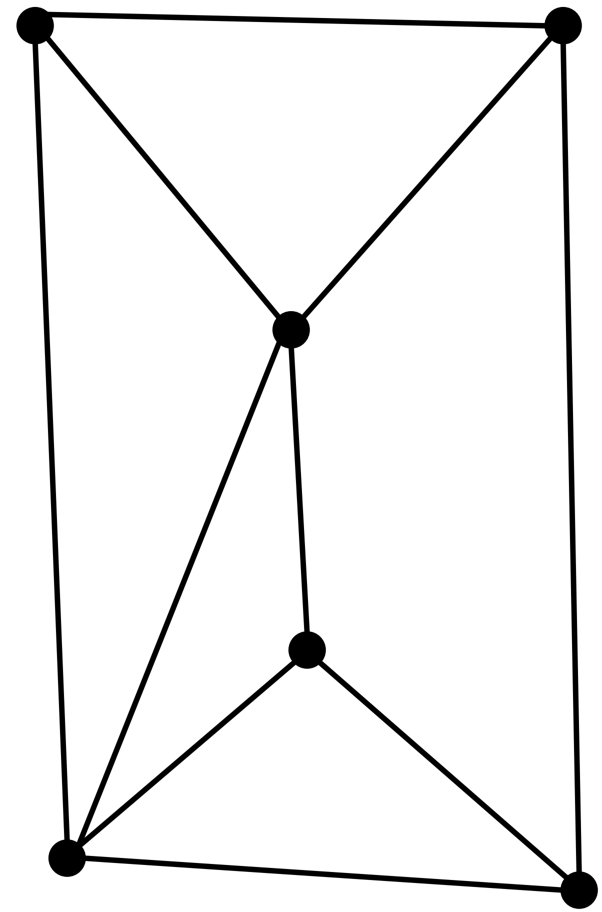

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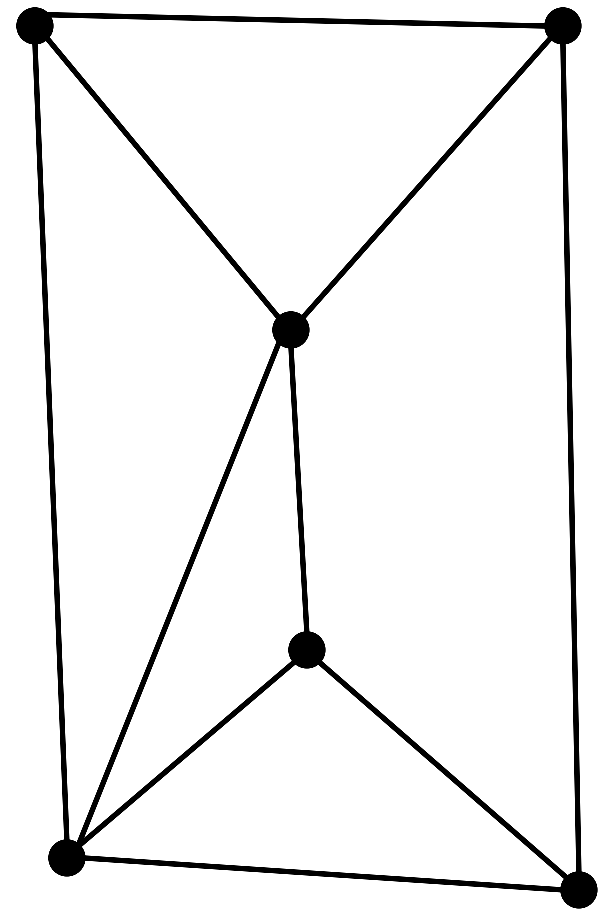

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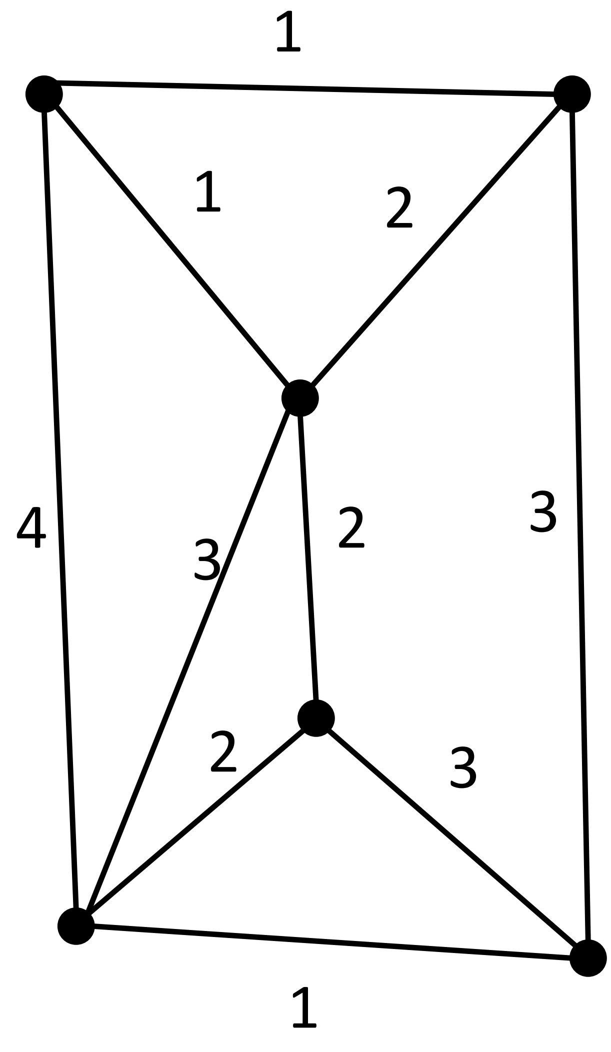

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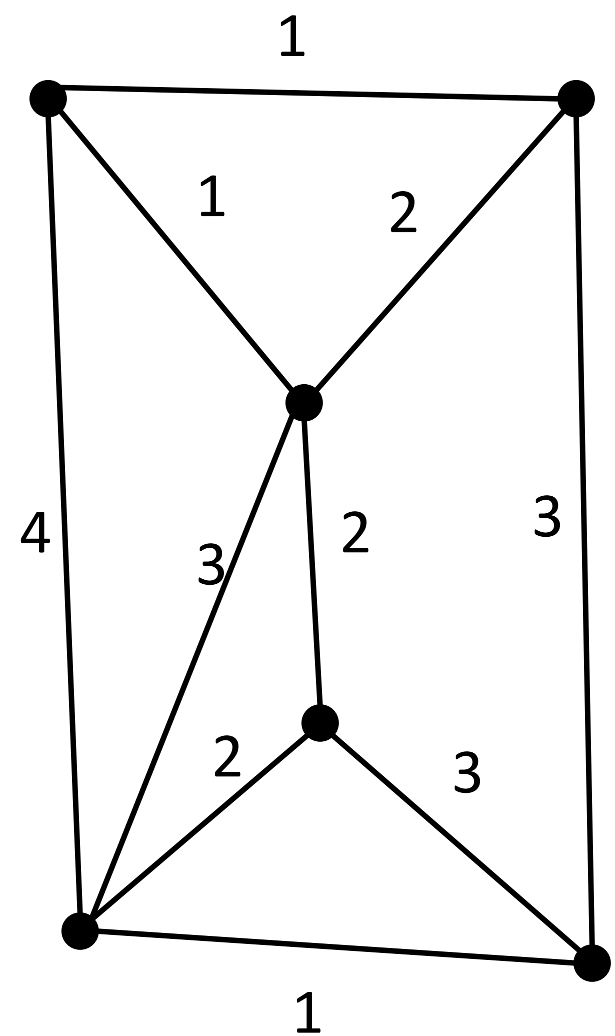

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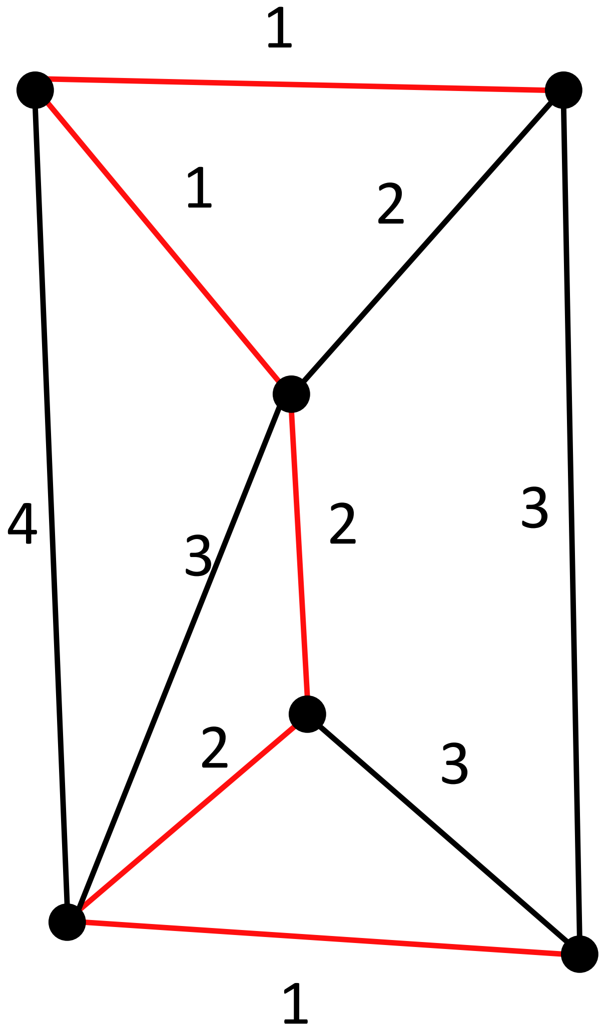

Minimum Spanning Trees

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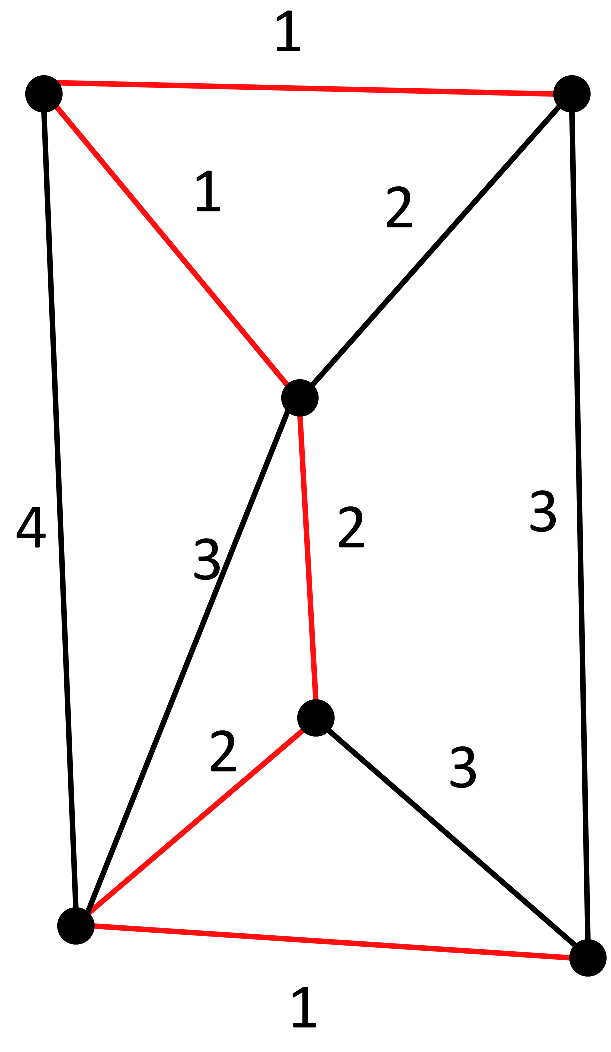
- What is the cheapest way to connect them? 1+1+1+2+2=7



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A <u>spanning tree</u> in a graph G, is a subset of the edges of G that connect all vertices and have no cycles.

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Definition: A tree is a connected graph, with no cycles.

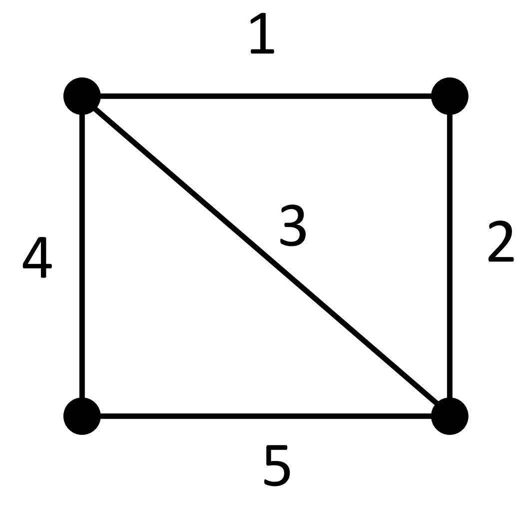
A spanning tree in a graph G, is a subset of the edges of G that connect all vertices and have no cycles.

If G has weights, a minimum spanning tree is a spanning tree whose total weight is as small as possible.

Question: MST

What is the weight of the minimum spanning tree of the graph below?

A) 5
B) 6
C) 7
D) 8
E) 9



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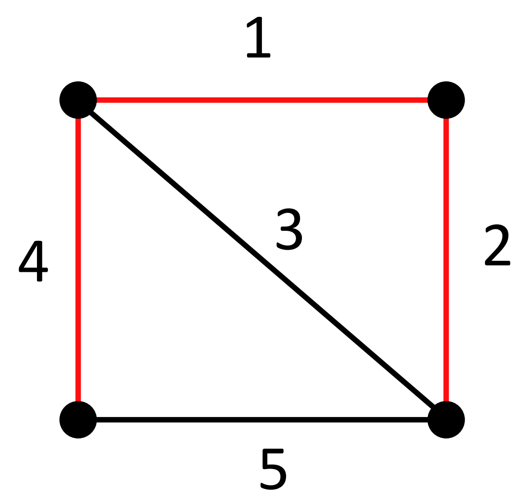

Basic Facts about Trees

<u>Lemma:</u> For an undirected graph G, any two of the below imply the third:

1. $|E| = |V|-1$
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Corollary: If G is a tree, then $|E| = |V|-1$.

Proof Idea

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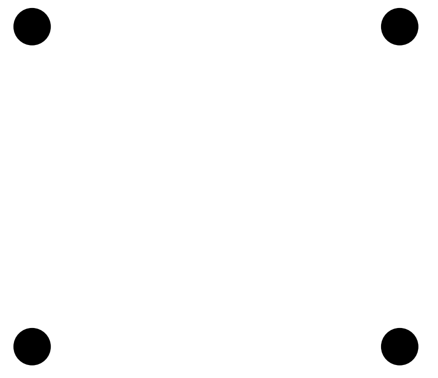

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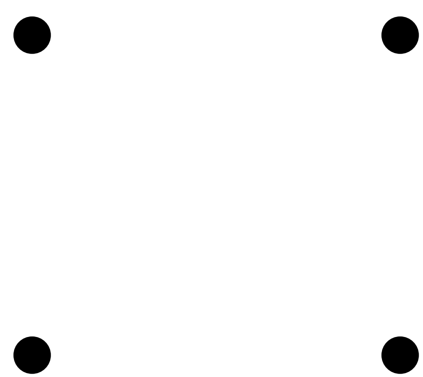

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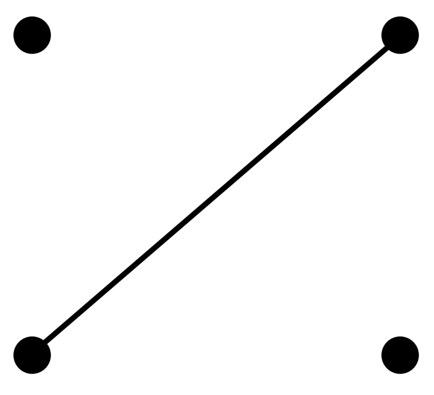

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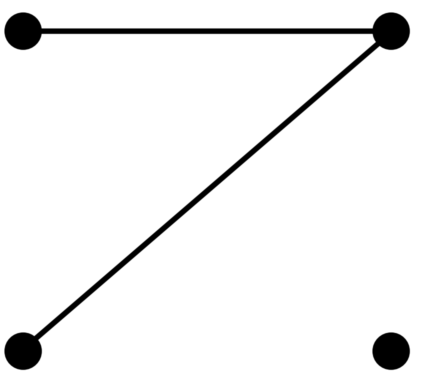

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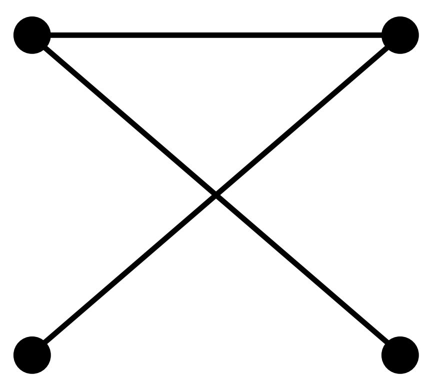
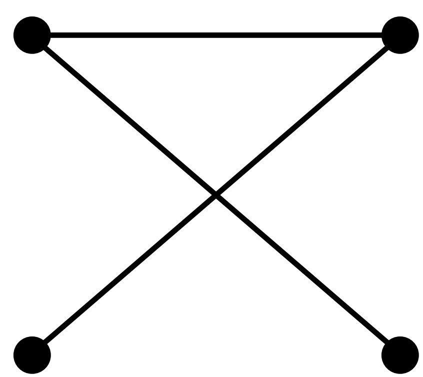


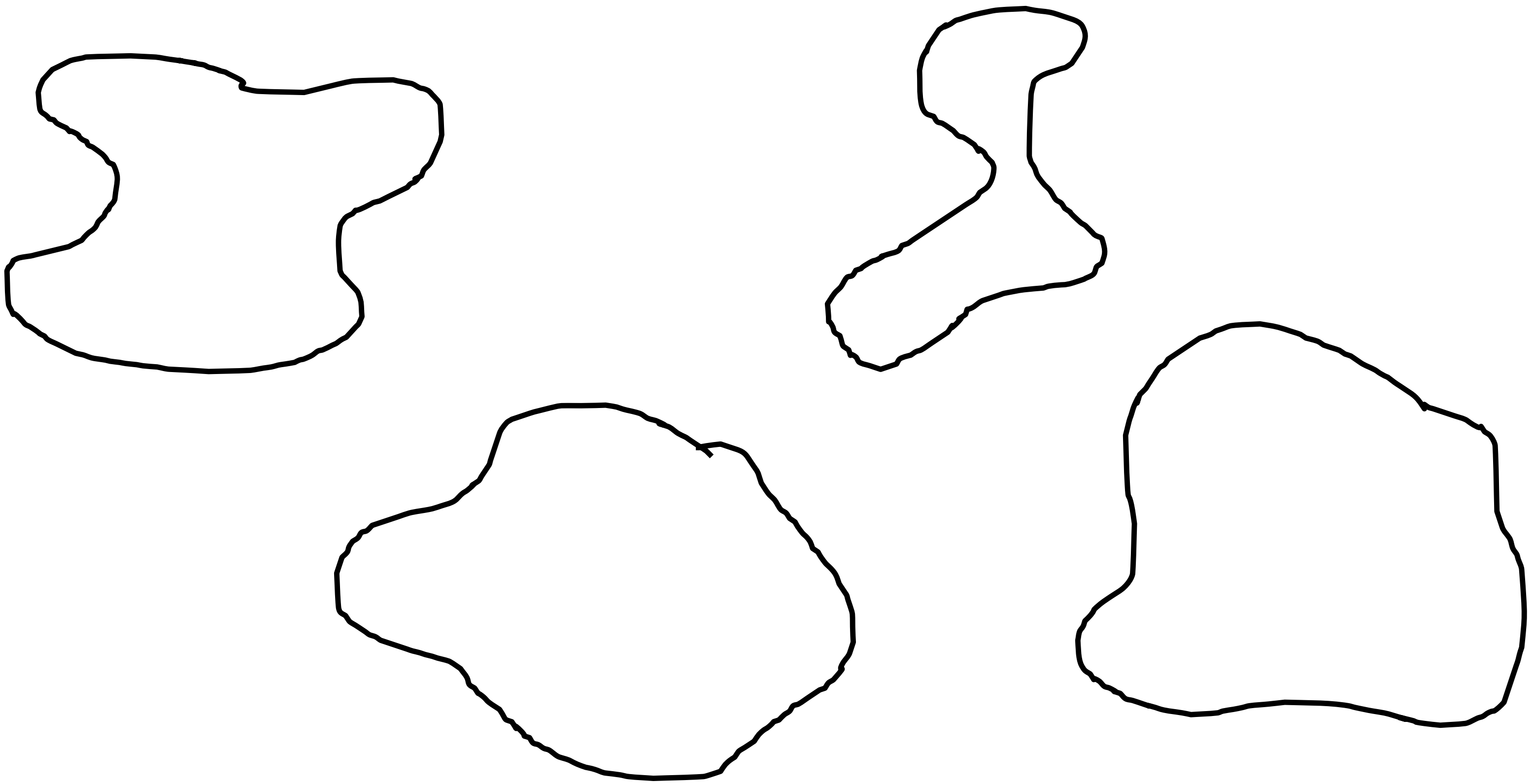
Proof Idea

- Start with a graph with no edges.
- Add edges one at a time.
- Count number of connected components.



Extra Edge

An extra edge decreases the number of CCs by 1 *unless* it creates a cycle.



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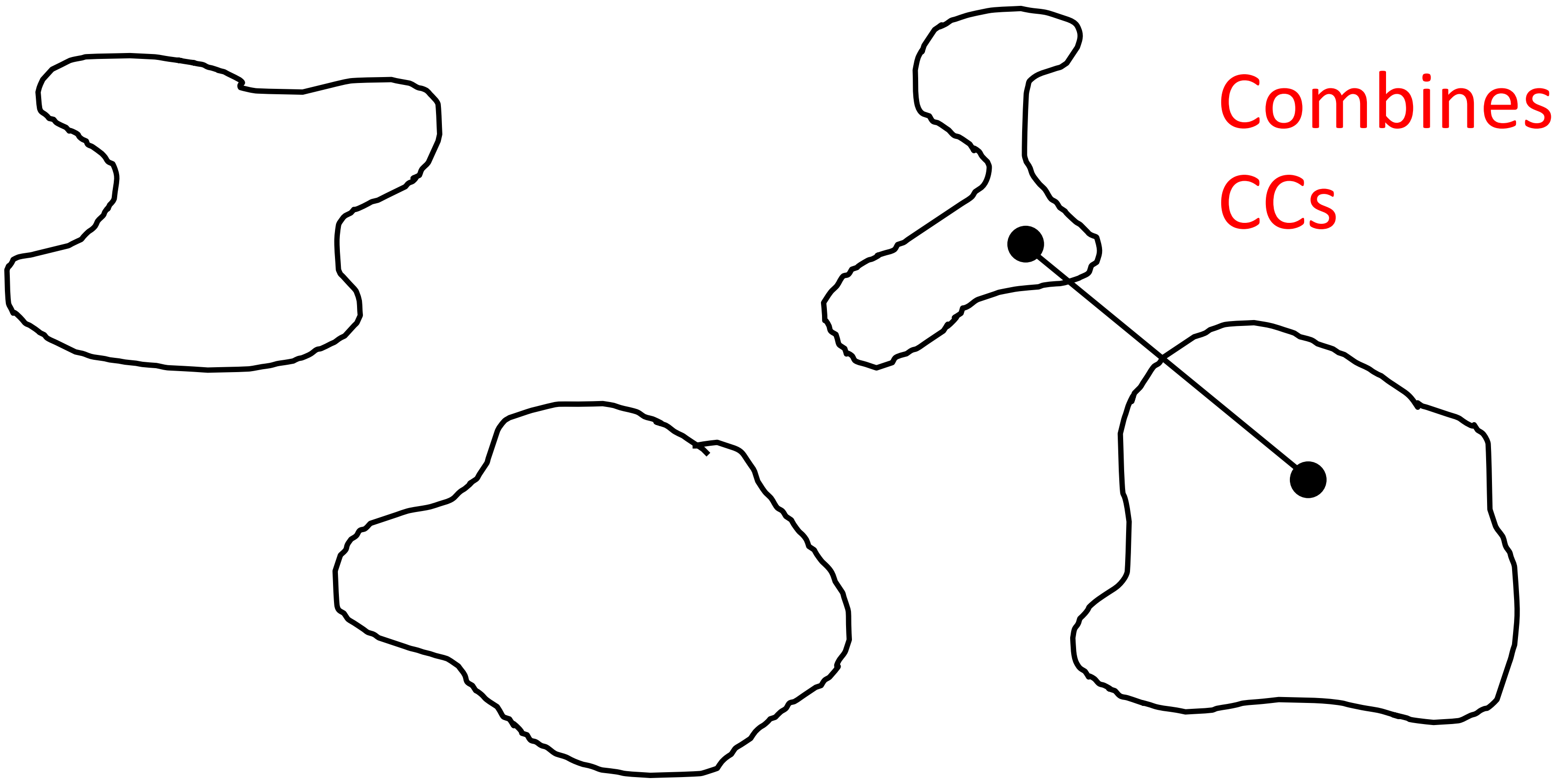

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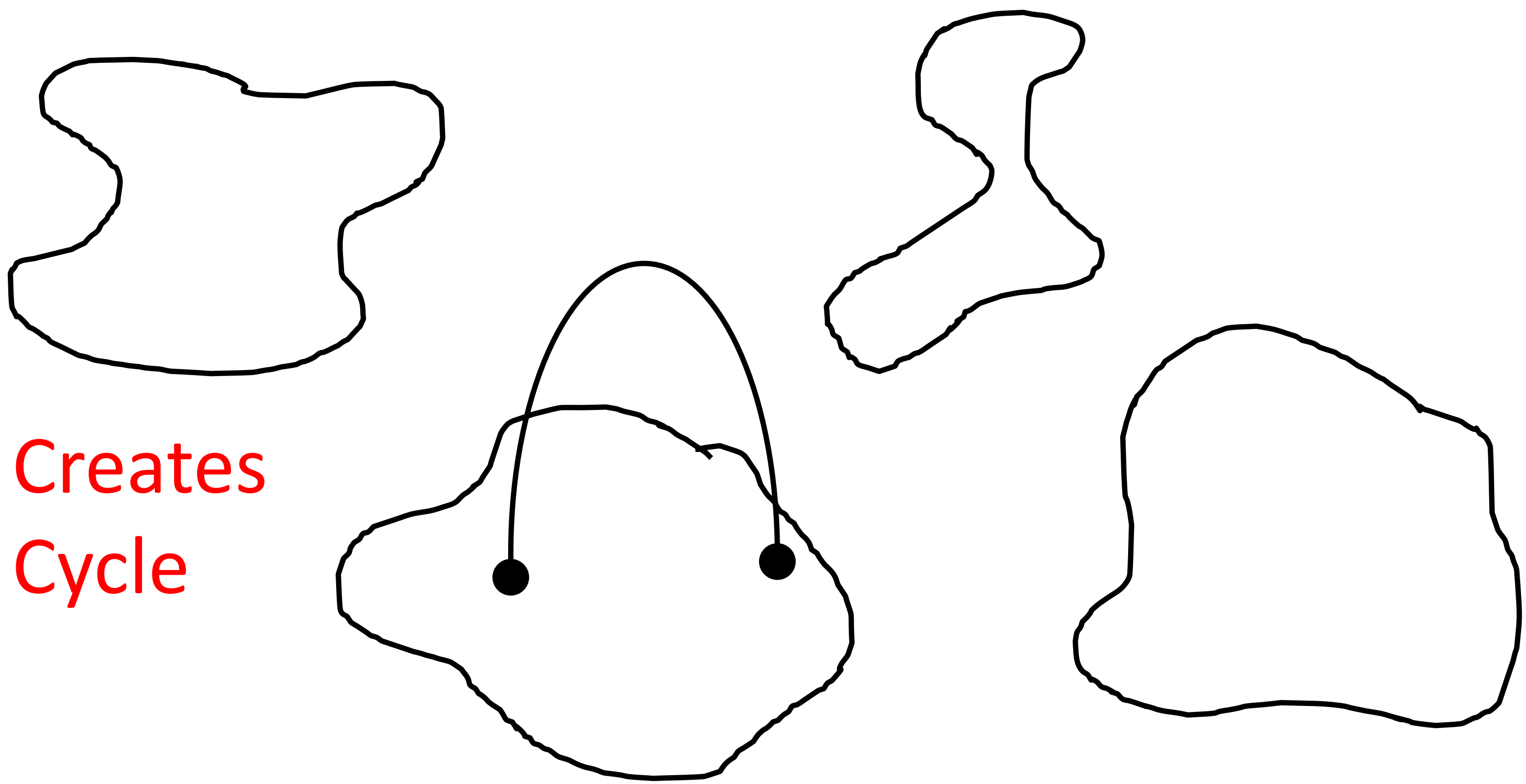

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If connected and no cycles, each edge decreases by 1, so must be |V|-1 edges.