Announcements

- No Homework 6

Last Time

- NP Complete/Hard
  - As hard as any problem in NP
- Circuit SAT, 3-SAT, Maximum Independent Set are NP Complete/Hard

Lemma

A 3-SAT instance is satisfiable if and only if it is possible to select one term from each clause without selecting both a variable and its negation.

Zero-One Equations

Problem: Given a matrix A with only 0 and 1 as entries and b a vector of 1s, determine whether or not there is an x with 0 and 1 entries so that

$$Ax = b.$$

Today

- NP Hardness of Knapsack and TSP

3-SAT → ZOE

Basic Idea:

- Use the one term from each clause formulation of 3-SAT.
- Create one variable for each term to denote whether or not it has been selected.
- Add equations to enforce exactly one term from each clause, no contradictory terms selected.

Example

$$(x \vee y \vee z) \wedge (\bar{x} \vee y) \wedge (\bar{y} \vee \bar{x})$$

x_1 x_2 x_3 x_4 x_5 x_6 x_7

One term per clause:

$x_1 + x_2 + x_3 = 1$

$x_4 + x_5 = 1$

$x_6 + x_7 = 1$

Replace

$a + b \leq 1$ with

$a + b + c = 1$

No Contradictions:

$x_1 + x_4 \leq 1$ ~~$x_8 = 1$~~

$x_1 + x_7 \leq 1$ ~~$x_9 = 1$~~

$x_2 + x_6 \leq 1$ ~~$x_{10} = 1$~~

$x_5 + x_6 \leq 1$ ~~$x_{11} = 1$~~

Not allowed inequalities

General Construction

- Create one variable per term
- For each clause, create one equation
- For each pair of contradictory term, create an equation with those two and a new variable

Another Way of Looking at ZOE

Recall if $A = [v_1\ v_2\ v_3\ \ldots\ v_n]$,

$Ax = x_1 v_1 + x_2 v_2 + x_3 v_3 + \ldots + x_n v_n$.

Example:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\begin{aligned} &x_1 * [\ 1\ 0\ 0\ 1\ ] + \\ &x_2 * [\ 0\ 0\ 1\ 1\ ] + \\ &x_3 * [\ 1\ 1\ 1\ 0\ ] \\ &\text{----------} \\ &= \quad [\ 1\ 1\ 1\ 1\ ] \end{aligned}$$

What if we treated these as numbers rather than vectors?

Reduction

If the numbers are represented in a large enough base that carrying is impossible, we have a solution to the vector equation if and only if we have a solution to the number equation.

Subset Sum

Problem: Given a set S of numbers and a target number C, is there a subset $T \subseteq S$ whose elements sum to C.

Alternatively: Can we find $x_y \in \{0,1\}$ so that

$$\sum_{y \in S} x_y y = C.$$

Reduction: ZOE → Subset Sum.

Knapsack

Subset Sum is pretty closely related to (non-repeated) knapsack.

Subset Sum wants to find a set of values so that the weights equal the capacity.

Knapsack wants to find a set of values so that the weights are at most the capacity and the value is large.

Subset Sum → Knapsack

- Create Knapsack problem where for each item Value(item) = Weight(item).
- Maximizing value is the same as maximizing weight (without going over capacity).
- We can achieve value = capacity if and only if there is a subset of the items with total weight equal to capacity.

Knapsack is NP-Hard

3-SAT → ZOE → Subset Sum → Knapsack

Wait. Didn't we have a polynomial time DP for knapsack?

Our algorithm was polynomial in the total weight, which in this case is exponential.

One Final Reduction

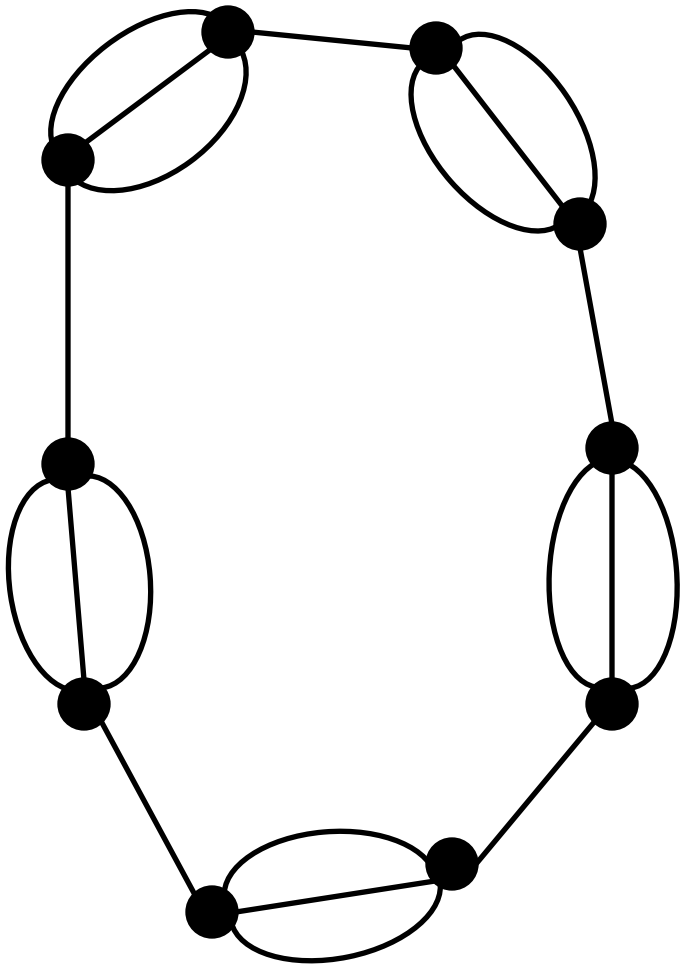
The last reduction we are going to show is ZOE → Hamiltonian Cycle. This will show that both Hamiltonian Cycle and TSP are NP-Complete/Hard.

Strategy

Often in order to show that a problem is NP-Complete, you want to be able to show that it can simulate logic somehow.

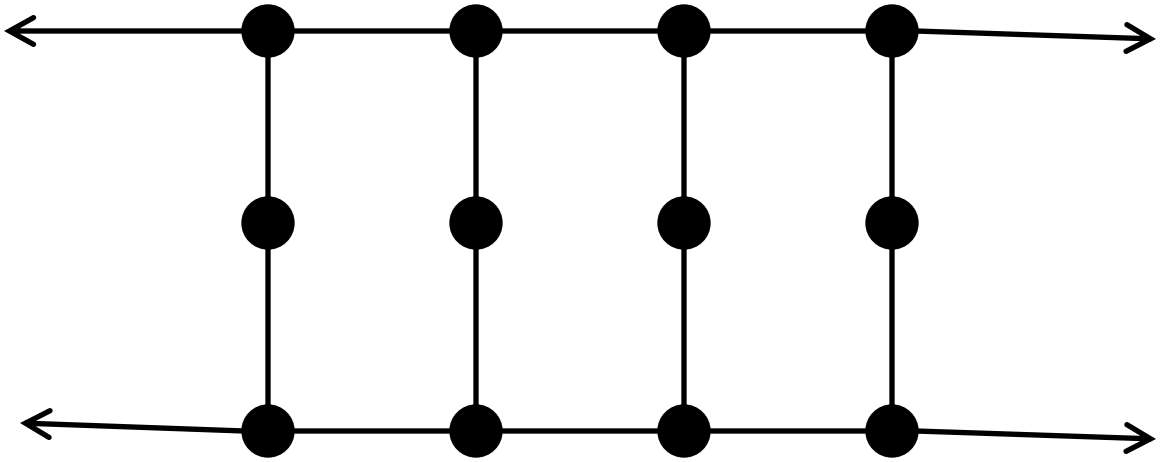
This is a bit difficult for Hamiltonian Cycle as most graphs have too many options, so we will want to find specific graphs with clear, binary choices.

Strategy



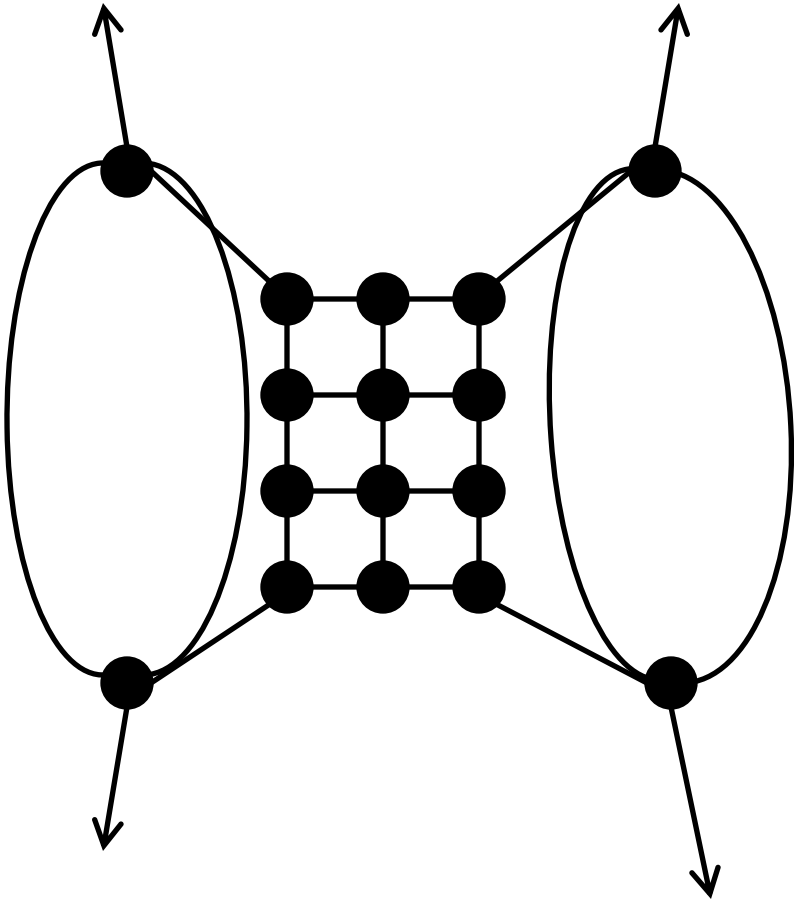
- Start with a cycle
- Double up some edges
- Cycle must pick one edge from each pair.
    - This provides a nice set of binary variables
- Need a way to add restrictions so that we can't just use <u>any</u> choices.

Gadget



- Must use these edges.
- Two ways to fill out.

Gadget Use



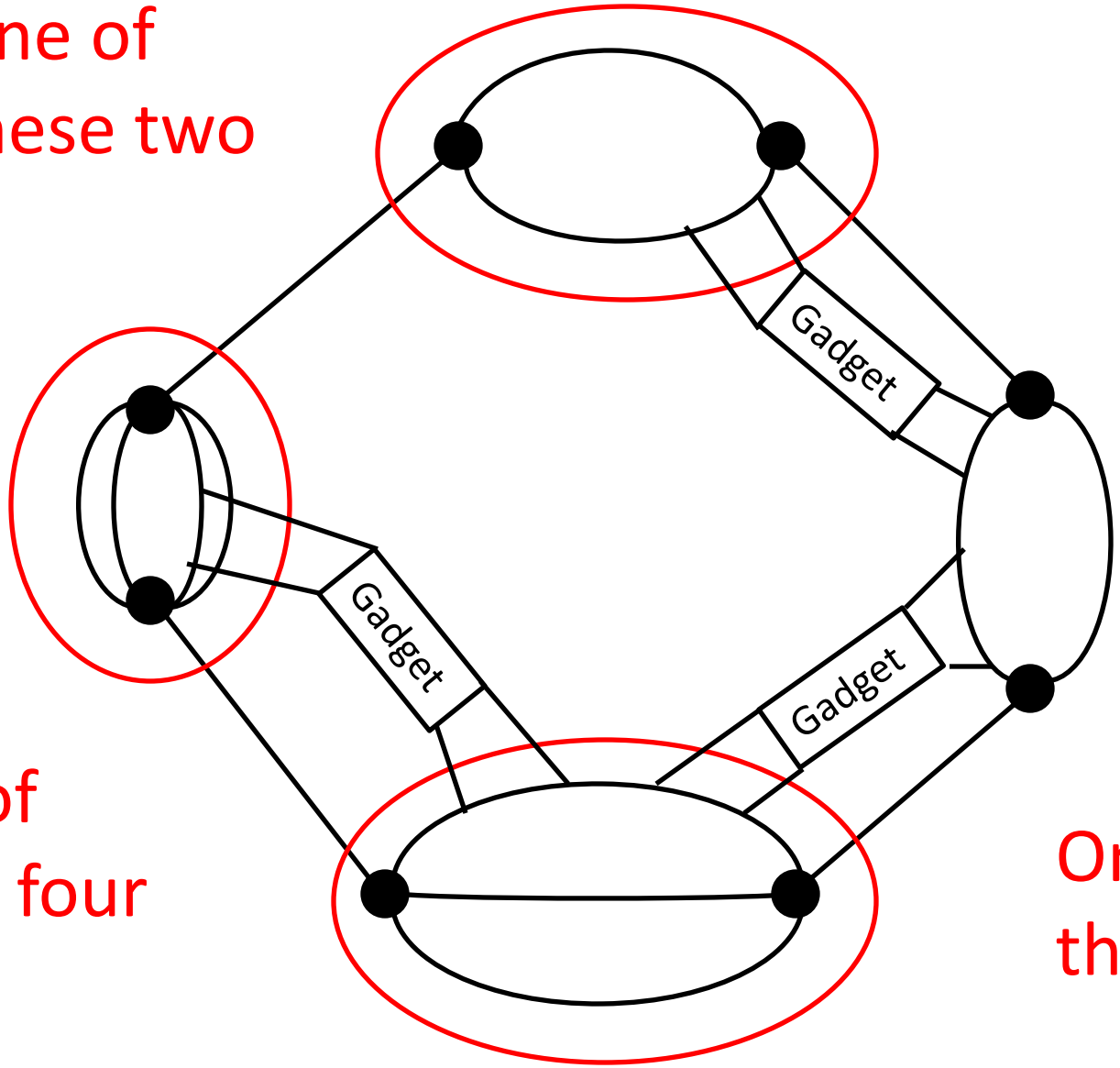
- Hook gadget up between a pair of edges.
- Hamiltonian Cycle must use exactly one of the connected edges.
- This allows us to force logic upon our choices.

Construction

By doing this for several pairs of edges we can construct Hamiltonian Cycle problems equivalent to the following:

- You are given a number of choices where you need to pick one from several options (of multi-edges).
- You have several constraints, that say of two choices you must have picked exactly one of them.

Example


One of
these two
Gadget
Gadget
Gadget
One of
these four
One of
these three

Full Construction

Choices:

- For each variable, choose either 0 or 1.
- For each equation, choose one variable.

Constraints:

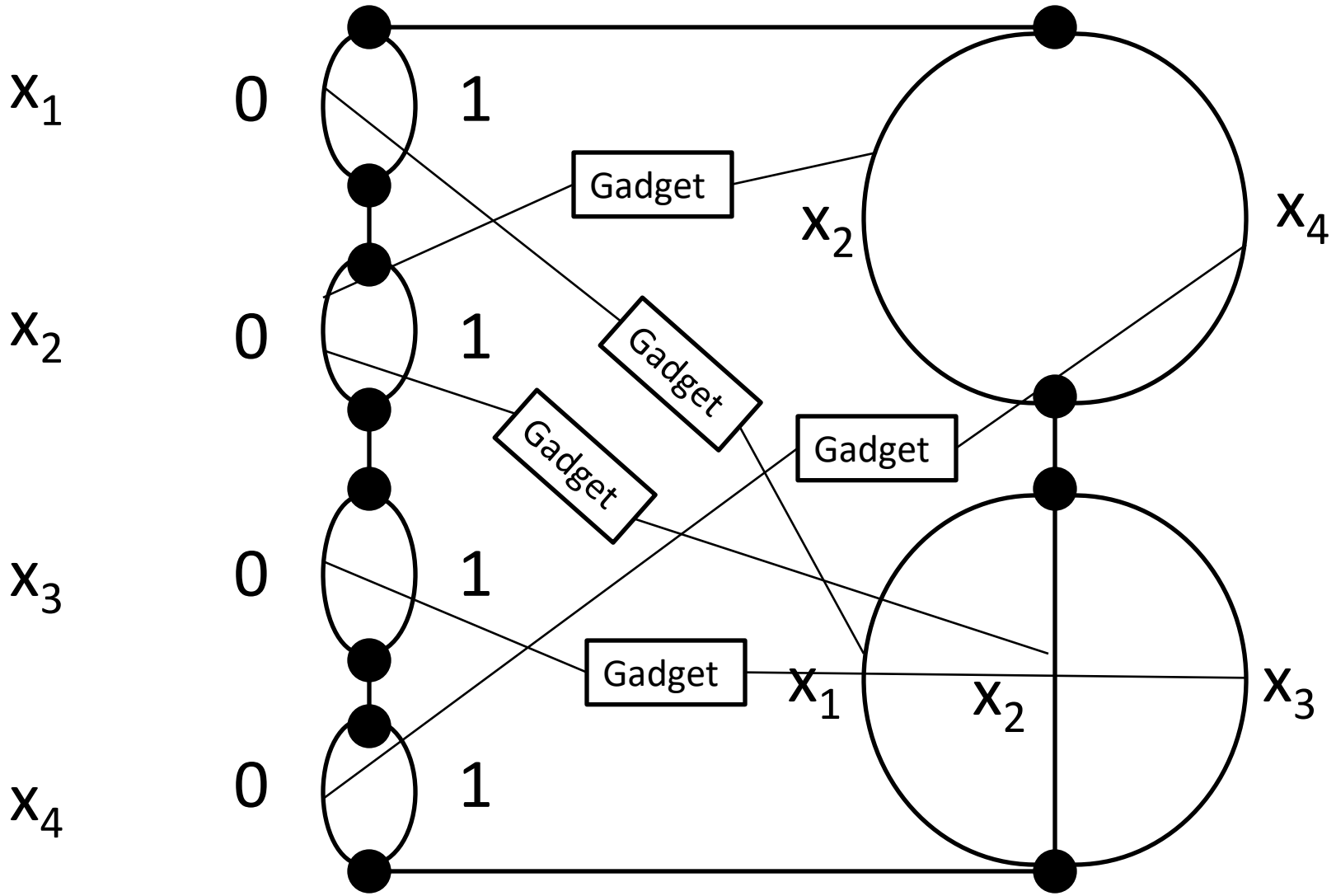
- For each variable that appears in an equation, exactly one of the following should be selected:
  - That variable in that equation
  - That variable equal to 0

$x_1=1\ x_2=0\ x_3=0\ x_4=1$

Example

x1+x2+x3=1

x2+x4=1



Analysis

If the ZOE has a solution, the Hamiltonian Cycle problem has a solution:

- Select the values of each variable and the variables equal to 1 in each equation.
- One for each and no conflicts.

If the Hamiltonian Cycle has a solution, so does the ZOE:

- Set variables equal to the values selected in the cycle.
- Gadgets mean that you can select a variable from an equation if and only if that variable is 1.
- Must have exactly one variable from each equation equal 1.
- Solution to ZOE.

Reduction Summary