This homework is due on gradescope Friday February 4th at 11:59pm on gradescope. Remember to justify your work even if the problem does not explicitly say so. Writing your solutions in \LaTeX is recommended though not required.

**Question 1** (Optimal Trade Routes, 40 points). Byron is the captain of a freighter ship. There are \( n \) ports that he can ship goods between, and for each ordered pair of ports \( x, y \), he has a profit \( p_{x,y} \geq 0 \) that he can make by hauling goods from one to the other, and a time \( t_{x,y} > 0 \) that the trip will take. Byron wants to find a trade route, that is a cycle of ports \( x_1, x_2, \ldots, x_m, x_1 \) so that his average profit per unit time is maximized. The profit per unit time is given by the formula:

\[
\frac{p_{x_1,x_2} + p_{x_2,x_3} + \ldots + p_{x_{m-1},x_m} + p_{x_m,x_1}}{t_{x_1,x_2} + t_{x_2,x_3} + \ldots + t_{x_{m-1},x_m} + t_{x_m,x_1}}.
\]

(a) Given a target profit \( P \), give an \( O(n^3) \) algorithm to determine whether or not Byron has a route that can get him average profit at least \( P \). [30 points]

Hint: You may want to find another way to write the condition that the average profit is at least \( P \).

(b) Given an \( \epsilon > 0 \), give an \( O(n^3 \log(1/\epsilon)) \) time algorithm that computes a number \( P^* \) so that \( P^* \) is within \( \epsilon \max_{x,y} p_{x,y}/t_{x,y} \) of the best achievable average profit \( P \). [10 points]

**Question 2** (Shortest Paths with Limited Negative Edges, 25 points). Let \( G \) be a directed, weighted graph where only edges between different strongly connected components have negative weights. Show how to compute shortest path lengths in \( G \) in \( O(|V| \log(|V|) + |E|) \) time.

**Question 3** (More Complicated Divide and Conquer Runtimes, 35 points). Suppose that we have that

\[
T(n) = T(\lfloor n/a \rfloor) + T(\lfloor n/b \rfloor) + O(n^d)
\]

for some real numbers \( a, b > 1 \) and \( d \geq 0 \).

(a) Show that there is a unique real number \( r > 0 \) so that \( a^{-r} + b^{-r} = 1 \). [5 points]

(b) Show that if \( d < r \) that \( T(n) = O(n^r) \). [15 points]

(c) Show that if \( d > r \) that \( T(n) = O(n^d) \). [15 points]

Note: Parts b and c above can be proved by induction, but you need to be careful. You need to show that there is some constant \( C \) so that \( T(n) \leq Cn^r \) or \( T(n) \leq Cn^d \) for all \( n \). You need to make sure that the value of \( C \) doesn’t change between rounds of induction.

**Question 4** (Extra credit, 1 point). Approximately how much time did you spend working on this homework?