# Harper's Theorem

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## 1 Introduction

The classical isoperimetric problem is, given $m \in \mathbb{R}^{\geq 0}$, to find a measurable subset $S \subseteq \mathbb{R}^n$ whose boundary has size $m$ and whose volume is as large as possible; or equivalently, to find a measurable $S \subseteq \mathbb{R}^n$ of volume $m$ and whose boundary is as small as possible. The solution of course is a sphere.

We now describe a discrete version, which has 2 flavors. Let $G = (V, E)$ be a graph. Define the *edge boundary* of $S \subseteq V$ as

$$\{\{x, y\} \in E \mid x \in S, y \notin S\},$$

and the *vertex boundary* as

$$\{y \in V \mid y \notin S, \exists x \in S \; \{x, y\} \in E\}.$$

In the *edge isoperimetry problem*, we are given $G = (V, E)$ and $m \in \mathbb{N}$ and wish to find some $S \subseteq V$ such that $|S| = m$ and the edge boundary is as small as possible. In the *vertex isoperimetry problem*, we wish to minimize the size of the vertex boundary of $S$.

An important special case is when $G$ is the Boolean cube in $n$ dimensions. In that case, the solutions are a cube and a sphere, respectively, the latter first proven by Harper (1966). Here we will present a proof, due to Frankl and Füredi (1981), that the solution to the vertex isoperimetry problem is a sphere.

The purpose of this essay is to fill in some of the missing cases in the proof given in [1], one of them a major case with a nontrivial analysis.

## 2 Notation

Let $X$ be a set of size $n$, $\mathcal{P}(X)$ be the power set of $X$. Define the Hamming distance function on $\mathcal{P}(X)$ as the size of the symmetric difference:

$$d(x, y) = |x \triangle y|.$$

A *set system* on $X$ is a subset of $\mathcal{P}(X)$. The Hamming ball of center $x \subseteq X$ and radius $r \in \mathbb{N}$ is

$$B_r(x) = \{y \subseteq X \mid d(x, y) \leq r).$$

A set system $\mathcal{A} \subseteq \mathcal{P}(X)$ is *a* Hamming ball of center $x \subseteq X$ and radius $r \in \mathbb{N}$ iff

$$B_{r-1}(x) \subseteq \mathcal{A} \subseteq B_r(x).$$

The distance between 2 nonempty set systems $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(X)$ is

$$d(\mathcal{A}, \mathcal{B}) = \min\{d(a, b) \mid a \in \mathcal{A}, b \in \mathcal{B}\}.$$

# 3 Main result

**Theorem 1.** *Let $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(X)$ be nonempty set systems. Then $\exists$ a Hamming ball $\mathcal{A}_0$ with center $X$ and a Hamming ball $\mathcal{B}_0$ with center $\emptyset$ such that $|\mathcal{A}_0| = |\mathcal{A}|, |\mathcal{B}_0| = |\mathcal{B}|$, and $d(\mathcal{A}_0, \mathcal{B}_0) \geq d(\mathcal{A}, \mathcal{B})$.*

*Proof.* Let $d = d(\mathcal{A}, \mathcal{B})$. Choose $\mathcal{A}_0, \mathcal{B}_0$ so that $|\mathcal{A}_0| = |\mathcal{A}|, |\mathcal{B}_0| = |\mathcal{B}|$, $d(\mathcal{A}_0, \mathcal{B}_0) \geq d$ and so that

$$\sigma(\mathcal{A}_0, \mathcal{B}_0) = \sum_{A \in \mathcal{A}_0} |A| - \sum_{B \in \mathcal{B}_0} |B|$$

is maximal. It remains to show that $\mathcal{A}_0, \mathcal{B}_0$ are Hamming balls centered at $X, \emptyset$. Suppose not. Then by symmetry, we may assume that $\mathcal{A}_0$ is not a Hamming ball centered at $X$. So it is not the case that $\forall A \in \mathcal{A}_0, C \subseteq X$ $(|C| > |A| \to C \in \mathcal{A}_0)$. So $\exists A_0 \in \mathcal{A}_0, A_0^* \notin \mathcal{A}_0$ $|A_0^*| > |A_0|$. We may choose a closest such pair. I.e. choose $A_0 \in \mathcal{A}_0, A_0^* \notin \mathcal{A}_0$ so that $|A_0^*| > |A_0|$ and $\forall C \in \mathcal{A}_0, C^* \notin \mathcal{A}_0$ with $|C^*| > |C|$ we have $d(C, C^*) \geq d(A_0, A_0^*)$.

Let $U = A_0 - A_0^*, V = A_0^* - A_0$. Then $|U| < |V|$ since $|U| + |A_0 \cap A_0^*| = |A_0| < |A_0^*| = |V| + |A_0 \cap A_0^*|$.

We first claim that $\forall F \supseteq A_0$ $F \in \mathcal{A}_0$. It is sufficient to show this is true when $|F - A_0| = 1$, since then the claim follows by induction. So suppose $|F - A_0| = 1$ and let $h \in F - A_0$. Define $\mathrm{up} : \mathcal{A}_0 \to \mathcal{P}(X)$ and $\mathrm{dn} : \mathcal{B}_0 \to \mathcal{P}(X)$ by

$$\mathrm{up}(A) = \begin{cases} A \cup \{h\} & \text{if } A \cup \{h\} \notin \mathcal{A}_0 \\ A & \text{else} \end{cases}$$

$$\mathrm{dn}(B) = \begin{cases} B - \{h\} & \text{if } B - \{h\} \notin \mathcal{B}_0 \\ B & \text{else} \end{cases}.$$

We claim that $\mathrm{up}, \mathrm{dn}$ are both injective.

To see this, suppose $\mathrm{up}(A_1) = \mathrm{up}(A_2)$. If $A_1 \cup \{h\}, A_2 \cup \{h\} \in \mathcal{A}_0$, then $A_1 = \mathrm{up}(A_1) = \mathrm{up}(A_2) = A_2$, and we are done. So now suppose $A_1 \cup \{h\}, A_2 \cup \{h\} \notin \mathcal{A}_0$. If $h \in A_1$ or $h \in A_2$, then we have the contradiction that $A_1 \cup \{h\} = A_1 \in \mathcal{A}_0$ or $A_2 \cup \{h\} = A_2 \in \mathcal{A}_0$. So $h \notin A_1 \cup A_2$. So

$$\begin{aligned} A_1 &= (A_1 \cup \{h\}) - \{h\} = \mathrm{up}(A_1) - \{h\} \\ &= \mathrm{up}(A_2) - \{h\} = (A_2 \cup \{h\}) - \{h\} = A_2, \end{aligned}$$

and we are done. (Notice that it is this part of the argument that requires $|F - A_0| = 1$.) Finally, suppose $A_1 \cup \{h\} \in \mathcal{A}_0, A_2 \cup \{h\} \notin \mathcal{A}_0$. Then we have the contradiction $A_2 \cup \{h\} = \mathrm{up}(A_2) = \mathrm{up}(A_1) = A_1 \in \mathcal{A}_0$.

Next, suppose $\mathrm{dn}(B_1) = \mathrm{dn}(B_2)$. If $B_1 - \{h\}, B_2 - \{h\} \in \mathcal{B}_0$, then $B_1 = \mathrm{dn}(B_1) = \mathrm{dn}(B_2) = B_2$ and we are done. So now suppose $B_1 - \{h\}, B_2 - \{h\} \notin \mathcal{B}_0$. If $h \notin B_1$ or $h \notin B_2$, then we have the contradiction that $B_1 - \{h\} = B_1 \in \mathcal{B}_0$ or $B_2 - \{h\} = B_2 \in \mathcal{B}_0$. So $h \in B_1 \cap B_2$. So

$$\begin{aligned} B_1 &= (B_1 - \{h\}) \cup \{h\} = \mathrm{dn}(B_1) \cup \{h\} \\ &= \mathrm{dn}(B_2) \cup \{h\} = (B_2 - \{h\}) \cup \{h\} = B_2, \end{aligned}$$

and we are done. Finally, if $B_1 - \{h\} \in \mathcal{B}_0, B_2 - \{h\} \notin \mathcal{B}_0$, then we have the contradiction $B_2 - \{h\} = \mathrm{dn}(B_2) = \mathrm{dn}(B_1) = B_1 \in \mathcal{B}_0$.

So indeed, $\mathrm{up}, \mathrm{dn}$ are injective. Let $\mathcal{A}_0' = \operatorname{ran} \mathrm{up}, \mathcal{B}_0' = \operatorname{ran} \mathrm{dn}$. Then $|\mathcal{A}_0'| = |\mathcal{A}_0|, |\mathcal{B}_0'| = |\mathcal{B}_0|$. We claim that $d(\mathcal{A}_0', \mathcal{B}_0') \geq d(\mathcal{A}_0, \mathcal{B}_0)$. Let $A \in \mathcal{A}_0, B \in \mathcal{B}_0, A' = \mathrm{up}(A), B' = \mathrm{dn}(B)$.

If $A \cup \{h\} \in \mathcal{A}_0, B - \{h\} \in \mathcal{B}_0$, then $A' = A, B' = B$ and we have $d(A', B') = d(A, B)$. If $A \cup \{h\} \notin \mathcal{A}_0, B - \{h\} \notin \mathcal{B}_0$, then $A' = A \cup \{h\}, B' = B - \{h\}, h \notin A, h \in B$, and so $d(A', B') = d(A, B)$. If $A \cup \{h\} \notin \mathcal{A}_0, B - \{h\} \in \mathcal{B}_0$, then $d(A', B') \geq d(A, B - \{h\})$. Finally, if $A \cup \{h\} \in \mathcal{A}_0, B - \{h\} \notin \mathcal{B}_0$, then $d(A', B') \geq d(A \cup \{h\}, B)$. So indeed, $d(\mathcal{A}_0', \mathcal{B}_0') \geq d(\mathcal{A}_0, \mathcal{B}_0)$.

If $F \notin \mathcal{A}_0$, then since $F = \mathrm{up}(A_0)$, we have $\sigma(\mathcal{A}_0', \mathcal{B}_0') > \sigma(\mathcal{A}_0, \mathcal{B}_0)$, contradicting the maximality of $\mathcal{A}_0, \mathcal{B}_0$. So indeed, $F \in \mathcal{A}_0$.

Believe it or not, all of that argument was simply to show that $U \neq \emptyset$, which we can now see by setting $F = A_0^*$. Let us use $U, V$ to construct strictly better set systems $\mathcal{A}_1, \mathcal{B}_1$. Define $\mathrm{Up} : \mathcal{A}_0 \to \mathcal{P}(X), \mathrm{Dn} : \mathcal{B}_0 \to \mathcal{P}(X)$ by

$$\begin{aligned} \mathrm{Up}(A) &= \begin{cases} A \cup V - U & \text{if } U \subseteq A \subseteq \bar{V}, A \cup V - U \notin \mathcal{A}_0 \\ A & \text{else} \end{cases} \\ \mathrm{Dn}(B) &= \begin{cases} B \cup U - V & \text{if } V \subseteq B \subseteq \bar{U}, B \cup U - V \notin \mathcal{B}_0 \\ B & \text{else} \end{cases}. \end{aligned}$$

Notice that $\cup, -$ have the same level of precedence and associate from left to right. As before, we claim that $\mathrm{Up}, \mathrm{Dn}$ are injective.

So suppose $\mathrm{Up}(A_1) = \mathrm{Up}(A_2)$. If $\neg(U \subseteq A_1 \subseteq \bar{V} \wedge A_1 \cup V - U \notin \mathcal{A}_0)$ and $\neg(U \subseteq A_2 \subseteq \bar{V} \wedge A_2 \cup V - U \notin \mathcal{A}_0)$, then $A_1 = \mathrm{Up}(A_1) = \mathrm{Up}(A_2) = A_2$, and we are done. If $U \subseteq A_1 \subseteq \bar{V} \wedge A_1 \cup V - U \notin \mathcal{A}_0$ and $U \subseteq A_2 \subseteq \bar{V} \wedge A_2 \cup V - U \notin \mathcal{A}_0$, then

$$\begin{aligned} A_1 &= (A_1 \cup V - U) \cup U - V = \mathrm{Up}(A_1) \cup U - V \\ &= \mathrm{Up}(A_2) \cup U - V = (A_2 \cup V - U) \cup U - V = A_2, \end{aligned}$$

and we are done. If $U \subseteq A_1 \subseteq \bar{V} \wedge A_1 \cup V - U \notin \mathcal{A}_0$ and $\neg(U \subseteq A_2 \subseteq \bar{V} \wedge A_2 \cup V - U \notin \mathcal{A}_0)$, then we have the contradiction $A_1 \cup V - U = \mathrm{Up}(A_1) = \mathrm{Up}(A_2) = A_2 \in \mathcal{A}_0$.

Next, suppose $\mathrm{Dn}(B_1) = \mathrm{Dn}(B_2)$. If $\neg(V \subseteq B_1 \subseteq \bar{U} \wedge B_1 \cup U - V \notin \mathcal{B}_0)$ and $\neg(V \subseteq B_2 \subseteq \bar{U} \wedge B_2 \cup U - V \notin \mathcal{B}_0)$, then $B_1 = \mathrm{Dn}(B_1) = \mathrm{Dn}(B_2) = B_2$, and we are done. If $V \subseteq B_1 \subseteq \bar{U} \wedge B_1 \cup U - V \notin \mathcal{B}_0$ and $V \subseteq B_2 \subseteq \bar{U} \wedge B_2 \cup U - V \notin \mathcal{B}_0$, then

$$\begin{aligned} B_1 &= (B_1 \cup U - V) \cup V - U = \mathrm{Dn}(B_1) \cup V - U \\ &= \mathrm{Dn}(B_2) \cup V - U = (B_2 \cup U - V) \cup V - U = B_2, \end{aligned}$$

and we are done. If $V \subseteq B_1 \subseteq \bar{U} \wedge B_1 \cup U - V \notin \mathcal{B}_0$ and $\neg(V \subseteq B_2 \subseteq \bar{U} \wedge B_2 \cup U - V \notin \mathcal{B}_0)$, then we have the contradiction $B_1 \cup U - V = \mathrm{Dn}(B_1) = \mathrm{Dn}(B_2) = B_2 \in \mathcal{B}_0$.

So indeed, $\mathrm{Up}, \mathrm{Dn}$ are injective. Let $\mathcal{A}_1 = \operatorname{ran} \mathrm{Up}, \mathcal{B}_1 = \operatorname{ran} \mathrm{Dn}$. Then $|\mathcal{A}_1| = |\mathcal{A}_0|, |\mathcal{B}_1| = |\mathcal{B}_0|$. Also $A_0^* = \mathrm{Up}(A_0)$, and so $\sigma(\mathcal{A}_1, \mathcal{B}_1) > \sigma(\mathcal{A}_0, \mathcal{B}_0)$. If we can show that $d(\mathcal{A}_1, \mathcal{B}_1) \geq d(\mathcal{A}_0, \mathcal{B}_0)$, then we will have our contradiction and complete the proof. So let $A \in \mathcal{A}_0, B \in \mathcal{B}_0, A' = \mathrm{Up}(A), B' = \mathrm{Dn}(B)$. We want to show that $\exists \tilde{A} \in \mathcal{A}_0, \tilde{B} \in \mathcal{B}_0\ d(A', B') \geq d(\tilde{A}, \tilde{B})$.

If $A' \in \mathcal{A}_0, B' \in \mathcal{B}_0$, then we are done. If $A' \in \mathcal{A}_1 - \mathcal{A}_0, B' \in \mathcal{B}_1 - \mathcal{B}_0$, then $A' = A \cup V - U, B' = B \cup U - V$, and so $d(A', B') = d(A, B)$, and we are done.

Now suppose $A' \in \mathcal{A}_1 - \mathcal{A}_0, B' \in \mathcal{B}_0$. Then $A' = A \cup V - U, U \subseteq A \subseteq \bar{V}$. If $V \subseteq B \subseteq \bar{U}$, then $B \cup U - V \in \mathcal{B}_0$, and so $B' = B$. So $d(A', B') = d(A', B) = d(A, B \cup U - V)$, and we are done. This case is illustrated in figure ??

Otherwise, since $U \neq \emptyset$, we have $\exists u \in U, v \in V\ (u \in B \vee v \notin B)$. Let $\tilde{A} = A \cup (V - \{v\}) - (U - \{u\}) = A' \cup \{u\} - \{v\}$. This case is illustrated in figure ?? Then $|\tilde{A}| = |A'| > |A|$. Also, $d(A, \tilde{A}) < |U| + |V| = d(A_0, A_0^*)$, which, by minimality of $A_0, A_0^*$, implies that $\tilde{A} \in \mathcal{A}_0$. But then

$$\begin{aligned} d(A', B') &= d(A', B) \\ &= d(\tilde{A}, B) + u \in B - v \in B - u \notin B + v \notin B \\ &= d(\tilde{A}, B) - (-1)^{u \in B} - (-1)^{v \notin B} \\ &\geq d(\tilde{A}, B), \end{aligned}$$

and we are done.

Finally, suppose $A' \in \mathcal{A}_0, B' \in \mathcal{B}_1 - \mathcal{B}_0$. Then $V \subseteq B \subseteq \bar{U}, B' = B \cup U - V \notin \mathcal{B}_0$. Also $A' = A$, else we would have the contradiction $A' = A \cup V - U \notin \mathcal{A}_0$. Let $\alpha = A - (U \cup V), \beta = B - (U \cup V)$. We will break this into 4 (not necessarily mutually exclusive) cases. For the first case, suppose $U \subseteq A \subseteq \bar{V}$. Then since $A' = A$, we must have $A \cup V - U \in \mathcal{A}_0$. So $d(A', B') = d(A, B') = d(A \cup V - U, B)$, and we are done.

For the second case, suppose $U \cup V \subseteq A$. Then

$$d(A', B') = d(\alpha, \beta) + |V| > d(\alpha, \beta) + |U| = d(A, B), \tag{1}$$

and we are done.

For the third case, suppose $|A \cap U| < |V| - |A \cap V|$. Then set $\tilde{A} = \alpha \cup V$. This case is illustrated in figure ?? Then $|\tilde{A}| > |A|, d(A, \tilde{A}) < |U| + |V| = d(A_0, A_0^*)$, otherwise, we would reduce to the first case. So $\tilde{A} \in \mathcal{A}_0$. So

$$d(A', B') \geq d(\alpha, \beta) = d(\tilde{A}, B),$$

and we are done.

For the fourth case, suppose $|A \cap U| \geq |V| - |A \cap V|$. Choose any $\gamma \subseteq A \cap U$ such that $|\gamma| = |A \cap U| - (|V| - |A \cap V|)$. We claim that $U - \gamma \neq \emptyset$. For suppose not. If $|V| - |A \cap V| = 0$, then we must have $U \cup V \subseteq A$ and we reduce to the second case. But otherwise $|V| - |A \cap V| > 0$, and so $\gamma \subset U$.

So let $u \in U - \gamma$. Set $\tilde{A} = \alpha \cup \gamma \cup V \cup \{u\}$. Then

$$\begin{aligned}
&|V| - |A \cap V| \leq |A \cap U| \leq |U| < |V| \\
\Rightarrow &|A \cap V| > 0 \\
\Rightarrow &d(A, \tilde{A}) < |U| + |V| = d(A_0, A_0^*)
\end{aligned}$$

and $|\tilde{A}| = |A| + 1$. So we must have $\tilde{A} \in \mathcal{A}_0$. Also

$$\begin{aligned}
d(A', B') &= d(\alpha, \beta) + |U| - |A \cap U| + |A \cap V| \\
d(\tilde{A}, B) &= d(\alpha, \beta) + |\gamma| + 1 \\
&= d(\alpha, \beta) + |A \cap U| + |A \cap V| - |V| + 1.
\end{aligned}$$

So

$$\begin{aligned}
d(A', B') - d(\tilde{A}, B) = &|V| + |U| - 2|A \cap U| - 1 \\
\geq &|V| - |U| - 1 \geq 0,
\end{aligned}$$

which completes the fourth case, and the proof. □

We can now easily show that a Hamming ball solves the vertex isoperimetry problem. Let $m \in [1, 2^n]$ and let $S \subseteq 2^n$ have size $m$. Let $\partial S \subseteq 2^n$ be the vertex boundary of $S$. Let $T = 2^n - (S \cup \partial S)$. Assume $T \neq \emptyset$. Then $d(S, T) \geq 2$. By the theorem, $\exists$ Hamming balls $S_0, T_0 \subseteq 2^n$ such that $|S_0| = |S|, |T_0| = |T|, d(S_0, T_0) \geq 2$. So $|\partial S_0| \leq 2^n - |S_0| - |T_0| = 2^n - |S| - |T| = |\partial S|$.

On the other hand, if $T = \emptyset$, then $\partial S = 2^n - S$ and so no subset of $2^n$ of size $m$ can have a boundary larger than that of $S$. In particular, a Hamming ball of size $m$ will have a boundary no larger than that of $S$.

# References


[1] B. Bollobás, Combinatorics: Set Systems, Hypergraphs, Families of Vectors, and Combinatorial Probability

Cambridge University Press, 126-128, 1986