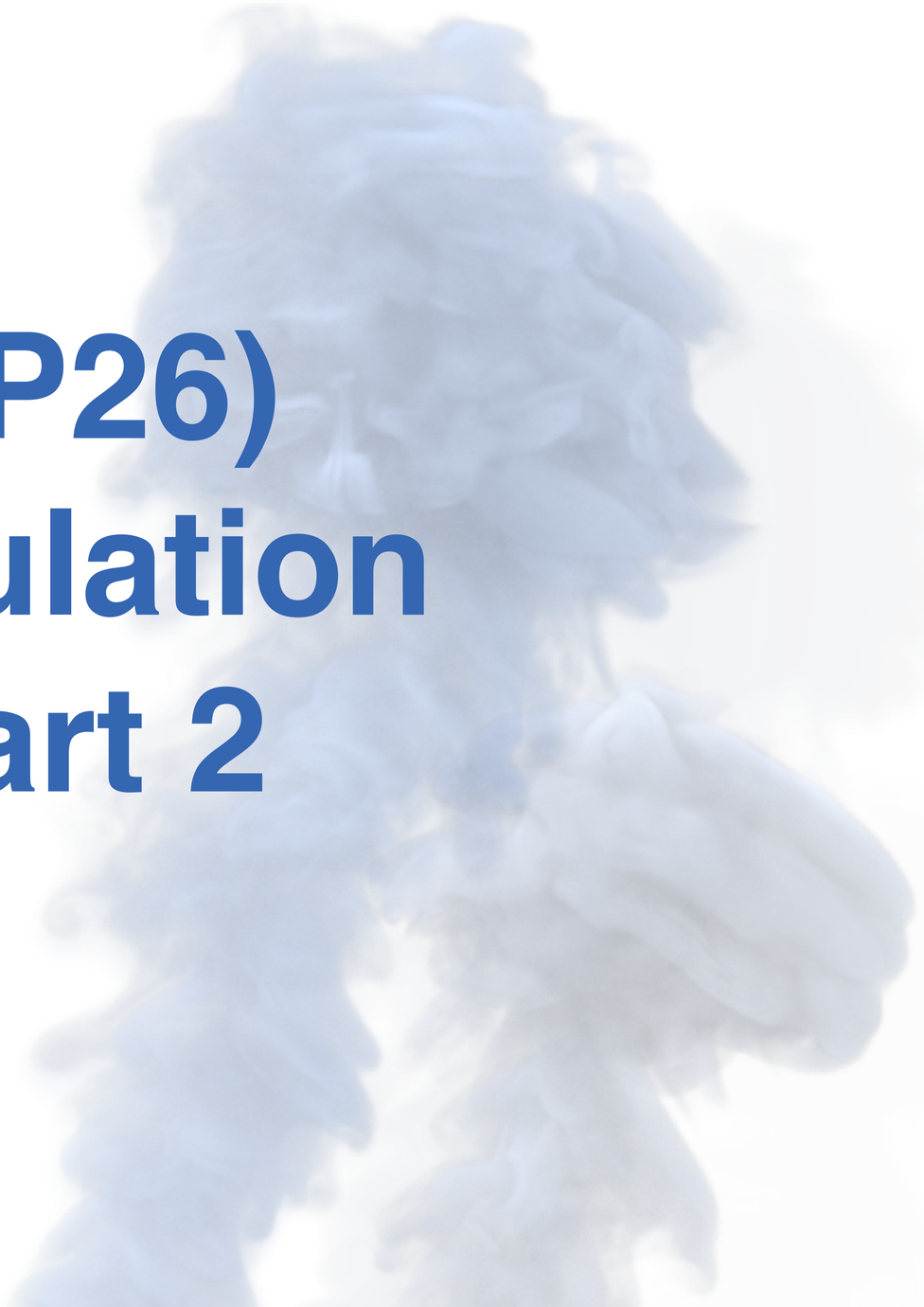
CSE 291 (SP26)
Physical Simulation
Tensors: Part 2

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Tensors for continuum mechanics

- Tensors for continuum mechanics
- Tensor duality
- Elasticity

State of deformable body is a map

- The setup for deformable body
  - Manifold M: **Material** coordinate, **Lagrangian** coordinate
  - Manifold W: **World** coordinate, **Eulerian** coordinate
  - A state of a deformable body is a map (called **flow map**)

$$\phi : M \to W$$

- The flow map assigns each material point its world position
- For a dynamical system, the flow map is time-dependent

$$\phi : \mathbb{R} \times M \to W \qquad \phi(t) : M \to W$$

Some assumptions on M, W

$$\phi(t)\colon M \to W$$

- To define inertia, we need a few structures on Material and World:
  - A time-independent n-form on material describing mass density

$$\rho_M \in \Omega^n(M)$$

$$\text{TotalMass}(\text{SomeRegion} \subset M) := \int_{\text{Region}} \rho_M$$

  - A time-independent metric on the world

$$\flat_W \in \Gamma(T^*W \odot T^*W)$$

Kinetic energy and potential energy

let's understand the tensor types of derivatives of flow maps

- Kinetic energy

$$\mathcal{K}(\phi, \dot{\phi}) := \frac{1}{2} \int_M |\dot{\phi}|^2_{\flat_W} \rho_M$$

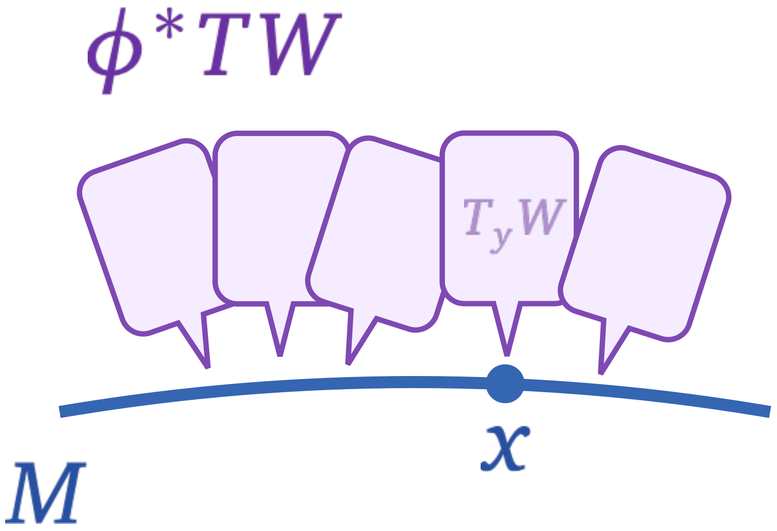
- Potential energy

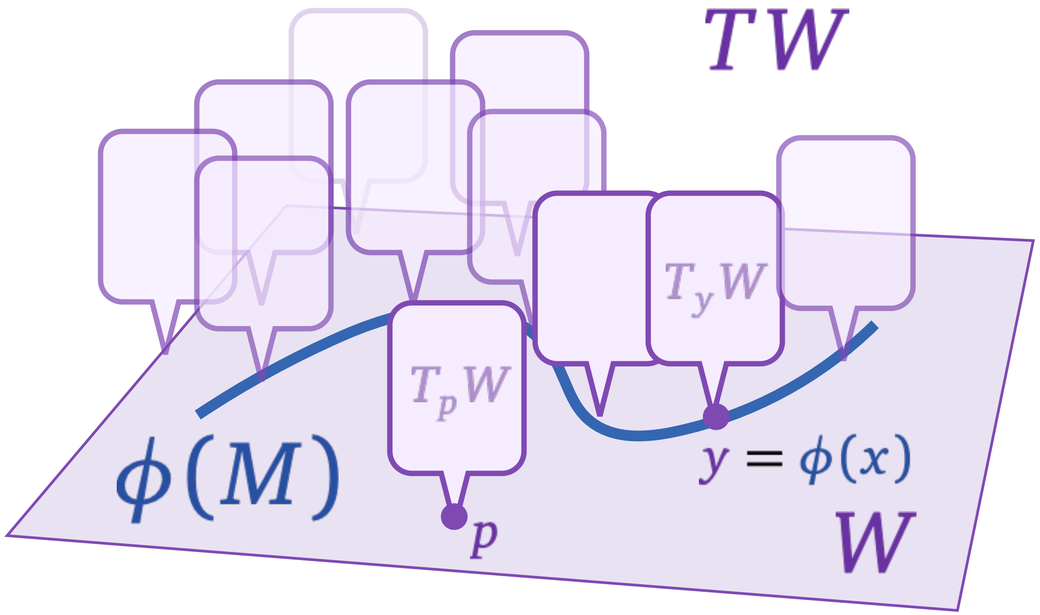
$$\mathcal{U}(\phi) := \int_M U(\phi, d\phi)$$

Pullback bundle

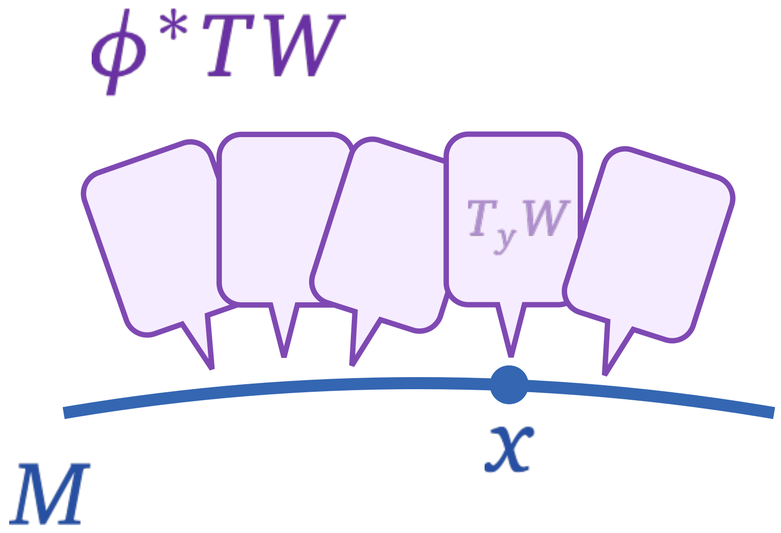
- Consider a bundle over W, say the tangent bundle TW
- Suppose there is a map into W, say a flow map $\phi : M \to W$
- Then we can construct a pullback bundle over M called $\phi^* TW$ so that the fibers are given by

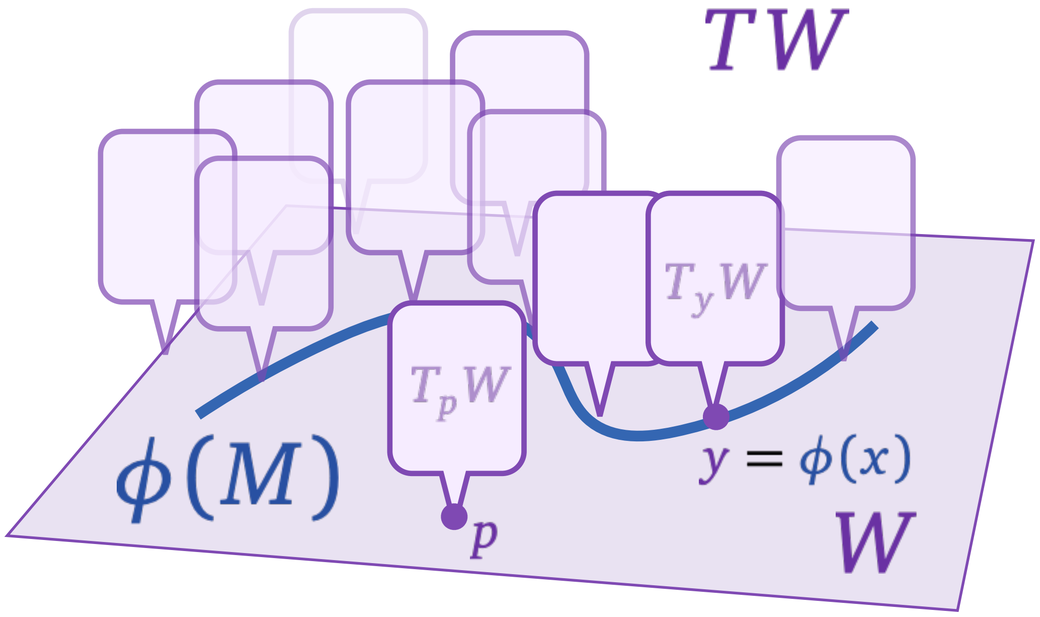
$$(\phi^* TW)_x = T_{\phi(x)} W$$





Pullback bundle





- We will also use the following notation $T_\phi W = \phi^*TW$

$$(T_\phi W)_x = T_{\phi(x)}W$$

Pullback bundle

- With $\phi : M \to W$
  - TM is a bundle over M
  - TW is a bundle over W
  - $T_\phi W$ is a bundle over M

Time derivative of flow map

- For a time dependent flow map $\phi(t): M \rightarrow W$

$$\dot{\phi} = \frac{\partial \phi}{\partial t} \in \ ?$$

Time derivative of flow map

- For a time dependent flow map $\phi(t): M \to W$

$$\dot{\phi} = \frac{\partial \phi}{\partial t} \in \Gamma(T_\phi W)$$

Differential (Jacobian) of flow map

- Given a flow map

$$\phi : M \to W$$

- Its differential, at each point, is

$$d\phi|_p : T_pM \xrightarrow{\text{linear}} T_{\phi(p)}W$$

- The differential is of type “(world)vector-valued 1-form over M”

$$\phi_* = d\phi \in ?$$

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$$\phi_* = d\phi \in \Gamma(T^*M \otimes T_\phi W)$$

$$= \Omega^1(M; T_\phi W) \quad \text{for short}$$

Two-point tensor

- In general, in continuum mechanics, **two-point tensors** are of type

$$\Gamma\Big(\underbrace{T^*M \otimes \cdots \otimes T^*M}_{p \text{ copies}} \otimes \underbrace{TM \otimes \cdots \otimes TM}_{q \text{ copies}} \otimes \underbrace{T^*_\phi W \otimes \cdots \otimes T^*_\phi W}_{\ell \text{ copies}} \otimes \underbrace{T_\phi W \otimes \cdots \otimes T_\phi W}_{m \text{ copies}} \Big)$$

- Term coined by Erickson 1960. See "*Mathematical Foundation of Elasticity*" by J. Marsden and T. Hughes 1983.
- Special shorthand:

$$\Gamma\Big(\underbrace{T^*M \wedge \cdots \wedge T^*M}_{k \text{ copies}} \otimes E \Big) = \Omega^k(M; E)$$

E-valued k-form

Two-point tensor

- Special shorthand: $$\Gamma\Big(\underbrace{T^*M \wedge \cdots \wedge T^*M}_{k \text{ copies}} \otimes E\Big) = \Omega^k(M;E)$$
  E-valued k-form

- For a tensor $\tau \in \Omega^k(M;E)$
  - As a k-form, it's to be measured/evaluated over infinitesimal k-dim geometries
  - The measurement is a vector/tensor of type *E*.

Tensor Duality

- Tensors for continuum mechanics
- Tensor duality
- Elasticity

Dual space of tensor product

- Dual of tensor:

$$(U \otimes V)^* = U^* \otimes V^*$$

$$\langle \alpha \otimes \beta | u \otimes v \rangle = \langle \alpha | u \rangle \langle \beta | v \rangle$$

- In basis, this looks like Frobenius inner product for matrices

$$\mathbf{A} : \mathbf{B} = \mathrm{tr}(\mathbf{A}^\intercal \mathbf{B}) = \sum_{ij} A_{ij} B_{ij}$$

- For tensor fields, the dual requires additionally tensoring an n-form

$$\Gamma(E)^* = \Gamma(E^* \otimes \wedge^n T^* M)$$

$$\langle\!\langle \tau | \sigma \rangle\!\rangle = \int_M \underbrace{\langle \tau | \sigma \rangle}_{\text{n-form}}$$

Dual space of tensor product

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  - Example

$$\Omega^0(M)^* = \Omega^n(M)$$

  - Example

$$\Gamma(T^*M \otimes T_\phi W)^* = \Gamma(TM \otimes T_\phi^* W \otimes \wedge^n T^*M)$$

Vector ⊗ n-form = (n-1)-form

- Let V be any vector space
- We have a canonical isomorphism

$$V \otimes (\wedge^n V^*) \cong \wedge^{n-1} V^*$$

$$\vec{v} \otimes \mu \mapsto i_{\vec{v}} \mu$$

  - This mapping is linearly bijective.

Vector ⊗ n-form = (n-1)-form

- More generally

$$(\wedge^k V) \otimes (\wedge^n V^*) \cong \wedge^{n-k} V^*$$

  - Example:

$$\begin{aligned}\Omega^k(M)^* &= \Gamma(\wedge^k T^*M)^* \\ &= \Gamma((\wedge^k TM) \otimes (\wedge^n T^*M)) \\ &= \Gamma(\wedge^{n-k} T^*M) \\ &= \Omega^{n-k}(M)\end{aligned}$$

$$\langle\!\langle \underbrace{\alpha}_{k\text{-form}} | \underbrace{\beta}_{(n-k)\text{-form}} \rangle\!\rangle = \int_M \underbrace{\alpha \wedge \beta}_{n\text{-form}}$$

More examples

- What is type of $d\phi$ and what is the dual object?

$$d\phi \in \Gamma(T^*M \otimes T_\phi W)$$

$$\Gamma(T^*M \otimes T_\phi W)^* = \Gamma(TM \otimes T_\phi^* W \otimes \wedge^n T^*M)$$

$$= \Gamma(\wedge^{n-1} T^*M \otimes T_\phi^* W)$$

momentum flux

which is stress

Elasticity

- Tensors for continuum mechanics
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Postulates

- The position of the body is described by a (time-dependent) map

$$\phi : M \to W$$

where M has a time-independent mass density $\rho_M \in \Omega^n(M)$

and W has a time-independent metric $\flat_W \in \Gamma(T^*W \odot T^*W)$

- The elastic potential energy for a map ϕ takes the form

$$\mathcal{U}(\phi) = \int_M U(\phi^* \flat_W)$$

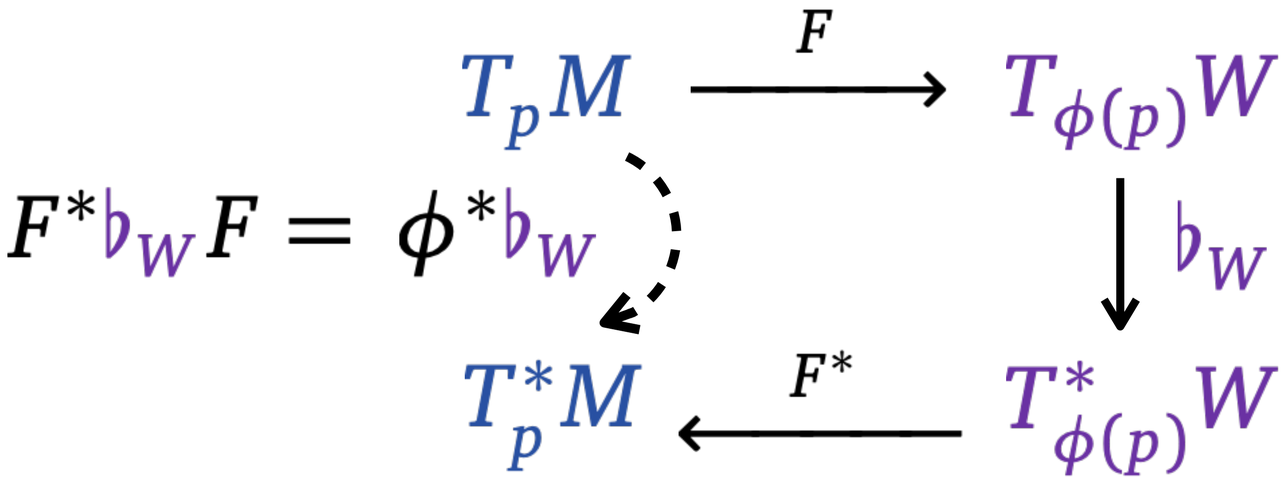
for some fiber-wise (nonlinear) mapping (depending on material)

$$U_p : T_p^*M \odot T_p^*M \xrightarrow{\text{nonlinear}} \wedge^n T_p^*M$$

i.e. the potential is only a function of the induced metric encoding its notion of distances in the world. (**Frame-indifference**)

Terminology

- We call $F = \phi_* = d\phi \in \Gamma(T^*M \otimes T_\phi W)$ **deformation gradient**.
- The induced metric $\phi^* \flat_W$ can be understood by the diagram



- The induced metric $C := F^* \flat_W F \in \Gamma(T^*M \odot T^*M)$ is called the (right)-**Cauchy–Green tensor**.

Notation in 3D Cartesian coordinate

- Let (X, Y, Z) denote the Cartesian coordinate for M and (x, y, z) the Cartesian coordinate for W
- Flow map $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \phi^1(X,Y,Z) \\ \phi^2(X,Y,Z) \\ \phi^3(X,Y,Z) \end{bmatrix}$
- Deformation gradient $\mathbf{F} = \begin{bmatrix} \frac{\partial \phi^1}{\partial X} & \frac{\partial \phi^1}{\partial Y} & \frac{\partial \phi^1}{\partial Z} \\ \frac{\partial \phi^2}{\partial X} & \frac{\partial \phi^2}{\partial Y} & \frac{\partial \phi^2}{\partial Z} \\ \frac{\partial \phi^3}{\partial X} & \frac{\partial \phi^3}{\partial Y} & \frac{\partial \phi^3}{\partial Z} \end{bmatrix}$
- Right Cauchy–Green tensor $\mathbf{C} = \mathbf{F}^{\mathsf{T}}\mathbf{F}$

Deriving elastic force

- Back to our potential energy $\mathcal{U}(\phi) = \int_M U(\phi^{*\flat_W}) = \int_M U(C)$

  given some model $U : \Gamma(T^*M \odot T^*M) \xrightarrow{\text{pointwise nonlinear}} \Gamma(\wedge^n T^*M)$

- The force is given by $-d\mathcal{U}_\phi$
- Compute it using backpropagation.

Deriving elastic force

$$\mathcal{U}(\phi) = \int_M U(\phi^* \flat_W) = \int_M U(C)$$

$$C^\infty(M;W) \xrightarrow{\phi} \xrightarrow{d} \Gamma(T^*M \otimes T_\phi W) \xrightarrow{F} \xrightarrow{\mathcal{G}} \Gamma(T^*M \odot T^*M) \xrightarrow{C = F^* \flat_W F} \xrightarrow{U} \Gamma(\wedge^n T^*M) \xrightarrow{U(C)} \xrightarrow{\int_M} \mathbb{R}$$

Sequence of linear maps on tangent spaces:

$$\Gamma(T_\phi W) \xrightarrow{d^{\nabla^W}} \Gamma(T^*M \otimes T_\phi W) \xrightarrow{d\mathcal{G}|_F} \Gamma(T^*M \odot T^*M) \xrightarrow{dU|_C} \Gamma(\wedge^n T^*M) \xrightarrow{\int_M} \mathbb{R}$$

Pullback:

$$\Gamma(\wedge^n T^*M \otimes T_\phi^* W) \xleftarrow[(d^\nabla)^*]{d\mathcal{U}_\phi} \Gamma(\wedge^{n-1} T^*M \otimes T_\phi^* W) \xleftarrow{\frac{\partial U}{\partial F}} \xleftarrow{d\mathcal{G}|_F^*} \Gamma(TM \odot TM \otimes \wedge^n T^*M) \xleftarrow{\frac{\partial U}{\partial C}} \xleftarrow{dU|_C^*} \Omega^0(M) \xleftarrow{\mathbb{1}} \mathbb{R}^* \xleftarrow{1}$$

Terminology

$$\Gamma(\wedge^{n-1}T^*M \otimes T^*_\phi W) \xleftarrow{\frac{\partial U}{\partial F}} \Gamma(TM \odot TM \otimes \wedge^n T^*M) \xleftarrow{\frac{\partial U}{\partial C}} \Omega^0(M) \xleftarrow{\mathbb{1}} \mathbb{R}^* \xleftarrow{1}$$

$$\Gamma(\wedge^n T^*M \otimes T^*_\phi W) \xleftarrow[(d^\nabla)^*]{d\mathcal{U}_\phi}$$

2nd Piola–Kirchoff stress (type is dual to right-Cauchy–Green)

$$S := 2\frac{\partial U}{\partial C} \in \Gamma(TM \odot TM \otimes \wedge^n T^*M)$$

1st Piola–Kirchoff stress tensor (type is dual to deformation gradient)

$$P := d\mathcal{G}|_F{}^* S = \frac{\partial U}{\partial F} \in \Gamma(\wedge^{n-1}T^*M \otimes T^*_\phi W)$$