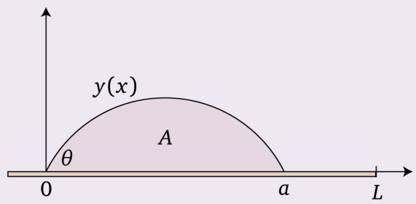
CSE 291 (SP 2024) Homework 2

2.1 Calculus of Variations

Exercise 2.1 — 10pt (+ 5 bonus).

Consider a water droplet on the table. For simplicity, we will consider a droplet in a 2D world resting on a 1D table. The shape of the droplet is described by the graph of a function $y(x)$ over $[0, a]$ with $y(0) = y(a) = 0$. We want to determine the shape y and the contact angle $\theta = \tan^{-1}(y'(0))$ (which will be the same as $-\tan^{-1}(y'(a))$) of the droplet.



The shape, the angle, and the length span a are determined by minimizing the following interfacial energy functionals that are proportional to the arclength of the interface. Let E_1 be the interfacial energy between air and water, E_2 be that between water and table, and E_3 be that between air and table:

$$E_1 = \int_0^a \gamma_{\mathrm{gl}} \sqrt{1 + y'(x)^2}\, dx \tag{1}$$

$$E_2 = \int_0^a \gamma_{\mathrm{sl}}\, dx = \gamma_{\mathrm{sl}} a \tag{2}$$

$$E_3 = \gamma_{\mathrm{sg}}(L - a) \tag{3}$$

for some gas–liquid, solid–liquid, solid–gas interfacial energy (surface tension) coefficients $\gamma_{\mathrm{gl}}, \gamma_{\mathrm{sl}}, \gamma_{\mathrm{sg}}$. We find y and a that minimizes $E_1 + E_2 + E_3$ subject to the constraint that the total area $\int_0^a y(x)\, dx = A$.

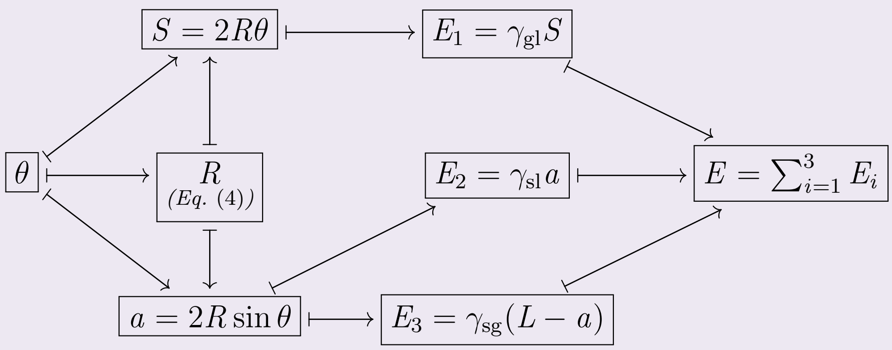
(a) The variational problem is easier and more familiar when a is fixed. For a fixed a, show that the optimal shape y is a portion of a circle.

(b) **(bounus)** Knowing that the curve is a circular arc, the variational problem reduces to an optimization for a reduced 1D parameter space of a (the parameter space can also be the contact angle θ or the radius of the circle *etc*). Show that the contact angle θ satisfies Young's wetting equation $\gamma_{\mathrm{gl}} \cos\theta + \gamma_{\mathrm{sl}} - \gamma_{\mathrm{sg}} = 0$.

Hint *Here is a possible change of variables. Let R be the radius of the circle and let θ be the contact angle. In terms of R, θ the variable a can be expressed as $a = 2R\sin\theta$, and the total arclength S of the arc (for $E_1 = \gamma_{\mathrm{gl}} S$) is $S = 2R\theta$. With a fixed area A the radius R and the angle θ are related by the relation $A = \theta R^2 - R^2 \sin\theta\cos\theta$ (sector of angle 2θ minus a triangle). That is,*

$$R = \sqrt{\frac{A}{\theta - \frac{1}{2}\sin(2\theta)}}. \tag{4}$$

In summary, the energies E_1, E_2, E_3, and thus the total energy $E = E_1 + E_2 + E_3$, are functions of just θ computed by the following computation graph:



2.2 Least Action Principle with Constraints

In a rigid body simulation, the positional variable $\mathbf{q}$ can be quite general and non-Cartesian; for a single body, it consists of its translation and rotation degrees of freedom; for a mechanical device with linkages the generalize coordinate may consist more angle variables. One recovers the physical world coordinate of every atom on the rigid body by composing Euclidean transformations (translations and rotations) parameterized by $\mathbf{q}$.

Formally, let $\mathbf{q}$ be the coordinate for a manifold Q. Every atom a ($a \in \mathcal{A}$ for some index set $\mathcal{A}$) has a physical position $\mathbf{x}_a \in \mathbb{R}^3$ given by $\mathbf{x}_a = f_a(\mathbf{q})$ for some transformation $f_a \colon Q \to \mathbb{R}^3$, $a \in \mathcal{A}$. Each movement $\dot{\mathbf{q}} \in T_{\mathbf{q}}Q$ is equipped with a kinetic energy $K(\mathbf{q}, \dot{\mathbf{q}})$ defined by adding the kinetic energies of all atoms $K(\mathbf{q}, \dot{\mathbf{q}}) = \int_{\mathcal{A}} \frac{1}{2} |(f_a)_*(\dot{\mathbf{q}})|^2_{\mathbb{R}^3}\, da$, which is a positive definite quadratic form of $\dot{\mathbf{q}}$. Each position $\mathbf{q}$ is also annotated with a potential energy $U(\mathbf{q}) = \int_{\mathcal{A}} u_a(f_a(\mathbf{q}))\, da$ for some potential function u_a for each atom. These functions $K \colon TQ \to \mathbb{R}$ and $U \colon Q \to \mathbb{R}$ can usually be derived explicitly or precomputed offline. Now with K, U, the dynamical system is modeled as the optimality condition for the action

$$S(\mathbf{q}) = \int_0^T \Big(K(\mathbf{q}, \dot{\mathbf{q}}) - U(\mathbf{q}) \Big)\, dt \quad \Longrightarrow \quad \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{\mathbf{q}}}\right) = \frac{\partial K}{\partial \mathbf{q}} - \frac{\partial U}{\partial \mathbf{q}}. \tag{5}$$

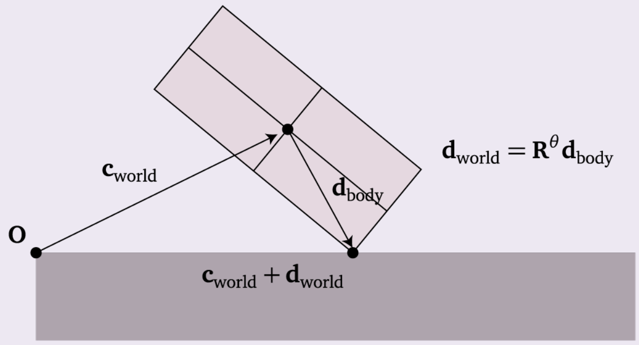
Now, when the rigid body interacts with the world, such as during collision, we add inequality constraints on the physical position of each atom. For example, the body should not penetrate the ground. In general, we can model the inequality constraint by describing an inequality condition for the physical position of each atom $h_a \leq 0$ for some $h_a \colon \mathbb{R}^3 \to \mathbb{R}$. By composition, they translate to constraints $(h_a \circ f_a) \leq 0$ on Q with $(h_a \circ f_a) \colon Q \to \mathbb{R}$. The KKT optimality condition is

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{\mathbf{q}}}\right) = \frac{\partial K}{\partial \mathbf{q}} - \frac{\partial U}{\partial \mathbf{q}} - (h_a \circ f_a)^* \mu_a(t) \tag{6}$$

for some Lagrange multipliers $\mu_a(t)$ indexed by both a and t so that $\mu_a(t) \geq 0$ and $\mu_a(t) = 0$ whenever $(h_a \circ f_a)(\mathbf{q}) \lneq 0$.

Physically, when the body is not in contact ($(h_a \circ f_a)(\mathbf{q}) \lneq 0$), we have no additional force from collision. When collision happens, the change of momentum can be added by an impulse. But there is a restriction on what type of impulse one can add: It must be in the image of the back propagation by $h_a \circ f_a$.

Exercise 2.2 — 10 pts.



A classic treatment for handling collision in a rigid body simulation is the following. Whenever a vertex of a moving body penetrate another object, "push" the positional variables $\mathbf{q}$ back to the feasible set and "readjust" the velocity so that the change in the momentum is in the range of the pullback of constraint on the world position.

In the plane, let $\mathbf{q} = \left[\begin{smallmatrix}\mathbf{c}_{\text{world}} \\ \theta\end{smallmatrix}\right]$ where $\mathbf{c}_{\text{world}} = \mathbf{c} = \left[\begin{smallmatrix}c_x \\ c_y\end{smallmatrix}\right]$ denotes the center of mass of a body, and θ be the rotation angle. Each vertex has a displacement vector $\mathbf{d}_{\text{body}} = \left[\begin{smallmatrix}d_x \\ d_y\end{smallmatrix}\right]$ in the body coordinate, representing its displacement from the center of the body. Note that the world coordinate for this vertex is given by

$$\begin{bmatrix} x \\ y \end{bmatrix}_{(\mathbf{c},\theta)} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{\mathbf{R}^\theta} \begin{bmatrix} d_x \\ d_y \end{bmatrix} + \underbrace{\begin{bmatrix} c_x \\ c_y \end{bmatrix}}_{\mathbf{c}}. \tag{7}$$

The kinetic energy energy is given by

$$K\left(\left[\begin{smallmatrix}\mathbf{c} \\ \theta\end{smallmatrix}\right], \left[\begin{smallmatrix}\dot{\mathbf{c}} \\ \dot\theta\end{smallmatrix}\right]\right) = \tfrac{m}{2}\left(\dot c_x^2 + \dot c_y^2\right) + \tfrac{m}{3}|\mathbf{d}_{\text{body}}|^2\dot\theta^2 = \begin{bmatrix}\dot{\mathbf{c}}^\intercal & \dot\theta\end{bmatrix} \underbrace{\begin{bmatrix} \frac{m}{2} & & \\ & \frac{m}{2} & \\ & & \frac{m|\mathbf{d}|^2}{3} \end{bmatrix}}_{\mathbf{M}/2} \begin{bmatrix} \dot{\mathbf{c}} \\ \dot\theta \end{bmatrix} \tag{8}$$

when the body is a uniformly distributed rectangle and when $\mathbf{d}_{\text{body}}$ is a corner. Note that the momentum associate to $\dot{\mathbf{q}} = \left[\begin{smallmatrix}\dot{\mathbf{c}} \\ \dot\theta\end{smallmatrix}\right]$ is given by $\mathbf{M}\dot{\mathbf{q}}$.

Let the constraint be that $-y(\mathbf{c},\theta) \le 0$ (as the ground is described by $y = 0$). And suppose the vertex with displacement $\mathbf{d}_{\text{body}}$ is currently in contact with the ground. What is the set of valid modification of the variable velocity $(\dot{\mathbf{q}}^{\text{after}} - \dot{\mathbf{q}}^{\text{before}})$ before and after the collision?

The solution should just be expanding $\{\mathbf{M}^{-1}(dy|^*_{\mathbf{q}})\mu \mid \mu \ge 0\}$.

(In practice, for perfectly elastic collision, one will also impose energy conservation and search for μ so that the kinetic energy after the collision equals to the one before the collision.) ■

2.3 Miniproject

Exercise 2.3 — 20 pts (+5 bonus). Design a mechanical system and derive its equation of motion using the least action principle. You can use ODE solver, *e.g.* explicit RK4 or implicit Euler (may require root-finding at every iteration), or derive a variational integrator, to integrate and march the animation. You are welcome to model simple collision, such as a procedure similar to **Exercise 2.2** or just put a soft barrier function in your potential energy. Similar to HW0, you will write a document to describe your system and the numerical algorithm you employed, as well as uploading a video of your animation. In this miniproject, we expect you to show the work of the mathematical derivation for arriving at the equation of motion in the written document.

Here are some examples

- A body bouncing in a bowl: `https://youtu.be/0zmWEzrtvJA`.
- Linkage system such as double (or more) pendulum.
- Dzhanibekov effect in 3D rigid rotations (`https://en.wikipedia.org/wiki/Tennis_racket_theorem`)

Bonus credits will be given to those with excellent exposition or complexity of the system. ■