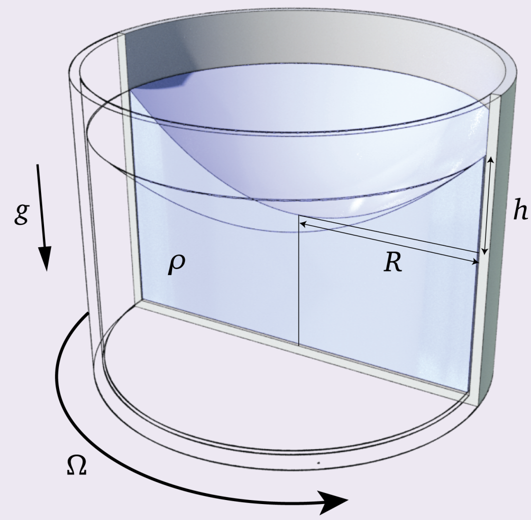
CSE 291 (SP 2024) Homework 1

1.1 Dimensional Analysis

Exercise 1.1 — 10 pts. Consider a liquid in a cylindrical container that is rotating at a certain frequency Ω (*e.g.*counted as rounds per minute). At equilibrium, the liquid surface will have an elevation difference h between the center and the rim. We postulate that this height h is a function of the rotating frequency Ω, the radius R of the container, the liquid's density ρ, and the gravitational acceleration g. Use dimensional analysis to answer the following questions.



(a) We already know that the aspect ratio $\Pi_1 = \frac{h}{R}$ is one of the dimensionless variables. Using the Buckingham Π Theorem, this aspect ratio is determined by some other dimensionless variables $\Pi_1 = f(\Pi_2, \ldots)$. How many and what are these other dimensionless variables? (Choose $\Pi_2, \ldots$ so that they do not involve h by a suitable linear combination of the null vectors of the dimension matrix.)

(b) Does the aspect ratio $\Pi_1 = \frac{h}{R}$ depend on the density ρ?

(c) Perform the rotating bucket experiment on both Earth and Mars using the same bucket size R. To achieve the same aspect ratio ($\frac{h}{R}$) for the liquid surfaces on both planets, how much faster or slower should the rotation frequency (Ω) on Mars be compared to the rotation frequency on Earth? (Earth's gravity is $9.806\,\mathrm{m/s^2}$, while Mars' gravity is $3.721\,\mathrm{m/s^2}$.)

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1.2 Vectors, Covectors, Differentials, and Adjoint

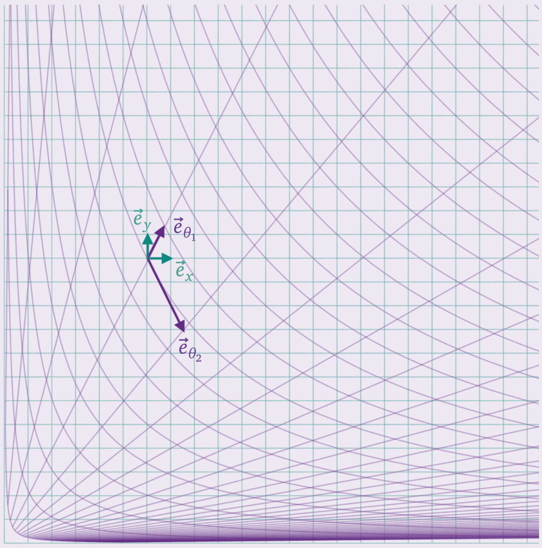
Exercise 1.2 — 20 pts. Consider a Cartesian–Euclidean xy-plane and an alternative coordinate system in the first quadrant defined by the following two coordinate functions:

$$\theta_1 = \sqrt{xy}, \quad \theta_2 = \sqrt{\frac{x}{y}}. \tag{1}$$

This can be expressed equivalently as:

$$x = \theta_1\theta_2, \quad y = \frac{\theta_1}{\theta_2} \tag{2}$$

assuming that θ_1, θ_2, x, and y are all positive. The differentials of the coordinate functions yield two covector bases, (dx, dy) and $(d\theta_1, d\theta_2)$. A coordinate vector basis is defined as the dual basis of a covector basis. Specifically, $(\vec{e}_x, \vec{e}_y)$ is the dual basis to (dx, dy), and $(\vec{e}_{\theta_1}, \vec{e}_{\theta_2})$ is the dual basis to $(d\theta_1, d\theta_2)$.



(a) Establish the transformation matrices for the change of covector bases. That is, what are the coefficients $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ in the following relationships?

$$dx = a_{11}\,d\theta_1 + a_{12}\,d\theta_2, \quad dy = a_{21}\,d\theta_1 + a_{22}\,d\theta_2 \tag{3}$$

$$d\theta_1 = b_{11}\,dx + b_{12}\,dy, \quad d\theta_2 = b_{21}\,dx + b_{22}\,dy. \tag{4}$$

More concisely,

$$\begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} d\theta_1 \\ d\theta_2 \end{bmatrix}, \quad \begin{bmatrix} d\theta_1 \\ d\theta_2 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}. \tag{5}$$

Please express both a_{ij}'s and b_{ij}'s in terms of θ_1 and θ_2.
Hint *Matrix inversion.*

(b) What are the coefficients $\mathbf{F} = \left[\begin{smallmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{smallmatrix}\right]$ and $\mathbf{G} = \left[\begin{smallmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{smallmatrix}\right]$ as expressions of θ_1, θ_2 that relate the coordinate vectors?

$$\begin{bmatrix} \vec{e}_{\theta_1} & \vec{e}_{\theta_2} \end{bmatrix} = \begin{bmatrix} \vec{e}_x & \vec{e}_y \end{bmatrix} \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}, \quad \begin{bmatrix} \vec{e}_x & \vec{e}_y \end{bmatrix} = \begin{bmatrix} \vec{e}_{\theta_1} & \vec{e}_{\theta_2} \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}. \tag{6}$$

Hint *Use the results from* ***(a)***.

(c) Consider the function $U = x^2 + y^2 = \theta_1^2\theta_2^2 + \frac{\theta_1^2}{\theta_2^2}$. Given that $dU = 2xdx + 2ydy = 2\theta_1\theta_2 dx + \frac{2\theta_1}{\theta_2} dy$ (since $\frac{\partial U}{\partial x} = 2x$ and $\frac{\partial U}{\partial y} = 2y$), compute the partial derivatives $\frac{\partial U}{\partial \theta_1}$ and $\frac{\partial U}{\partial \theta_2}$ of U with respect to the (θ_1, θ_2) coordinate system.
Hint *By definition* $dU = \frac{\partial U}{\partial \theta_1} d\theta_1 + \frac{\partial U}{\partial \theta_2} d\theta_2$. *Apply one of the transformation matrices from* ***(a)***, ***(b)***.

(d) Consider again $U = x^2 + y^2 = \theta_1^2\theta_2^2 + \frac{\theta_1^2}{\theta_2^2}$. What is the gradient vector $\vec{v} = \operatorname{grad} U = \_\_\vec{e}_{\theta_1} + \_\_\vec{e}_{\theta_2}$ written in the (θ_1, θ_2) coordinate system? Note that $\vec{e}_x, \vec{e}_y$ are orthonormal, so $\vec{v} = \operatorname{grad} U = 2x\vec{e}_x + 2y\vec{e}_y$.

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Derivative of Functions on Matrices

Later in elasticity, we will encounter many matrix operations like $\mathbf{A} \mapsto \mathbf{A}^\intercal\mathbf{A}$. This is a change of coordinate and we want to know what is the differential (Jacobian) of this operation, and what is the pullback through this operation.

Let us first clarify what are transposes and dual space of space of linear operators.

Let U, V be two vector spaces. The space of all linear maps from U to V is denoted by $\mathrm{Hom}(U; V) = \{\mathbf{A} \colon U \xrightarrow{\text{linear}} V\}$ (where "Hom" is the the abbreviation of "homomorphism"). The *transpose* (or *adjoint*) $\mathbf{A}^*$ of an operator $\mathbf{A} \in \mathrm{Hom}(U; V)$ has type $\mathbf{A}^* \in \mathrm{Hom}(V^*; U^*)$:

$$\begin{tikzcd} U \arrow[r, "\mathbf{A}"] \arrow[d, dotted, leftrightarrow, "\text{dual}"'] & V \arrow[d, dotted, leftrightarrow, "\text{dual}"] \\ U^* & V^* \arrow[l, "\mathbf{A}^*"'] \end{tikzcd}$$

defined by $(\mathbf{A}^*\beta)[\![\vec{u}]\!] = \beta[\![\mathbf{A}\vec{u}]\!]$ for all $\beta \in V^*$ and $\vec{u} \in U$.

Observe that the dual space $\mathrm{Hom}(U; V)^*$ is given by

$$\mathrm{Hom}(U; V)^* = \mathrm{Hom}(U^*; V^*)$$

where the dual pairing $\langle \mathbf{Y} | \mathbf{X} \rangle$ for $\mathbf{Y} \in \mathrm{Hom}(U^*; V^*)$ and $\mathbf{X} \in \mathrm{Hom}(U; V)$ is given by

$$\langle \mathbf{Y} | \mathbf{X} \rangle = \mathrm{tr}(\mathbf{Y}^*\mathbf{X}) = \mathrm{tr}(\mathbf{X}^*\mathbf{Y}) = \mathrm{tr}(\mathbf{X}\mathbf{Y}^*) = \mathrm{tr}(\mathbf{Y}\mathbf{X}^*), \tag{7}$$

where the trace operator is defined for operators of type $\mathrm{Hom}(U; U)$, $\mathrm{Hom}(V; V)$ etc that map from a space to the same space. (The trace of an operator $\mathbf{C} \in \mathrm{Hom}(U; U)$

is given by $\operatorname{tr}(\mathbf{C}) = \sum_i \langle dx^i | \mathbf{C}\vec{e}_i \rangle$ where (dx^i) and $(\vec{e}_j)$ are dual basis for U^* and U. This is basically saying that the trace is the sum of the diagonal entries of a matrix. The trace $\operatorname{tr}(\mathbf{C})$ is also the sum of the eigenvalues of $\mathbf{C}$. In the following exercise, we don't need the definition of trace beyond the properties of trace in the hint.)

Exercise 1.3 — 10 pts. Let $\mathbf{B} \in \operatorname{Hom}(V, V^*)$ be some linear map. Consider the following (nonlinear) function:

$$f\colon \operatorname{Hom}(U, V) \to \operatorname{Hom}(U, U^*), \quad f(\mathbf{A}) := \mathbf{A}^*\mathbf{B}\mathbf{A}. \tag{8}$$

(a) What is the differential $df|_{\mathbf{A}}\colon \operatorname{Hom}(U, V) \xrightarrow{\text{linear}} \operatorname{Hom}(U, U^*)$ of f at $\mathbf{A}$? That is, for a given perturbation direction $\mathbf{E} \in \operatorname{Hom}(U, V)$, compute $df|_{\mathbf{A}}[\![\mathbf{E}]\!] = \frac{d}{d\epsilon} f(\mathbf{A} + \epsilon\mathbf{E})|_{\epsilon=0}$.

(b) What is the adjoint $(df|_{\mathbf{A}})^*\colon \operatorname{Hom}(U^*, U) \xrightarrow{\text{linear}} \operatorname{Hom}(U^*, V^*)$ of the linear map $df|_{\mathbf{A}}$ from part **(a)**? That is, for a given $\mathbf{S} \in \operatorname{Hom}(U^*, U)$, compute $(df|_{\mathbf{A}})^*[\![\mathbf{S}]\!]$, which should satisfy $\langle (df|_{\mathbf{A}})^*[\![\mathbf{S}]\!] | \mathbf{E} \rangle = \langle \mathbf{S} | (df|_{\mathbf{A}})[\![\mathbf{E}]\!] \rangle$ for all $\mathbf{E} \in \operatorname{Hom}(U, V)$.

Hint *The trace operator satisfies linearity* $\operatorname{tr}(\mathbf{A}_1 + \mathbf{A}_2) = \operatorname{tr}(\mathbf{A}_1) + \operatorname{tr}(\mathbf{A}_2)$, *cyclic permutability* $\operatorname{tr}(\mathbf{A}_1\mathbf{A}_2 \cdots \mathbf{A}_n) = \operatorname{tr}(\mathbf{A}_2 \cdots \mathbf{A}_n\mathbf{A}_1)$, *and* $\operatorname{tr}(\mathbf{A}) = \operatorname{tr}(\mathbf{A}^*)$. ■