Continuum mechanics: The study of the statics & dynamics for deformable bodies.

Examples: Elastic solid, fluids, elastoplastic bodies, viscoelastic fluids, ferromagnetic fluids, plasmas, ...

Elastic: Potential energy is a function of how the body is deformed.
$\Rightarrow$ Restoration force

Fluids: Potential is a function of how the volume is changed.
$\Rightarrow$ No restoration force to shearing deformation.

Plastic: Referenced undeformed shape can change over time.

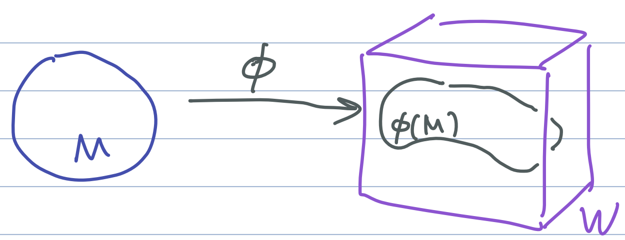
Viscous: Incremental potential is a function of rate of change of deformation.

Ferro: Fluids made of little magnets

Plasma: Fluids made of electrically conducting material.

Setup for deformable body

- Material space M (Lagrangian coordinate)
- World space $W = \mathbb{R}^3$ Euclidean space (Eulerian coordinate)
- State of the body is a map $\phi: M \to W$.
- Material space is equipped with a mass measure dm
- World space is equipped with a metric $|\cdot|_W^2$ or $\flat_W$



For $\underline{\text{elastic body}}$, the kinetic energy is given by

$$\mathcal{K}(\dot{\phi}) = \frac{1}{2}\int_M |\dot{\phi}|_W^2 \, dm_M$$

Potential energy takes the form

$$\mathcal{U}(\phi) = \int_M U(\underbrace{\phi^* b_W}_{\substack{\text{bilinear form} \\ \text{induced inner product on } M}})$$

b_W is a symmetric bilinear form

$b_W(\vec{u})(\vec{v}) = \langle \vec{u}, \vec{v} \rangle_W$

$\underset{\text{bilinear form}}{\phi^* b_W}$ becomes a bilinear form on M

$(\phi^* b_W)(\vec{X}, \vec{Y}) = \langle \phi_{*\,\text{vec}} \vec{X}, \phi_{*\,\text{vec}} \vec{Y} \rangle$

$$U : \begin{Bmatrix} \text{symmetric} \\ \text{linear} \\ \text{forms} \end{Bmatrix} \longrightarrow \begin{Bmatrix} \text{energy} \\ \text{density} \end{Bmatrix}$$ ← space of measures

Lagrangian $\mathcal{L}(\phi, \dot{\phi}) = \mathcal{K}(\dot{\phi}) - \mathcal{U}(\phi)$

EL eq: $\rho_M \ddot{\phi} = -\dfrac{\delta \mathcal{U}}{\delta \phi}$. where $\rho \, dV = dm$

$C(M) = (M \xrightarrow{\text{conti}} \mathbb{R})$

$C(M)^* = \{\text{space of measures}\}$

$\underset{\substack{\cap \\ C(M)^*}}{\langle\!\langle dm} | \underset{\substack{\cap \\ C(M)}}{f \rangle\!\rangle} = \int_M f \, dm$

Let us denote the coordinate of M by (X^1, X^2, X^3)

" W by (x^1, x^2, x^3)

$$\vec{\phi}(\vec{X}) = \begin{bmatrix} \phi^1(\vec{X}) \\ \phi^2(\vec{X}) \\ \phi^3(\vec{X}) \end{bmatrix}$$

Define $F(\vec{x}) := d\vec{\phi}(\vec{X}) = \begin{bmatrix} d\phi^1 \\ d\phi^2 \\ d\phi^3 \end{bmatrix}$ or $\begin{bmatrix} \frac{\partial \phi^1}{\partial X^1} & \frac{\partial \phi^1}{\partial X^2} & \frac{\partial \phi^1}{\partial X^3} \\ & \vdots & \\ \frac{\partial \phi^3}{\partial X^1} & \frac{\partial \phi^3}{\partial X^2} & \frac{\partial \phi^3}{\partial X^3} \end{bmatrix}$

$F = F^i_a \frac{\partial}{\partial x^i} dX^a$

$F^i_a = \dfrac{\partial \phi^i}{\partial X^a}$

If $\vec{v} \in T_{\vec{X}}M$, $\vec{v} = v^a \dfrac{\partial}{\partial X^a}$,

then $(\phi_{*\,\text{vec}} \vec{v}) = (F^i_a v^a) \dfrac{\partial}{\partial x^i}$.

We call F the $\underline{\text{deformation gradient}}$ in continuum mechanics.

Next let us understand the concept of pullback metric.

Let $\langle \, , \, \rangle_W$ be the inner product structure in W.

Using $\frac{\partial}{\partial x^i}$ basis we have inner product matrix

$$\flat_{ij} := \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle$$

$$\flat_{ij} v^i = (\flat v)_j \quad \text{lowers the index}$$

$\flat : T_x W \to T_x^* W$ is the flat operator

$$\langle \flat(\vec{u}) | \vec{v} \rangle = \langle \vec{u}, \vec{v} \rangle_W .$$

We also have F that relates $T_{\vec{X}} M$ with $T_{\vec{x}} W$, $\vec{x} = \vec{\phi}(\vec{X})$.

$$\begin{array}{ccc} T_{\vec{X}} M & \xrightarrow{F} & T_{\vec{x}} W \\ & & \downarrow \flat \\ T^*_{\vec{X}} M & \xleftarrow{F^*} & T^*_{\vec{x}} W \end{array}$$

where F^* is the adjoint of F.

By composing the maps we have

$$(\phi^*_{\text{metric}} \flat) := F^* \flat F : T_{\vec{X}} M \to T^*_{\vec{X}} M$$

an inner product structure for $T_{\vec{X}} M$:

$$\langle \vec{u}, \vec{v} \rangle_{\phi^*_{\text{metric}} \flat} = \langle (\phi^* \flat) \vec{u} \, | \, \vec{v} \rangle = \langle F^* \flat F \vec{u} \, | \, \vec{v} \rangle$$

$$= \langle \flat F \vec{u} \, | \, F \vec{v} \rangle$$

$$= \langle F\vec{u}, F\vec{v} \rangle_W .$$

We call $C := \phi^*_{\text{metric}} \flat = F^* \flat F$ the right-Cauchy-Green tensor.

If the coord on W is orthonormal, then C in matrix form is

$$C = F^T F.$$

In general:

$$C = C_{ab}\, dX^a \otimes dX^b, \quad C_{ab} = F^i_a\, b_{ij}\, F^j_b$$

Back to potential energy

$$\mathcal{U}(\phi) = \int_M U(C) \quad \text{given some model } U : \left\{\begin{smallmatrix}\text{inner}\\ \text{products}\\ \text{on } M\end{smallmatrix}\right\} \to \{\text{measure}\}$$

Computation diagram

$$\begin{tikzcd}
\underset{\text{flow maps}}{\boxed{\phi}} \arrow[r, "F = d\phi", "d"'] & \underset{\text{deformation gradients}}{\boxed{F}} \arrow[r, "C = F^* b F", "g"'] & \boxed{C} \arrow[r, "U"] & \boxed{U(C)} \arrow[r, "\int_M"] & \boxed{\mathcal{U}(\phi)} \\
(M \to W) & \Gamma(T^*M \otimes TW) & \Gamma(T^*M \otimes T^*M) & C(M)^* & \mathbb{R}
\end{tikzcd}$$

Now we pullback the unit covector $1\,dz \in T^*_{\mathcal{U}(\phi)}\mathbb{R}$

$$\frac{\partial \mathcal{U}}{\partial U} = \Big(\int_M\Big)^*_{\text{cov}}(1\,dz)$$

$$\frac{\partial \mathcal{U}}{\partial C} = \frac{\partial U}{\partial C}\frac{\partial \mathcal{U}}{\partial U} = (U)^*_{\text{cov}}\Big(\int_M\Big)^*_{\text{cov}}(1\,dz) \qquad \boxed{S := 2\frac{\partial \mathcal{U}}{\partial C} \text{ is called 2nd Piola-Kirchhoff stress}}$$

$$\frac{\partial \mathcal{U}}{\partial F} = \frac{\partial C}{\partial F}\frac{\partial U}{\partial C}\frac{\partial \mathcal{U}}{\partial U} = \underset{\text{cov}}{g^*}\,\underset{\text{cov}}{U^*}\,\underset{\text{cov}}{\int_M^*}(1\,dz) \qquad \boxed{P := \frac{\partial \mathcal{U}}{\partial F} \text{ is called 1st Piola-Kirchhoff stress}}$$

$$\frac{\partial \mathcal{U}}{\partial \phi} = \frac{\partial F}{\partial \phi}\frac{\partial C}{\partial F}\frac{\partial U}{\partial C}\frac{\partial \mathcal{U}}{\partial U} = \underset{\text{cov}}{d^*}\,\underset{\text{cov}}{g^*}\,\underset{\text{cov}}{U^*}\,\underset{\text{cov}}{\int_M^*}\,1\,dz. \qquad \boxed{f = -\frac{\partial \mathcal{U}}{\partial \phi} \text{ force}}$$

$$\boxed{\phi} \xrightarrow[d]{F = dF} \boxed{F} \xrightarrow[g]{C = F^{*} b F} \boxed{C} \xrightarrow{U} \boxed{U(C)} \xrightarrow{\int_M} \boxed{\mathcal{U}(\phi)}$$

$$(M \to W) \qquad \Gamma(T^{*}M \otimes TW) \qquad \Gamma(T^{*}M \otimes_{sym} T^{*}M) \qquad C(M)^{*} \qquad \mathbb{R}$$

push forward

$$T_\phi(M \to W) \xrightarrow{d} \Gamma(T^{*}M \otimes TW) \xrightarrow{dg} \Gamma(T^{*}M \otimes_{sym} T^{*}M) \xrightarrow{dU} C(M)^{*} \xrightarrow{\int_M} \mathbb{R}$$

$$= \Gamma(T_\phi W)$$

$$= \left\{ \vec{v} : M \to TW, \ \vec{v}(X) \in T_{\phi(X)} W \right\}$$

adjoint

$$\Gamma(T_\phi^{*} W) \xleftarrow{d^{*}} \Gamma(TM \otimes T^{*}W) \otimes C(M)^{*} \xleftarrow{(dg)^{*}} \Gamma(TM \otimes_{sym} TM) \otimes C(M)^{*} \xleftarrow{(dU)^{*}} C(M) \xleftarrow{(\int_M)^{*}} \mathbb{R}^{*}$$

$$\mathbb{R}^{*} = \mathbb{R}$$

$$\boxed{-f} \qquad \boxed{P} \qquad \boxed{\tfrac{1}{2} S} \qquad \boxed{?}$$

e.g. S in index notation should be

$$P = P^a_i \frac{\partial}{\partial x^a} dx^i \, dX^1 dX^2 dX^3 \qquad S = S^{ab} \frac{\partial}{\partial x^a} \otimes_{sym} \frac{\partial}{\partial x^b} \otimes dX^1 dX^2 dX^3$$

What is $(\int_M^{*})(1\,dz) \in C(M)$?

If $h = (\int_M^{*})(1\,dz)$ then

$$\langle\!\langle h \mid dm \rangle\!\rangle = \langle\!\langle \textstyle\int_M^{*}(1\,dz) \mid dm \rangle\!\rangle = \langle 1\,dz \mid \textstyle\int_M dm \rangle$$

$$\langle\!\langle h \mid dm \rangle\!\rangle = \int h \, dm, \qquad dm \in C(M)^{*}$$

$$= \int dm$$

$$\forall dm$$

$$\Rightarrow h = \mathbb{1} \quad \text{const. one function.}$$

What is $(dU)^{*} \mathbb{1} = \frac{1}{2} S$?

$$\langle\!\langle \tfrac{1}{2}S \mid \mathring{C} \rangle\!\rangle = \langle\!\langle (dU)^{*}\mathbb{1} \mid \mathring{C} \rangle\!\rangle = \langle\!\langle \mathbb{1} \mid (dU)[\![\mathring{C}]\!] \rangle\!\rangle$$

$$\int \tfrac{1}{2}S\,\mathring{C} \qquad = \int_M \left\langle \frac{\partial U}{\partial C} \middle| \mathring{C} \right\rangle = \int_M \operatorname{tr}\left(\left(\frac{\partial U}{\partial C}\right)^{*} \mathring{C}\right)$$

$$\Rightarrow S = 2\frac{\partial U}{\partial C}.$$

The mapping $S = 2\left.\frac{\partial U}{\partial C}\right|_C =: R(C)$ is called a stress-strain relation

<u>Example</u>: Suppose there is a rest metric b_M on the material

$E := \frac{1}{2}(\#_M C - \mathrm{id})$ measures the deviation between $C = \phi^* b$ and b_M.

Green-Lagrange or Green-St. Venant strain

$(\#_M = b_M^{-1})$

St Venant – Kirchhoff model:

$$U(C) = \left(\frac{\lambda}{2}\operatorname{tr}(E)^2 + \mu \operatorname{tr}(E^2)\right) dX^1 dX^2 dX^3,$$

where λ, μ are called the Lamé parameters.

$\lambda + \frac{2}{3}\mu$: bulk modulus

μ : shear modulus > 0

$$S = 2\frac{\partial U}{\partial C} = \left(\lambda \operatorname{tr}(E)\#_M + 2\mu E \#_M\right) \otimes dX^1 dX^2 dX^3.$$

This is the linear stress-strain model

What is $P = (dg)^*(\frac{1}{2}S)$?

$$g : F \longmapsto F^* b_W F$$

$$dg|_F[\mathring{F}] = \mathring{F}^* b_W F + F^* b_W \mathring{F} = \text{Symmetric part of } 2F^* b_W \mathring{F}.$$

$$\langle\!\langle dg|_F[\mathring{F}] | \tfrac{1}{2}S \rangle\!\rangle = \langle\!\langle \mathring{F} | (dg^*) \tfrac{S}{2} \rangle\!\rangle$$

$$" \tfrac{1}{2}\int_M tr(S^*(\mathring{F}^* b_W F + F^* b_W \mathring{F}))$$

$$= \tfrac{1}{2}\int_M tr[\mathring{F}^* b_W F S^*] + tr[\mathring{F}^* b_W F S]$$

$$= \int_M \langle \mathring{F} | b_W F (\tfrac{S+S^*}{2}) \rangle$$

$$= \langle\!\langle \mathring{F} | b_W F (\tfrac{S+S^*}{2}) \rangle\!\rangle$$

$$\Rightarrow \boxed{P = b_W F(\tfrac{S+S^*}{2}).}$$

(In $S = 2\frac{\partial U}{\partial C}$, we only have U defined for symmetric C, so we only have info of the sym pat of $\frac{\partial U}{\partial C}$.

The skew-sym part of $\frac{\partial U}{\partial C}$ could be anything depending on how we artificially extend the definition of U to skew matrices.

Note that $P = b_W F(\frac{S+S^*}{2})$ only use the sym part of S.

Without loss of generality we can always assume S being symmetric. Then $\boxed{P = b_W F S.}$)

Textbook notations: $\quad ((b_W)_{ij} = \delta_{ij})$

$$F^i_a = \frac{\partial \phi^i}{\partial X^a}, \quad C = F^T F, \quad E = \tfrac{1}{2}(C - I)$$

$$= \begin{array}{cc} & a \rightarrow \\ i \downarrow & \begin{bmatrix} \quad \end{bmatrix} \end{array}$$

$S = \lambda \operatorname{tr}(E) + 2\mu E$ 2nd PK stress

$P = FS$ 1st PK stress

Final step: d^*

$$\langle\langle d^* P \mid \mathring{\phi} \rangle\rangle = \langle\langle P \mid d\mathring{\phi} \rangle\rangle$$

$$= \int_M \operatorname{tr}(P^* d\mathring{\phi}) = \int_M P^a_i \frac{\partial \mathring{\phi}^i}{\partial X^a} dX^1 dX^2 dX^3$$

$$\overset{\text{div}}{\underset{\text{thm}}{=}} - \int_M \frac{\partial P^a_i}{\partial X^a} \mathring{\phi}^i dX^1 dX^2 dX^3$$

$$+ \oint_{\partial M} P^a_i \mathring{\phi}^i n_a dA$$

$$\Rightarrow d^* P = -\frac{\partial P^a_i}{\partial X^a} dx^i dX^1 dX^2 dX^3 \quad = -\operatorname{div} P \quad \text{in the interior}$$

There is a boundary force $P^a_i n_a dx^i$

or Pn.

$$\begin{array}{cc} & a \rightarrow \\ i \downarrow & \begin{bmatrix} P \end{bmatrix} \begin{bmatrix} n \end{bmatrix} \end{array}$$

P maps a normal to a force

Equation of motion: $\rho_M \ddot{\phi} = \operatorname{div} P$ in the interior

$\frac{dm}{dA} \ddot{\phi} = Pn$ on the boundary.

Cauchy stress: σ is the change of coord of P to the W coord.

$\sigma n_{\text{world}} = f_{\text{world}}$

$$P = P^a_i \frac{\partial}{\partial X^a} dx^i \, dX^1 dX^2 dX^3 \qquad P = \phi^* \sigma$$

vec-measure

$$\sigma = \sigma_i^j \frac{\partial}{\partial x^j} dx^i \, dx^1 dx^2 dx^3$$

$$\begin{array}{cc} & j \rightarrow \\ i \downarrow & \begin{bmatrix} \sigma \end{bmatrix} \end{array}$$

$$\underset{vec}{\phi^*}\left(u^i \frac{\partial}{\partial x^i}\right) = \left[(F^{-1})^a_i u^i\right] \frac{\partial}{\partial X^a}$$

$$\underset{measure}{\phi^*}\left(\rho\, dx^1 \cdots dx^n\right) = \det(F)\, \rho\, dX^1 \cdots dX^n$$

$$\underset{\substack{vec-\\meas}}{\phi^*}\left(u^i \frac{\partial}{\partial x^i} dx^1 \cdots dx^n\right) = \det(F) (F^{-1})^a_i u^i \frac{\partial}{\partial X^a} dX^1 \cdots dX^n.$$

$$\underset{\substack{vec\\meas}}{\phi^*}\left(\sigma_i^{\ j} \underbrace{\frac{\partial}{\partial x^j}}_{vec} \overbrace{dx^i}^{\text{force covector}} \underbrace{dx^1 \cdots dx^n}_{meas}\right) = \det(F) (F^{-1})^a_j \sigma_i^{\ j} \underbrace{\frac{\partial}{\partial X^a}}_{vec} \overbrace{dx^i}^{\text{force}} \underbrace{dX^1 \cdots dX^n}_{meas}$$

$$= P_i^{\ a} \frac{\partial}{\partial X^a} dx^i dX^1 \cdots dX^n.$$

$$\Rightarrow P_i^{\ a} = (F^{-1})^a_j\, \sigma_i^{\ j} \det(F)$$ Let $J = \det(F)$.

$$\Leftrightarrow P = J \sigma F^{-T}$$ as a matrix.

$$\Leftrightarrow P = \sigma \operatorname{cof}(F).$$

<u>Thm</u> (Piola idenity)

$$\underset{\substack{\uparrow\\ \text{diverge}}}{d^*}\left(\underset{\substack{vec\\meas}}{\phi^*} \sigma\right) = \underset{\substack{vec\\meas}}{\phi^*} d^* \sigma$$ or $\nabla \cdot (\sigma \operatorname{cof}(F)) = J\, \nabla \cdot \sigma$

$\Rightarrow$ In equilibrium $\operatorname{div} P = 0$ then $\operatorname{div} \sigma = 0$.

<u>Thm</u> $\tilde{\sigma} = \# \sigma$ is symmetric

$$\tilde{\sigma}^{ij} = \underset{\substack{\uparrow\\ \text{world space metric}}}{\#^{ik}} \sigma_k^{\ j}$$

(pf) $\mathcal{P} = b_W F\left(\frac{S+S^*}{2}\right)$

$\sigma = J\mathcal{P}F^* = b_W F\left(\frac{S+S^*}{2}\right)F^*$

$\tilde{\sigma} = \frac{\sigma}{b_W} = F\left(\frac{S+S^*}{2}\right)F^*$ symmetric.