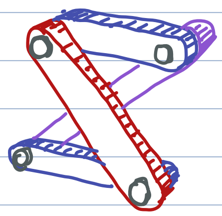
Consider a mechanical system with (generalized) position space Q.

A <u>holonomic constraint</u> is an equality constraint on the positions.

e.g.

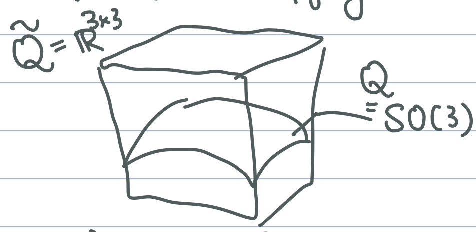
① Linkage system



② Rigid body

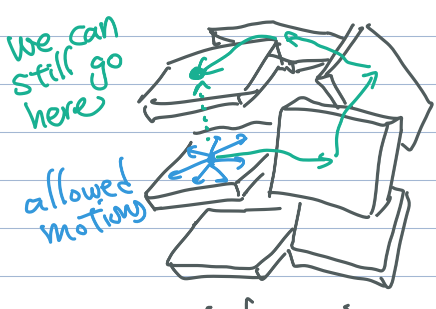
$\tilde{Q}$ = All 3×3 matrices.

$Q \subset \tilde{Q}$ constrained to matrices $R \in \tilde{Q}$ satisfying $R^T R = \mathrm{id}$.

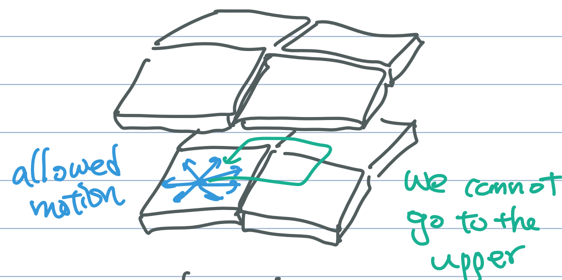


Holonomic constraints restrict the position to a lower dimensional space.

A <u>non-holonomic constraint</u> is a constraint on the velocity in such a way that fails to restrict the position to a lower dimensional space.



non-holonomic



holonomic.

An assignment of a subspace $\mathcal{A}_p \subset T_pQ$ for each tangent space T_pQ is called a <u>subspace field</u> or called a <u>distribution</u>.

- A distribution $\mathcal{A} \subset TQ$ can represent constraints on velocity.
- A distribution $\mathcal{A}$ is <u>integrable</u> if $\exists$ family submanifolds $S \subset Q$ (with the same dimension as $\mathcal{A}$) so that $\mathcal{A}_p$ is the tangent space T_pS.
- <u>Integrable</u> $\Leftrightarrow$ <u>constraint is holonomic</u>.

- How to decide if a given distribution $\mathcal{A}$ is integrable?
- A vector field $\vec{u} \in \Gamma(TQ) = \{\vec{u} : Q \to TQ \mid \vec{u}(p) \in T_pQ\}$ generate a flow $\phi^t_{\vec{u}} : Q \to Q$

$$\begin{cases} \frac{\partial}{\partial t}\phi^t_{\vec{u}}(p) = \vec{u}\left(\phi^t_{\vec{u}}(p)\right) & \text{for all } p \\ \phi^0_{\vec{u}}(p) = p \end{cases}$$

<u>Thm Frobenius integrability condition</u> A distribution $\mathcal{A}$ is integrable iff for any vector fields $\vec{a}, \vec{b} \in \Gamma(\mathcal{A}) = \{\vec{u} : Q \to TQ \mid \vec{u}(p) \in \mathcal{A}_p\}$

$$\underbrace{(\phi^t_{\vec{a}})}_{\text{flow generated by } \vec{a}}\underbrace{{}_*}_{\text{push forward}}\vec{b} \quad \left(= (d\phi^t_{\vec{a}})\vec{b}\right) \quad \text{remain} \in \Gamma(\mathcal{A}).$$

<u>Lie derivative</u>: Derivative of pullback via a flow.

<u>Def</u> Let $\phi^t_{\vec{u}} : Q \to Q$ $\left(\phi^0_{\vec{u}} = \mathrm{id}_Q,\ \frac{\partial \phi^t_{\vec{u}}}{\partial t} = \vec{u} \circ \phi^t_{\vec{u}}\right)$ be the flow map generated by a velocity field $\vec{u} \in \Gamma(TQ)$. Let $(\phi^t_{\vec{u}})^*$ be pullback

$$\begin{cases} \underset{\text{fcn}}{(\phi^t_{\vec{u}})^*} : (Q \to \mathbb{R}) \to (Q \to \mathbb{R}) & \underset{\text{fcn}}{(\phi^t_{\vec{u}})^*} g = g \circ \phi^t_{\vec{u}} \\ \underset{\text{vec}}{(\phi^t_{\vec{u}})^*} : \Gamma(TQ) \to \Gamma(TQ) & \underset{\text{vec}}{(\phi^t_{\vec{u}})^*} \vec{w} = (d\phi^t_{\vec{u}})^{-1} \vec{w} \\ \underset{\text{covec}}{(\phi^t_{\vec{u}})^*} : \Gamma(T^*Q) \to \Gamma(T^*Q) & \underset{\text{covec}}{(\phi^t_{\vec{u}})^*} \alpha = (d\phi^t_{\vec{u}})^{\overset{\text{adjoint}}{*}} \alpha \end{cases}$$

Define the <u>Lie derivative</u> $\mathcal{L}_{\vec{u}}$ as

$$\underset{\text{fcn}}{\mathcal{L}_{\vec{u}}} : (Q \to \mathbb{R}) \to (Q \to \mathbb{R}) \qquad \underset{\text{fcn}}{\mathcal{L}_{\vec{u}}} g := \frac{\partial}{\partial t}\Big|_{t=0} \left[\underset{\text{fcn}}{(\phi^t_{\vec{u}})^*} g\right]$$

$$\underset{\text{vec}}{\mathcal{L}_{\vec{u}}} : \Gamma(TQ) \to \Gamma(TQ) \qquad \underset{\text{vec}}{\mathcal{L}_{\vec{u}}} \vec{w} = \frac{\partial}{\partial t}\Big|_{t=0} \left[\underset{\text{vec}}{(\phi^t_{\vec{u}})^*} \vec{w}\right]$$

and so on for any object which has pullback.

Frobenius integrability thm $\mathcal{A}$ is integrable iff $\mathcal{L}_{\vec{a}}\vec{b} \in \Gamma(\mathcal{A})$ $\forall \vec{a},\vec{b} \in \Gamma(\mathcal{A})$.

Now let's unpack what $\underset{\text{vec}}{\mathcal{L}_{\vec{u}}}$ is.

Thm $\underset{\text{fcn}}{\mathcal{L}_{\vec{u}}} g = \langle dg | \vec{u} \rangle = \vec{u} \triangleright g$.

(pf) $\underset{\text{fcn}}{\mathcal{L}_{\vec{u}}} g = \frac{\partial}{\partial t}\Big|_{t=0} (\underset{\text{fcn}}{\phi_{\vec{u}}^{t\,*}} g) = \frac{\partial}{\partial t}(g \circ \phi_{\vec{u}}^t) = \langle dg | \frac{\partial \phi_{\vec{u}}^t}{\partial t} \rangle$

$= \langle dg | \vec{u} \circ \phi_{\vec{u}}^t \rangle \big|_{t=0} = \langle dg | \vec{u} \rangle$.

Now recall $(\underset{\text{vec}}{\phi_*} \vec{v}) \triangleright g = \vec{v} \triangleright (\underset{\text{fcn}}{\phi^*} g)$. and $\underset{\text{vec}}{\phi^*} = (\underset{\text{vec}}{\phi_*})^{-1}$.

$\Rightarrow (\underset{\text{vec}}{\phi^*} \vec{w}) \triangleright (\underset{\text{fcn}}{\phi^*} g) = \underset{\text{fcn}}{\phi^*} (\vec{w} \triangleright g)$.

$\Rightarrow \frac{\partial}{\partial t}\Big|_{t=0} \left((\phi^*_{\text{vec}} \vec{w}) \triangleright (\underset{\text{fcn}}{\phi^*} g) \right) = \frac{\partial}{\partial t}\Big|_{t=0} \left(\underset{\text{fcn}}{\phi^*} (\vec{w} \triangleright g) \right)$

$\Rightarrow (\underset{\text{vec}}{\mathcal{L}_{\vec{u}}} \vec{w}) \triangleright g + \vec{w} \triangleright \underset{\text{fcn}}{\mathcal{L}_{\vec{u}}} g = \underset{\text{fcn}}{\mathcal{L}_{\vec{u}}} (\vec{w} \triangleright g)$

$\Rightarrow \underset{\text{vec}}{\mathcal{L}_{\vec{u}}} \vec{w} = \vec{u} \triangleright \vec{w} \triangleright g - \vec{w} \triangleright \vec{u} \triangleright g$.

Def For $\vec{u}, \vec{w} \in \Gamma(TQ)$ Let $[\vec{u}, \vec{w}] \in \Gamma(TQ)$ defined so that $[\vec{u}, \vec{w}] \triangleright g := \vec{u} \triangleright (\vec{w} \triangleright g) - \vec{w} \triangleright (\vec{u} \triangleright g)$.

Thm $\underset{\text{vec}}{\mathcal{L}_{\vec{u}}} \vec{w} = [\vec{u}, \vec{w}]$.

Calculate in coordinate. Let $x^1, \cdots, x^n : Q \to \mathbb{R}$ be coordinate.
$dx^1, \cdots, dx^n$ basis for T^*Q. $\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}$ be dual basis for TQ.

Write $\vec{u} = u^i \frac{\partial}{\partial x^i}$. $\vec{w} = w^j \frac{\partial}{\partial x^j}$.

$[\vec{u}, \vec{w}] = z^i \frac{\partial}{\partial x^i}$. Want to know $z^i = ?$

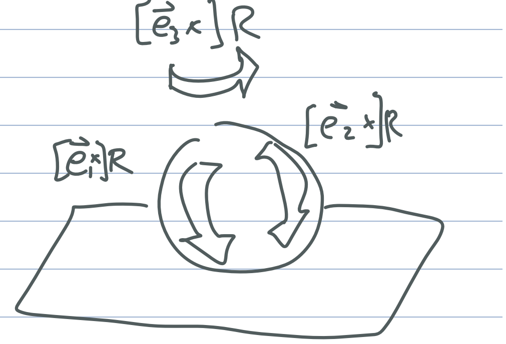
$$[\vec{u},\vec{w}]\triangleright g = \underbrace{\vec{u}\triangleright\vec{w}\triangleright g}_{u^i\partial_i(w^j\partial_j g) = u^i(\partial_i w^j)\partial_j g + \cancel{u^i w^j(\partial_i\partial_j g)}} - \vec{w}\triangleright\vec{u}\triangleright g = w^j(\partial_j u^i)\partial_i g + \cancel{u^i w^j(\partial_i\partial_j g)}$$

$$\Rightarrow [\vec{u},\vec{w}]\triangleright g = \underbrace{\left(u^j\frac{\partial w^i}{\partial x^j} - w^j\frac{\partial u^i}{\partial x^j}\right)}_{z^i}\frac{\partial g}{\partial x^i}$$

Or in short

$$[\vec{u},\vec{w}] = d_{\vec{u}}\vec{w} - d_{\vec{w}}\vec{u}$$ where $d_{(\text{direction})}(\text{vector})$ is the component-wise directional derivative.

<u>Example</u> <u>Rolling ball</u>



$Q = SO(3)$. $\quad T_R M = \{[\vec{\omega}\times]R \mid \vec{\omega}\in\mathbb{R}^3\}$

$\mathcal{A}_R := \mathrm{span}\{[\vec{e}_1\times]R,\ [\vec{e}_2\times]R\}$

Check $\mathcal{A}$ is not integrable (non-holonomic):

Check $\exists$ vector fields $X, Y \in \Gamma(\mathcal{A})$, but $[X,Y]\notin\Gamma(\mathcal{A})$.

Pick $X_R = [\vec{e}_1\times]R$. $\quad Y_R = [\vec{e}_2\times]R$.

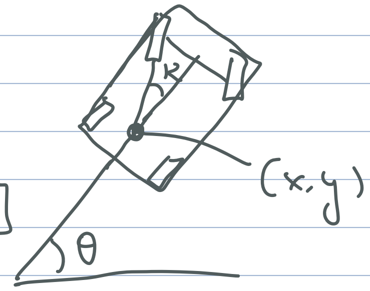
$$(d_X Y)_R = [\vec{e}_2\times]([\vec{e}_1\times]R)$$

$$(d_Y X)_R = [\vec{e}_1\times]([\vec{e}_2\times]R)$$

$$\Rightarrow [X,Y]_R = -[(\vec{e}_1\times\vec{e}_2)\times]R = -[\vec{e}_3\times]R \in \mathcal{A}_R.$$

We can generate the $[\vec{e}_3\times]R$ rotation by cycling X_R & Y_R!

Example Driving a car.



$$Q = \{(x,y,\theta,\kappa)\} = \mathbb{R}^2 \times \overset{\text{circle}}{S^1} \times [-a,a]$$

$$TQ_{(x,y,\theta,\kappa)} = \{(\dot{x},\dot{y},\dot{\theta},\dot{\kappa})\} = \mathbb{R}^4$$

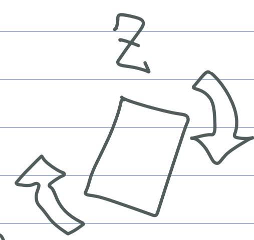
Allowed motion

$$\mathcal{A}_{(x,y,\theta,\kappa)} = \text{span}\left\{ \underbrace{\overset{=:X}{\begin{bmatrix}\cos\theta \\ \sin\theta \\ \kappa \\ 0\end{bmatrix}}}_{\text{move forward}}, \underbrace{\overset{=:Y}{\begin{bmatrix}0\\0\\0\\1\end{bmatrix}}}_{\text{steering}} \right\}$$

Check that $\mathcal{A}$ is non-holonomic.

$$[X,Y] = d_X Y - d_Y X = [\underset{=0}{dY}][X] - \underbrace{\begin{bmatrix}0 & 0 & -\sin\theta & 0 \\ 0 & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0\end{bmatrix}}_{dX}\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

$$= \begin{bmatrix}0\\0\\-1\\0\end{bmatrix} =: Z$$



Parallel parking

$$[X,Z] = -\begin{bmatrix}0 & 0 & -\sin\theta & 0 \\ 0 & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}0\\0\\-1\\0\end{bmatrix} = \begin{bmatrix}-\sin\theta \\ \cos\theta \\ 0 \\ 0\end{bmatrix}$$