Goal Derive & understand motion of a freely rotating 3D object.

- Derivation using least action principle.
- "Pseudo-velocity"
- Understand the Dzhanibekov effect
- Poinsot ellipsoid
- A Lie group integrator for rigid body rotation.

Recall If a Lagrangian takes the form $L(\vec{q},\dot{\vec{q}}) = \underbrace{K(\vec{q},\dot{\vec{q}})}_{\text{Kinetic energy}} - \underbrace{U(\vec{q})}_{\text{potential energy}}$

the EL-eq for minimizing $\int_0^T L(\vec{q},\dot{\vec{q}})\,dt$ is

$$\underbrace{\frac{d}{dt}\frac{\partial K}{\partial \dot{\vec{q}}}}_{\text{change of momentum}} = \underbrace{\frac{\partial K}{\partial \vec{q}}}_{\text{fictitious force}} - \underbrace{\frac{\partial U}{\partial \vec{q}}}_{\text{force from potential}}$$

- A pure inertial motion (free motion) is the case where $U=0$.
- In the following example (rigid body) the velocity variable is not exactly $\dot{\vec{q}}$ but some other proxy (angular velocity) generally called a pseudovelocity. $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$ doesn't hold; we need to rederive it using calculus of variation.

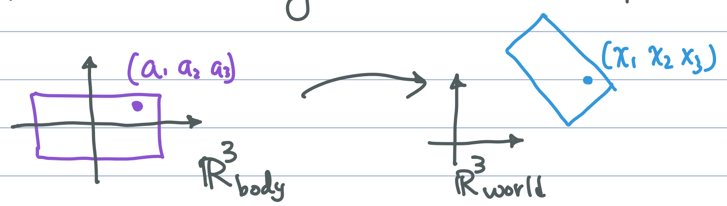
Rigid body kinematics

For a rigid body, the degree of freedom is the space of Euclidean transformations:

$$Q = SE(3) = \left\{ \left[\begin{array}{c|c} R_{3\times3} & \vec{c}_{3\times1} \\ \hline 0\ 0\ 0 & 1 \end{array}\right] \;\middle|\; \begin{array}{l} \left.\begin{array}{l} R^TR = id, \\ \det(R) = 1 \end{array}\right\} \text{R is a 3×3 rotation matrix} \\ \vec{c} \in \mathbb{R}^3 \;\} \text{c is the translation vector} \end{array} \right\}$$

$$= \underbrace{\left\{ R_{3\times3} \;\middle|\; \begin{array}{l} R^TR = id \\ \det(R) = 1 \end{array} \right\}}_{SO(3)} \underset{\text{semi direct product}}{\rtimes} \underbrace{\left\{ \vec{c} \;\middle|\; \vec{c} \in \mathbb{R}^3 \right\}}_{\mathbb{R}^3}$$

Each element $\left[\begin{array}{c|c} R & \vec{c} \\ \hline 0 & 1 \end{array}\right] \in SE(3)$ is a transformation that tells where every pt in the body coordinate is positioned in the world coordinate.



$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 1 \end{bmatrix} = \begin{bmatrix} R_{3\times3} & c \\ 0\ 0\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ 1 \end{bmatrix}$$

Fact: If we pick the origin of $\mathbb{R}^3_{body}$ at the center of mass, then the (total kinetic energy) Lagrangian $L(R, \dot{R}, \vec{c}, \dot{\vec{c}})$ decouples into

$$L = L_1(R, \dot{R}) + L_2(c, \dot{c}).$$

Rotation dynamics & translational dynamics don't interfere!

So we can just limit our attention to the dynamics of the 3×3 rotation matrix R.

$$Q = SO(3) = \{ R \in \mathbb{R}^{3\times 3} \mid \underbrace{R^T R = id,\ \det(R) = 1}_{\text{6D constraints on 9D space}} \}.$$

or $RR^T = id$ or $R^T = R^{-1}$.

Let's check what is $T_R Q$. Take variation on $R^T R = id$

$$\mathring{R}^T R + R^T \mathring{R} = 0$$

$$\Rightarrow (R^T \mathring{R})^T + R^T \mathring{R} = 0$$

$\Rightarrow$ $\boxed{\text{For every } \mathring{R} \in T_R Q,\ R^T \mathring{R} \text{ must be skew symmetric}}$

Similarly $RR^T = id \Rightarrow$ $\boxed{\mathring{R} R^T \text{ must be skew symmetric}}$

Lemma Every 3×3 skew symmetric matrix is a cross product with a 3D vector. (See the following construction)

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Def $$[\vec{a}\times] = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}.$$

Notation in Lie algebra theory: $[\vec{a}\times] = ad_{\vec{a}}$

Prop $[\vec{a}\times][\vec{b}\times] - [\vec{b}\times][\vec{a}\times] = [(\vec{a}\times\vec{b})\times]$.

Now we know how to parametrize $T_R Q$:

Every velocity in rotation $\dot{R}$ must take the form:

$$\dot{R} = R[\vec{\Omega}\times] = [\vec{\omega}\times]R$$

$\vec{\Omega} \in \mathbb{R}^3_{body}$ body angular velocity

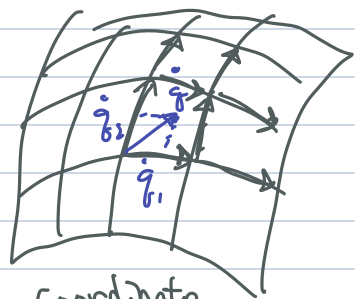
$\vec{\omega} \in \mathbb{R}^3_{world}$ world angular velocity.

- $\boxed{\vec{\omega} = R\vec{\Omega}}$ (Check!)
- If $\vec{\omega}$ is time independent then $R(\Delta t) = \underbrace{e^{\Delta t[\vec{\omega}\times]}}_{\text{rotation}} R(0)$

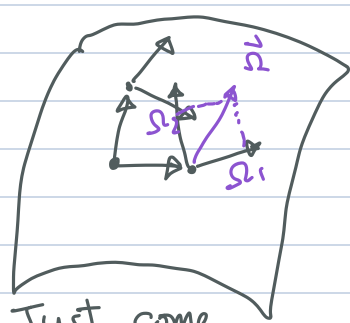
  $\vec{\Omega}$ 〃 〃 〃 $\therefore R(\Delta t) = R(0)\, e^{\Delta t[\vec{\Omega}\times]}$

So, indeed they generate axis-angle rotation in the body/world coordinates.

<u>Note</u> that $\vec{\Omega}, \vec{\omega} \in \mathbb{R}^3$ are just proxies for $\dot{R}$. Their components are not the rate of change of any generalized coordinate.



Coordinate basis vector



Just some basis for each tangent space.

<u>Kinetic energy</u>:

$$K(R,\dot{R}) = \int_{\text{Body}} \frac{1}{2} |\dot{R}\vec{r}_a|^2 \, dm_a$$

$$= \int \frac{\vec{r}_a^{\,T}}{2} [\vec{\Omega}\times]^T \underbrace{R^T R}_{id} [\vec{\Omega}\times] \vec{r}_a \, dm_a$$

= Some quadratic form on $\Omega_1, \Omega_2, \Omega_3$.

$$= \frac{1}{2} \underset{1\times 3}{\vec{\Omega}^T} [I_{\text{body}}] \underset{3\times 1}{\vec{\Omega}}$$

$\uparrow$ symmetric positive definite 3×3 matrix.

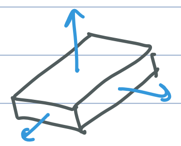
- I_{body} is called <u>moment of inertia</u> in body coord.
- $\boxed{I_{\text{body}} \text{ is time-independent!}}$

By $\vec{\omega} = R\vec{\Omega}$ we can also write

$$K(R,\dot{R}) = \frac{1}{2}\vec{\omega}^T \underbrace{R\, I_{body}\, R^T}_{=:\, I_{world}} \vec{\omega}$$

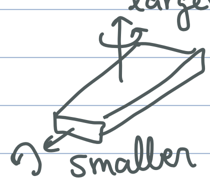
($I_{world}(t)$ is time dependent. It co-rotates with the body.)

- Eigenvectors of I_{body} are <u>principal axes of rotation</u>

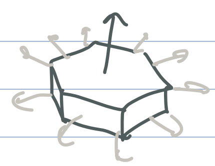


(They are orthogonal to each other)

Eigenvalues are the kinetic energies when rotating about that pricipal axes.



When two eigenvalues coincide, we have spinning top:



Derive equation of motion for $S(R(\cdot)) = \int_0^T K(R,\dot{R})\,dt$

$$= \int_0^T \frac{1}{2}\vec{\Omega}^T(t)\, I_{body}\, \vec{\Omega}(t)\,dt.$$

Perturb $R(t)$ in $\mathring{R}(t)$ direction. $\mathring{R} = R[\vec{V}\times] = [\vec{v}\times]R$

How does $\vec{\Omega}$ (and $\vec{\omega}$) get perturbed?

$$\dot{R} = R[\vec{\Omega}\times] \qquad \mathring{\dot{R}} = \mathring{R}[\vec{\Omega}\times] + R[\mathring{\vec{\Omega}}\times]$$

$$\mathring{R} = R[\vec{V}\times] \quad \Rightarrow \quad = R[\vec{V}\times][\vec{\Omega}\times] + R[\mathring{\vec{\Omega}}\times]$$

$$\dot{\mathring{R}} = \dot{R}[\vec{V}\times] + R[\dot{\vec{V}}\times]$$

$$= R[\vec{\Omega}\times][\vec{V}\times] + R[\dot{\vec{V}}\times]$$

$$\Rightarrow [\mathring{\vec{\Omega}}\times] = [\dot{\vec{V}}\times] + \underbrace{[\vec{\Omega}\times][\vec{V}\times] - [\vec{V}\times][\vec{\Omega}\times]}_{[(\vec{\Omega}\times\vec{V})\times]}$$

$$\Rightarrow \boxed{\mathring{\vec{\Omega}} = \dot{\vec{V}} + \vec{\Omega}\times\vec{V}}$$

Now $$\mathring{S} = \int_0^T \vec{\Omega}^T I_{body}\, \mathring{\vec{\Omega}}\, dt$$

$$= \int_0^T \vec{\Omega}^T I_{body}(\dot{\vec{V}} + \vec{\Omega}\times\vec{V})\, dt$$

$$= \int_0^T \left(-\dot{\vec{\Omega}}^T I_{body}\vec{V} + \vec{\Omega}^T I_{body}[\vec{\Omega}\times]\vec{V}\right) dt$$

$=0$ for all $\vec{V}:[0,T]\to\mathbb{R}^3$ (optimality condition)

$$\Rightarrow -\dot{\vec{\Omega}}^T I_{body} + \vec{\Omega}^T I_{body}[\vec{\Omega}\times] = 0$$

Transpose $\Rightarrow$ $(\vec{\Omega}\times)^T = -(\vec{\Omega}\times)$

$$\boxed{I_{body}\dot{\vec{\Omega}} + \vec{\Omega}\times(I_{body}\vec{\Omega}) = 0}$$

EL eq: in body coord
$$\begin{cases} \dot{R} = R[\vec{\Omega}\times] \\ I_{body}\dot{\vec{\Omega}} + \vec{\Omega}\times(I_{body}\vec{\Omega}) = 0 \end{cases}$$

Call $\vec{L} = I_{body}\vec{\Omega}$ the body angular momentum.

$$\boxed{\dot{\vec{L}} + (I_{body}^{-1}\vec{L})\times\vec{L} = 0}$$

Let's try world coord derivation:

$$K(R,\dot{R}) = \int_0^T \frac{1}{2}\vec{\omega}^T \underbrace{I_{world}}_{R\text{-depend}}\vec{\omega}\, dt = \int_0^T \frac{1}{2}\vec{\omega}^T R I_{body} R^T\vec{\omega}\, dt$$

Under $\mathring{R} = [\vec{v}\times]R$ we have $\mathring{\vec{\omega}} = \dot{\vec{v}} + \vec{v}\times\vec{\omega}$

$$\mathring{S} = \int_0^T (\vec{\omega}^T R I_{body})(R^T\vec{\omega})^\circ\, dt$$

$$= \int_0^T (\vec{\omega}^T R I_{body})(\mathring{R}^T\vec{\omega} + R^T\mathring{\vec{\omega}})\, dt$$

$$= \int_0^T (\vec{\omega}^T R I_{body})\left(R^T[\vec{v}\times]^T\vec{\omega} + R^T(\dot{\vec{v}} + \vec{v}\times\vec{\omega})\right) dt$$

cancel

$$= \int_0^T \omega^T R I_{body} R^T\dot{\vec{v}} = \int_0^T -(\omega^T R I_{body} R^T)^\cdot\, \vec{v}\, dt = 0 \quad \forall \vec{v}:[0,T]\to\mathbb{R}^3$$

$$\Rightarrow \frac{d}{dt}\left(R\, I_{body}\, R^T \vec{\omega}\right) = 0$$

or simply $\boxed{\frac{d}{dt}\left(I_{world}\,\vec{\omega}\right) = 0}$

We call $\boxed{\vec{l} = I_{world}\vec{\omega}}$ the (world) angular momentum.

Check $\boxed{\vec{l} = R\vec{L}}$

Expand $\underbrace{\dot{R}}_{[\omega\times]R} I_{body} R^T\vec{\omega} + R I_{body} \underbrace{\dot{R}^T}_{R^T(\omega\times)^T} \vec{\omega} + R I_{body} R^T \dot{\vec{\omega}} = 0$

$\vec{\omega}\times\vec{\omega} = 0$

$$\Rightarrow \boxed{I_{world}\dot{\vec{\omega}} + \vec{\omega}\times\left(I_{world}\vec{\omega}\right) = 0}$$

EL eq in world

$$\begin{cases} \dot{R} = [\vec{\omega}\times]R \\ I_{world}\dot{\vec{\omega}} + \vec{\omega}\times\left(I_{world}\vec{\omega}\right) = 0 \quad \text{or simply} \quad \frac{d}{dt}\big(\overbrace{I_{world}\vec{\omega}}^{\vec{l}}\big) = 0 \\ \text{where } I_{world} = R I_{body} R^T \end{cases}$$

Lie group integrator: Instead of forward/backward Euler, RK4 etc that numerically solves ODE with linear combination, we use matrix multiplication

(rotation matrix is closed under multiplication, not addition)

Simple 1-st order integrator

Given $R^{(0)}, \vec{\omega}^{(0)}$ compute $\vec{l} = R^{(0)} I_{body}\, \vec{\omega}^{(0)}$.

Loop

- $I_{world}^{(n)} = R^{(n)} I_{body} R^{(n)T}$
- $\vec{\omega}^{(n)} = I_{world}^{(n)\,-1}\,\vec{l}$
- $R^{(n+1)} = \text{Rotate}\Big(\underbrace{\Delta t|\omega^{(n)}|}_{\text{angle}}, \underbrace{\frac{\omega^{(n)}}{|\omega|}}_{\text{axis}}\Big)\, R^{(n)}$

2nd order Buss' method (Samual Buss 2001)

- $I_{world}^{(n)} = R^{(n)} I_{body} R^{(n)T}$
- $\vec{\omega}^{(n)} = I_{world}^{(n)\,-1} \vec{l}$
- $\vec{\alpha} = - I_{world}^{(n)} (\vec{\omega}^{(n)} \times \vec{l})$
- $\vec{\omega}^{(n+\frac{1}{2})} = \vec{\omega}^{(n)} + \frac{\Delta t}{2} \vec{\alpha} + \frac{(\Delta t)^2}{12} (\vec{\alpha} \times \vec{\omega})$
- $R^{(n+1)} = \mathrm{Rotate}\left(\Delta t |\vec{\omega}^{(n+\frac{1}{2})}|, \frac{\vec{\omega}^{(n+\frac{1}{2})}}{|\vec{\omega}^{(n+\frac{1}{2})}|}\right) R^{(n)}$

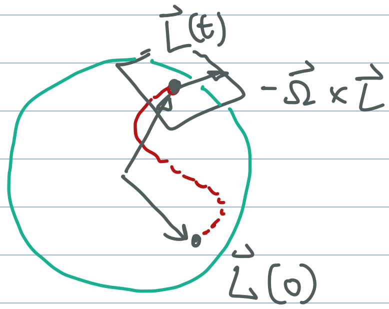
Poinsot ellipsoid

Recall the body-coord. angular momentum $\vec{L} = I_{body} \vec{\Omega} = R^{-1} \vec{l}$ satisfies

$$\dot{\vec{L}} + \vec{\Omega} \times \vec{L} = 0$$

This implies that $\vec{L}(t)$ lies on a sphere (since $\vec{\Omega} \times \vec{L} \perp \vec{L}$)

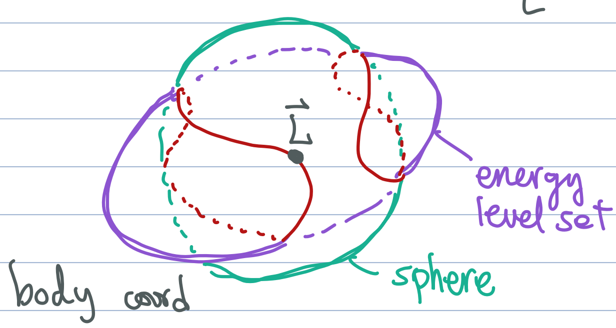
(The sphere is called a coadjoint orbit)



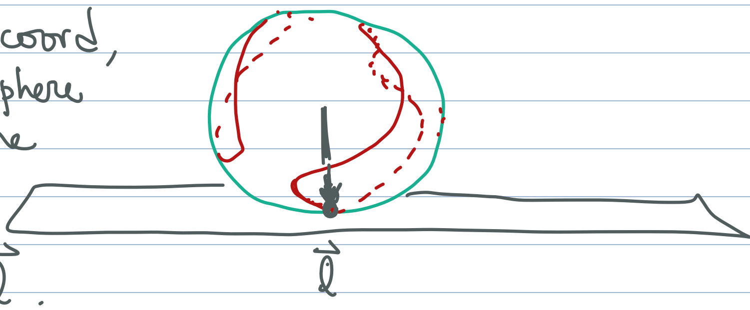
The trajectory lies on a levelset of energy

$$K = \frac{1}{2} \vec{\Omega}^T I_{body} \vec{\Omega} = \frac{1}{2} \vec{L}^T I_{body}^{-1} \vec{L}.$$

Level sets of $\frac{1}{2} \vec{L}^T \begin{bmatrix} \frac{1}{\lambda_1} & & \\ & \frac{1}{\lambda_2} & \\ & & \frac{1}{\lambda_3} \end{bmatrix} \vec{L}$ are ellipsoids with axes $\sqrt{\lambda_1}, \sqrt{\lambda_2}, \sqrt{\lambda_3}$.



In world coord, the corotated sphere rolls along the red track to maintain a constant $\vec{\ell}$.



The energy level set on the (coadjoint orbit) sphere looks like