In calculus, we take derivatives of functions of a few variables

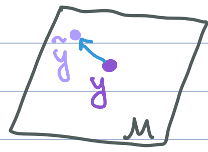
In calculus of variations, we take derivatives of $\underbrace{\text{functions of functions}}_{\text{functionals}}$

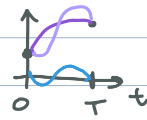
- Main purpose: Formulate optimization problem over function spaces and derive the optimality condition (e.g. the KKT condition)
- These KKT conditions are often ODEs or PDEs (called Euler–Lagrange eqs)
- Most physical eqs arise as the Euler-Lagrange eq
- Physical modeling can be reduced to designing one functional.

Example 1

Let $\mathcal{M} = \{ y : [0,T] \to \mathbb{R} \mid y(0) = y_0,\ y(T) = y_T \}$

$T_y\mathcal{M} = \{ \mathring{y} : [0,T] \to \mathbb{R} \mid \mathring{y}(0) = \mathring{y}(T) = 0 \}$





Consider functional $E : \mathcal{M} \to \mathbb{R}$

$$E(y) := \int_0^T \left(\frac{1}{2} y'(t)^2 + \cos(y(t)) \right) dt$$

Derive the optimality condition $dE|_y[\![\mathring{y}]\!] = 0 \quad \forall \mathring{y} \in T_y\mathcal{M}$.

$$dE|_y[\![\mathring{y}]\!] \underset{\text{def of directional derivative}}{=} \frac{d}{d\varepsilon}\Big|_{=0} E(y + \varepsilon\mathring{y}) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int_{t=0}^{T} \left[\frac{1}{2}\left(y'(t) + \varepsilon \mathring{y}'(t)\right)^2 + \cos\left(y(t) + \varepsilon\mathring{y}(t)\right) \right] dt$$

$$= \int_{t=0}^{T} \left[\left(y'(t) + \varepsilon\mathring{y}'(t)\right) \mathring{y}'(t) - \sin\left[y(t) + \varepsilon\mathring{y}(t)\right] \mathring{y}(t) \right]_{\varepsilon=0} dt$$

$$= \int_0^T y'(t)\,\mathring{y}'(t) - \sin(y(t))\,\mathring{y}(t)\, dt$$

$$\underset{\text{integration by parts}}{=} y'(t) \underbrace{\mathring{y}(t)}_{=0} \Big|_{t=0}^{T} - \int_0^T y''(t)\,\mathring{y}(t)\, dt - \int_0^T \sin(y(t))\,\mathring{y}(t)\, dt$$

$$= \int_0^T \underbrace{-\left(y''(t) + \sin(y(t))\right)}_{\text{We call this } \frac{\delta E}{\delta y}(y) \text{ (functional differential)}} \mathring{y}(t)\, dt$$

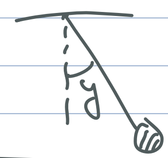
A faster derivation:

$$(E(y))^\circ = \left[\int_0^T \left(\tfrac{1}{2} y'^2 + \cos(y)\right) dt\right]^\circ = \int_0^T \left(y' \mathring{y}' - \sin(y)\, \mathring{y}\right) dt$$

$$= \int_0^T -y'' \mathring{y} - \sin(y)\, \mathring{y} = \int_0^T \frac{\delta E}{\delta y}\, \mathring{y}\, dt$$

$$\Rightarrow \frac{\delta E}{\delta y} = y'' + \sin(y).$$

Optimality condition: $y'' + \sin y = 0$ (pendulum eq.)



Common setup: $\mathcal{M} = \{ \vec{y} : [a,b] \to \mathbb{R}^m \mid \vec{y}(a) = \vec{y}_a,\ \vec{y}(b) = \vec{y}_b \}$

Consider functionals taking the form

$$S(y) = \int_a^b L(t, \vec{y}(t), \vec{y}'(t))\, dt$$

where $L : \underbrace{\mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^m}_{\text{denote coord by } (t, y^1, \dots, y^m, v^1, \dots, v^m)} \to \mathbb{R}$ is a given function called Lagrangian.

Under variation $\vec{y} + \varepsilon \mathring{\vec{y}}$, $S(y)$ varies by

$$\mathring{S} = \left[\int_a^b L(t, \vec{y}(t), \vec{y}'(t))\, dt\right]^\circ = \int_a^b \left(\frac{\partial L}{\partial y^i} \mathring{y}^i + \frac{\partial L}{\partial v^i} \mathring{y}^{i\prime} \right) dt$$

$$= \int_a^b \frac{\partial L}{\partial y^i} \mathring{y}^i - \frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\right) \mathring{y}^i\, dt$$

$$\Rightarrow \left(\frac{\delta S}{\delta \vec{y}}\right)^i = \frac{\partial L}{\partial y^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\right)\Bigg|_{\vec{v} = \vec{y}'}.$$

EL-eq:

$$\begin{cases} \dfrac{\partial L}{\partial y^1} - \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial v^1}\right) = 0 \\ \quad \vdots \\ \dfrac{\partial L}{\partial y^m} - \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial v^m}\right) = 0. \end{cases}$$

If $L(t, \vec{y}(t), \vec{v}(t)) = \underbrace{\boxed{\frac{1}{2} \vec{v}'(t)^T M \vec{v}(t)}}_{\text{Kinetic energy}} - \underbrace{\boxed{U(\vec{y}(t))}}_{\text{potential energy}}$ $M \in \mathbb{R}^{m\times m}$ inertia matrix

$U: \mathbb{R}^m \to \mathbb{R}$ potential energy

Then the EL eqs: $\left(\frac{\partial L}{\partial \vec{v}} = M\vec{v}(t), \quad \frac{\partial L}{\partial \vec{y}} = -\frac{\partial U}{\partial \vec{y}} = -dU \right)$

$$-dU - \frac{d}{dt}(M\vec{v}(t)) = 0 \Big|_{\vec{v} = \vec{y}'}$$

$$\Rightarrow \underbrace{\frac{d}{dt}\underbrace{(M\vec{y}'(t))}_{\text{momentum}}}_{\text{ch of momentum}} = \underbrace{-dU(\vec{y}(t))}_{\text{force}} \quad \text{— Newton's Law}$$

S is called the $\boxed{\text{action}}$

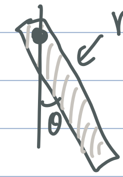
This derivation is called Hamilton's $\boxed{\text{Least Action Principle}}$

- Why calculus of var's instead of writing $F = ma$?

Main reason: Reduced order modeling

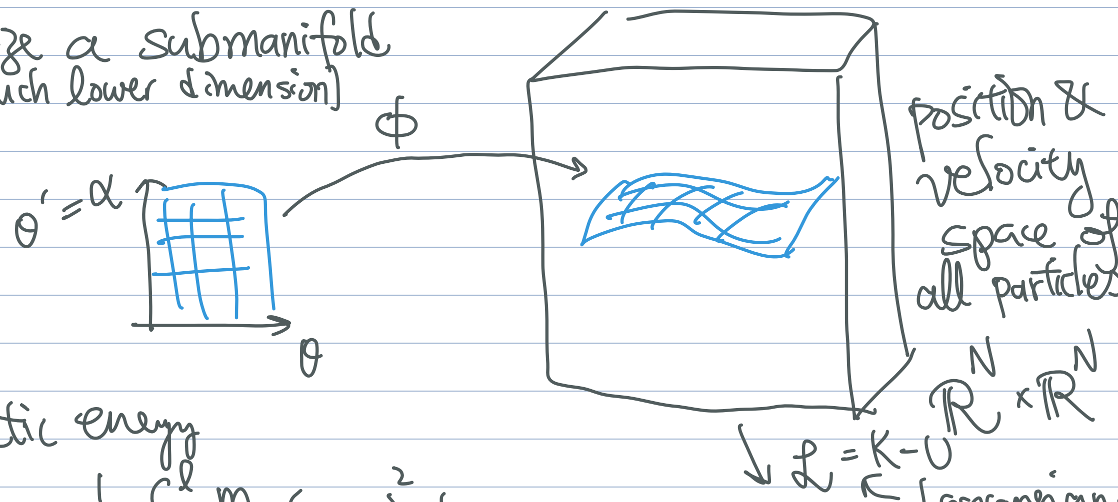
Example 2

Suppose we want to model a pendulum like



Applying Newton's law we need to model position & acceleration of every particle, and their interaction forces...

Let us parametrize a submanifold by the angle (much lower dimension)



Lagrangian = Total Kinetic energy − Potential energy.

Pullback Kinetic energy

$$K(\theta, \alpha) = \frac{1}{2}\int_0^l \frac{m}{l}(r\alpha)^2 \, dr = \frac{ml^2}{6}\alpha^2$$

$$U(\theta, \alpha) = \int_0^l -\frac{m}{l} g r \cos\theta \, dr = -\frac{mgl}{2}\cos\theta$$

$$L(\theta, \theta') = \frac{ml^2\theta'^2}{6} + \frac{mgl}{2}\cos\theta \Rightarrow \text{EL eq}: \quad \theta'' = -\frac{3g}{2l}\sin\theta$$

Noether's Thm of time independence

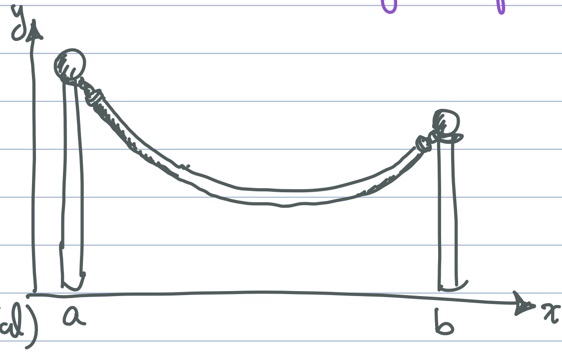
If $L(t, y, y') = L(y, y')$ no dependence on time in its expression.

Then EL $\quad \frac{\partial L}{\partial y^i} - \frac{d}{dt}\frac{\partial L}{\partial y^{i\prime}} = 0 \quad$ will imply

$$\sum_i \frac{\partial L}{\partial y^{i\prime}} y^{i\prime} - L = \text{const indep of time.}$$

(pf) $$\frac{d}{dt}\left(\frac{\partial L}{\partial y^{i\prime}} y^{i\prime} - L\right) = \underbrace{\frac{d}{dt}\left(\frac{\partial L}{\partial y^{i\prime}}\right) y^{i\prime} + \cancel{\frac{\partial L}{\partial y^{i\prime}} y^{i\prime\prime}} - \frac{\partial L}{\partial y^i} y^{i\prime}}_{=0 \text{ by EL eq}} - \cancel{\frac{\partial L}{\partial y^{i\prime}} y^{i\prime\prime}} = 0 \qquad \square$$

Example 3 Hanging Rope



$$E(y) = \int_a^b \rho g\, y \sqrt{1+y'^2}\, dx$$

(total gravitational potential)

$$G(y) = \int_a^b \sqrt{1+y'^2}\, dx \quad \text{(total length)}$$

minimise $E(y)$ s.t. $G(y) - l = 0$

$y : [a,b] \to \mathbb{R}$

$y(a) = y_a$

$y(b) = y_b$

KKT condition: $\frac{\delta E}{\delta y} + \lambda \frac{\delta G}{\delta y} = 0 \iff \tilde{E}(y) = \int_a^b L(y, y')\, dx$

where $L = (\rho g y + \lambda)\sqrt{1+y'^2}$.

L is x-independent. Therefore by Noether's Thm of time independence:

$$\frac{\partial L}{\partial y'} y' - L = c \text{ (const)}$$

$$\Rightarrow (\rho g y + \lambda)\left(\frac{y'^2}{\sqrt{1+y'^2}} - \sqrt{1+y'^2}\right) = c$$

$$\Rightarrow y' = \pm \frac{1}{c}\sqrt{(\rho g y + \lambda)^2 - c^2} \Rightarrow \frac{dy}{\sqrt{(\rho g y + \lambda)^2 - c^2}} = \pm \frac{dx}{c}$$

$$\frac{1}{\rho g}\cosh^{-1}\left(\frac{\rho g y + \lambda}{c}\right) = \pm\frac{x}{c} + C_1$$

$$\Rightarrow y = -\frac{\lambda}{\rho g} + \frac{c}{\rho g}\cosh\left(\frac{\rho g x}{c} + C_2\right)$$

λ, c, C_2 are constant.

In classical mechanics

$$L(\underbrace{\vec{q}}_{\text{(generalized) position}}, \dot{\vec{q}}) = \overbrace{K(\vec{q}, \dot{\vec{q}})}^{\text{Kinetic energy}} - \underbrace{U(\vec{q})}_{\text{potential energy.}}$$

$$\vec{p} := \frac{\partial L}{\partial \dot{\vec{q}}} = \frac{\partial K}{\partial \dot{\vec{q}}}$$ is called (generalized) momentum.

EL eq reads
$$\begin{cases} \frac{d}{dt}\vec{p} = -\partial \vec{U}(\vec{q}) - \frac{\partial K}{\partial q} \\ \vec{p} = \frac{\partial K}{\partial \dot{\vec{q}}}(\vec{q}, \dot{\vec{q}}). \end{cases}$$

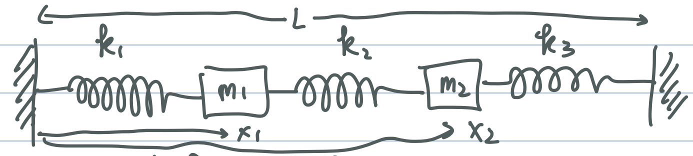
Noether:
$$\underbrace{\frac{\partial L}{\partial \dot{q}}\dot{q} - L}_{} = \text{const}$$

$$\frac{\partial K}{\partial \dot{q}}\dot{q} - K(q, \dot{q}) + U$$

If K is quadratic in $\dot{q}$ then
$$\frac{\partial K}{\partial \dot{q}}\dot{q} - K(q, \dot{q})$$
$$= K(q, \dot{q})$$

const $= K + U$ which is total energy.

Example 4



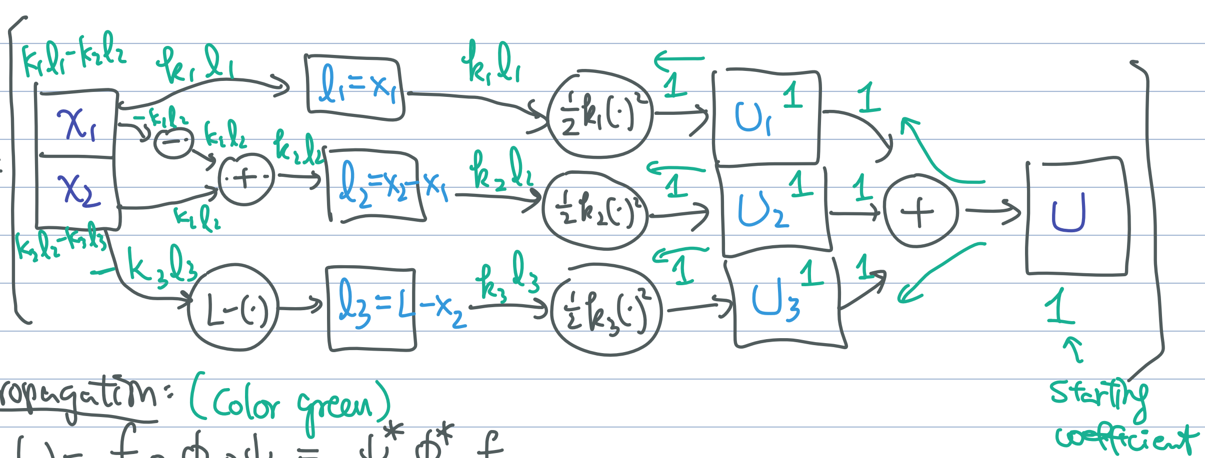
rest lengths of springs $= 0$.

$$K\left(\begin{pmatrix}x_1\\x_2\end{pmatrix},\begin{pmatrix}\dot{x}_1\\\dot{x}_2\end{pmatrix}\right)=\frac{1}{2}m_1\dot{x}_1^2+\frac{1}{2}m_2\dot{x}_2^2$$

$$U\left(\begin{pmatrix}x_1\\x_2\end{pmatrix}\right)=\frac{1}{2}k_1x_1^2+\frac{1}{2}k_2(x_2-x_1)^2+\frac{1}{2}k_3(L-x_2)^2.$$

$$p_1=\frac{\partial K}{\partial\dot{x}_1}=m_1\dot{x}_1\qquad p_2=\frac{\partial K}{\partial\dot{x}_2}=m_2\dot{x}_2$$

U is a composition of several operations:



Backpropagation: (Color green)

If $U=f\circ\phi\circ\psi=\underset{\text{fcn}}{\psi^*}\,\underset{\text{fcn}}{\phi^*}f$

$$dU=\underset{\text{covec}}{\psi^*}\,\underset{\text{covec}}{\phi^*}\,df=\underset{\text{covec}}{\psi^*}\,\underset{\text{covec}}{\phi^*}\,\underset{\text{covec}}{f^*}\,\underbrace{dz}_{}$$

unit covector on the final real line

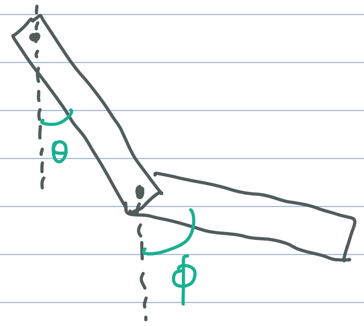
dz has coefficient 1. Pullback coefficient is by transpose of Jacobian.

$$\Rightarrow\quad\frac{\partial U}{\partial x_1}=k_1l_1-k_2l_2=k_1x_1-k_2(x_2-x_1)$$

$$\frac{\partial U}{\partial x_2}=k_2(x_2-x_1)-k_3(L-x_2)$$

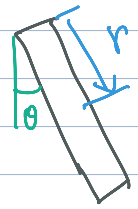
$$\Rightarrow \text{EL eq:}\quad\begin{cases}m_1\ddot{x}_1=-k_1x_1+k_2(x_2-x_1)\\ m_2\ddot{x}_2=k_2(x_2-x_1)+k_3(L-x_2).\end{cases}$$

Example 5 Double pendulum



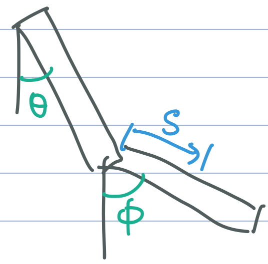
m : mass of rod (uniform density)
l : length of rod

World space Position of first rod



$$\begin{bmatrix} x_r \\ y_r \end{bmatrix} = \begin{bmatrix} r\sin\theta \\ -r\cos\theta \end{bmatrix}$$

" second rod



$$\begin{bmatrix} x_{2s} \\ y_{2s} \end{bmatrix} = \begin{bmatrix} l\sin\theta \\ -l\cos\theta \end{bmatrix} + \begin{bmatrix} s\sin\phi \\ -s\cos\phi \end{bmatrix}$$

World space velocity

$$\begin{bmatrix} \dot{x}_{1r} \\ \dot{y}_{1r} \end{bmatrix} = \begin{bmatrix} r\cos(\theta)\,\dot{\theta} \\ r\sin(\theta)\,\dot{\theta} \end{bmatrix}, \quad \begin{bmatrix} \dot{x}_{2s} \\ \dot{y}_{2s} \end{bmatrix} = \begin{bmatrix} l\cos(\theta)\,\dot{\theta} + s\cos(\phi)\,\dot{\phi} \\ l\sin(\theta)\,\dot{\theta} + s\sin(\phi)\,\dot{\phi} \end{bmatrix}$$

Kinetic energy $K(\theta,\phi,\dot{\theta},\dot{\phi}) = \int_0^l \frac{1}{2}\frac{m}{l}(\dot{x}_{1r} + \dot{y}_{1r})^2 dr + \int_0^l \frac{1}{2}\frac{m}{l}(\dot{x}_{2s} + \dot{y}_{2s})^2 ds$

$$U(\theta,\phi) = \int_0^l \frac{mg}{l} y_{1r}\, dr + \int_0^l \frac{mg}{l} y_{2s}\, ds$$

$$K = \frac{ml^2}{6}\dot{\theta}^2 + \frac{ml^2}{2}\dot{\theta}^2 + \frac{ml^2}{2}\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + \frac{ml^2}{6}\dot{\phi}^2$$

$$= \boxed{ml^2\left(\frac{2}{3}\dot{\theta}^2 + \frac{1}{2}\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + \frac{1}{6}\dot{\phi}^2\right)}$$

$$U = -\frac{mgl}{2}\cos\theta - mgl\cos\theta - \frac{mgl}{2}\cos\phi$$

$$\boxed{= -mgl\left(\frac{3}{2}\cos\theta + \frac{1}{2}\cos\phi\right)}$$

Lagrangian

$$\boxed{L = K - U = \frac{ml^2}{2}\begin{bmatrix}\dot\theta & \dot\phi\end{bmatrix}\begin{bmatrix}\frac{4}{3} & \frac{1}{2}\cos(\theta-\phi)\\ \frac{1}{2}\cos(\theta-\phi) & \frac{1}{3}\end{bmatrix}\begin{bmatrix}\dot\theta\\ \dot\phi\end{bmatrix} + mgl\left(\frac{3}{2}\cos\theta + \frac{1}{2}\cos\phi\right)}$$

$$\frac{\partial L}{\partial\begin{pmatrix}\dot\theta\\ \dot\phi\end{pmatrix}} = ml^2\begin{bmatrix}\frac{4}{3} & \frac{1}{2}\cos(\theta-\phi)\\ \frac{1}{2}\cos(\theta-\phi) & \frac{1}{3}\end{bmatrix}\begin{bmatrix}\dot\theta\\ \dot\phi\end{bmatrix} =: \underbrace{ml^2}_{\text{optional choice}}\begin{bmatrix}p_1\\ p_2\end{bmatrix} \quad \text{—— } (1)$$

$$\frac{\partial L}{\partial\begin{pmatrix}\theta\\ \phi\end{pmatrix}} = \begin{bmatrix}-\frac{ml^2}{2}\sin(\theta-\phi)\,\dot\theta\,\dot\phi - \frac{3mgl}{2}\sin\theta\\ \frac{ml^2}{2}\sin(\theta-\phi)\,\dot\theta\,\dot\phi - \frac{mgl}{2}\sin\phi\end{bmatrix} \quad \text{—— } (2)$$

EL-eq

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\begin{pmatrix}\dot\theta\\ \dot\phi\end{pmatrix}}\right) = \frac{\partial L}{\partial\begin{pmatrix}\theta\\ \phi\end{pmatrix}} \Rightarrow \frac{d}{dt}ml^2\begin{bmatrix}p_1\\ p_2\end{bmatrix} = \begin{bmatrix}(2)\end{bmatrix}$$

express $\dot\theta\ \dot\phi$ also by p_1, p_2

(1) also implies $$\begin{bmatrix}\dot\theta\\ \dot\phi\end{bmatrix} = \begin{bmatrix}\frac{4}{3} & \frac{1}{2}\cos(\theta-\phi)\\ \frac{1}{2}\cos(\theta-\phi) & \frac{1}{3}\end{bmatrix}^{-1}\begin{bmatrix}p_1\\ p_2\end{bmatrix}$$

Final ODE:

$$\frac{d}{dt}\begin{pmatrix}\theta\\ \phi\\ p_1\\ p_2\end{pmatrix} = \begin{pmatrix}\begin{pmatrix}\frac{4}{3} & \frac{1}{2}\cos(\theta-\phi)\\ \frac{1}{2}\cos(\theta-\phi) & \frac{1}{3}\end{pmatrix}^{-1}\begin{pmatrix}p_1\\ p_2\end{pmatrix}\\ \hdashline \cdots\end{pmatrix}$$