- Consider a function $f: M \to \mathbb{R}$ defined on an n-dim domain $M \subseteq \mathbb{R}^n$. Suppose M has a coordinate system $\theta_1, \cdots, \theta_n$.
- We can regard f as an expression $f(\theta_1, \cdots, \theta_n)$ of $\theta_1, \cdots, \theta_n$.
- What is the gradient of f?
  - Most would tell you $\nabla f = \begin{bmatrix} \partial f/\partial\theta_1 \\ \vdots \\ \partial f/\partial\theta_n \end{bmatrix}$

    and say that this is the vector of steepest ascend.
  - This is not true in general! (e.g. when coord lines are not orthogonal)
- What's true is

$$df = \frac{\partial f}{\partial\theta_1} d\theta_1 + \cdots + \frac{\partial f}{\partial\theta_n} d\theta_n$$

Let's understand what are these "d of functions."

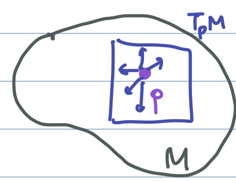
For each vector space $V = \{\vec{v} \in V\}$ (vectors) we can consider the dual vector space $V^* := \{\alpha : V \xrightarrow{\text{linear}} \mathbb{R}\}$. (covectors or dual vectors)

- We can plug in $\alpha[\![\vec{v}]\!] \in \mathbb{R}$. We call it dual pairing, sometimes denoted by $\langle \alpha | \vec{v} \rangle$.
- Can check that V^* is also a vector space (linear combinations of covectors are covectors) and $\dim(V^*) = \dim(V)$. $V^{**} = V$.

$\boxed{\text{Differentials of functions are covectors}}$

For each point $p \in M$, let $T_pM := \{\vec{v} \text{ based at } p\}$

"T_pM: vector space of velocities at which p can move."



$T_p^*M :=$ dual space of T_pM

Let $f: M \to \mathbb{R}$ be a scalar field.

Define $df_p \in T_p^*M$ as $df_p[\![\vec{v}]\!] := \lim_{\varepsilon \to 0} \frac{f(p + \varepsilon\vec{v}) - f(p)}{\varepsilon}$

directional derivative.

Basis Let $\vec{e}_1, \dots, \vec{e}_n$ be a basis for V. Then any vector $\vec{v} \in V$ can be written as:

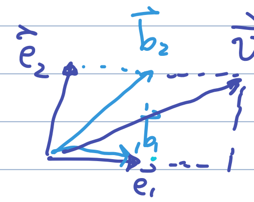
$$\vec{v} = v^1 \vec{e}_1 + v^2 \vec{e}_2 + \dots + v^n \vec{e}_n$$

$$= \sum_i v^i \vec{e}_i = v^i \vec{e}_i \quad \text{(Einstein summation convention)}$$

$$= \underbrace{[\vec{e}_1 \ \cdots \ \vec{e}_n]}_{\text{basis}} \underbrace{\begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix}}_{\text{coefficients}}$$

Change of basis? Let $\vec{b}_1 = \vec{e}_1$

$\vec{b}_2 = \vec{e}_1 + \vec{e}_2$



$$\vec{v} = [\vec{e}_1 \ \vec{e}_2] \begin{bmatrix} 2 \\ 1 \end{bmatrix} = [\vec{b}_1 \ \vec{b}_2] \begin{bmatrix} ? \\ ? \end{bmatrix}$$

Express change of basis $[\vec{b}_1 \ \vec{b}_2] = [\vec{e}_1 \ \vec{e}_2] \overbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}^{A}$

$$\begin{bmatrix} ? \\ ? \end{bmatrix} = A^{-1} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$$\boxed{\text{The components of vector transform by the inverse of the basis transf.}}$$

Dual basis thm Let $\vec{b}_1, \dots, \vec{b}_n$ be a basis for V.

Then there is a unique basis $\beta^1, \dots, \beta^n$ for V^* such that

$$\beta^i [\![\vec{b}_j]\!] = \delta^i_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$$

Example

$$[\vec{b}_1 \ \vec{b}_2] = [\vec{e}_1 \ \vec{e}_2] \begin{bmatrix} \boxed{\begin{matrix} 1 \\ 0 \end{matrix}} & \boxed{\begin{matrix} 1 \\ 1 \end{matrix}} \end{bmatrix}$$

coef of $\vec{b}_2$

coef of $\vec{b}_1$

Suppose dx^1, dx^2 is the dual basis of $\vec{e}_1, \vec{e}_2$.

and β^1, β^2 dual basis of $\vec{b}_1 \ \vec{b}_2$

Find $\beta^1 = \underline{?}\, dx^1 + \underline{?}\, dx^2$

$\beta^2 = \underline{?}\, dx^1 + \underline{?}\, dx^2$

i.e. $\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} \ ? \ \end{bmatrix} \begin{bmatrix} dx^1 \\ dx^2 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\shortparallel$$

Write $\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} [\vec{b}_1 \ \vec{b}_2] = \begin{bmatrix} \ ? \ \end{bmatrix} \overbrace{\begin{bmatrix} dx^1 \\ dx^2 \end{bmatrix} [e_1 \ e_2]}^{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} ? \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{array}{l} \text{coef of } \beta_1 \\ \text{coef of } \beta_2 . \end{array}$$

Let $df = c_1\, dx^1 + c_2\, dx^2 = c_i\, dx^i$ (in fact $c_i =: \frac{\partial f}{\partial x^i}$)

$= \omega_1 \beta^1 + \omega_2 \beta^2$ where $\begin{cases} dx^1\, dx^2 \text{ are dual basis to } e_1\, e_2 \\ \beta^1 \beta^2 \text{ are dual basis to } b_1\, b_2 \end{cases}$

and $[b_1\ b_2] = [e_1\ e_2] \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{A}$

i.e. $df = [c_1\ c_2] \begin{bmatrix} dx^1 \\ dx^2 \end{bmatrix} = [\omega_1\ \omega_2] \underbrace{\begin{bmatrix} \beta^1 \\ \beta^2 \end{bmatrix}}_{A^{-1} \begin{bmatrix} dx^1 \\ dx^2 \end{bmatrix}}$

Then $\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} = \begin{bmatrix} A^T \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ or $[\omega_1\ \omega_2] = [c_1\ c_2] \begin{bmatrix} A \end{bmatrix}$.

$\boxed{\text{The components of covectors are transformed by how basis is transformed}}$

In physics, Components of $\begin{cases} \text{vectors} \\ \text{covectors} \end{cases}$ are called $\begin{cases} \text{contravariant} \\ \text{covariant} \end{cases}$ vectors.

A <u>coordinate system</u> for an n-dim domain $M (\subseteq \mathbb{R}^n)$ is n functions

$x^1, \cdots, x^n : M \to \mathbb{R}$ s.t. $dx^1_p, \cdots, dx^n_p$ is a basis for $T_p^* M$ ($\forall p \in M$.)

Define $\vec{e}_1, \cdots, \vec{e}_n$ be its dual basis (called coord vectors)

In fact, we also use symbolically $\vec{e}_i = \frac{\partial}{\partial x^i}$

(A vector $\vec{v} = v^i \frac{\partial}{\partial x^i}$ can differentiate a function as direction derivative $\vec{v} \triangleright f = v^i \frac{\partial f}{\partial x^i}$)

$$\boxed{df\big|_p = \frac{\partial f}{\partial x^i}\Big|_p dx^i_p}$$

Partial derivatives $\frac{\partial f}{\partial x^i}$ are coeff of df under basis dx^i.

They are also $\frac{\partial f}{\partial x^i} = \langle df \mid \vec{e}_i \rangle = \frac{\partial}{\partial x^i} \triangleright f$.

Change of coordinates

Let M be an m-dim domain with coord $x^1, \dots, x^m$
N " " n " $y^1, \dots, y^n$.

Let $\phi : M \to N$ be a map.

$$M \xrightarrow{\phi} N$$

① Pullback of a function:
Through composing with ϕ we can pull a function $g : N \to \mathbb{R}$ back as $\tilde{g} : M \to \mathbb{R}$

$$\tilde{g} := g \circ \phi \qquad \tilde{g}(x^1, \dots, x^m) := g(\phi^1(x^1, \dots, x^m), \dots, \phi^n(x^1, \dots, x^m))$$

Let $\underset{\text{fcn}}{\phi^*} : (N \to \mathbb{R}) \xrightarrow{\text{lin}} (M \to \mathbb{R})$ $\quad \boxed{\underset{\text{fcn}}{\phi^*} g := g \circ \phi.}$

② Pushforward vector:

$$\underset{\text{vec}}{\phi_*} : T_pM \xrightarrow{\text{lin}} T_{\phi(p)}N, \quad \boxed{(\underset{\text{vec}}{\phi_*} \underset{\substack{\cap \\ TM}}{\vec{u}}) \triangleright \underset{\substack{\cap \\ (N\to\mathbb{R})}}{g} := \vec{u} \triangleright (\underset{\text{fcn}}{\phi^*} g)}$$

$$\vec{u} \triangleright \phi^* g = u^a \frac{\partial}{\partial x^a}(g \circ \phi) = u^a \frac{\partial y^i}{\partial x^a} \frac{\partial g}{\partial y^i}$$

$$\Rightarrow \boxed{\underset{\text{vec}}{\phi_*} \vec{u} = \left(\frac{\partial y^i}{\partial x^a} u^a \right) \frac{\partial}{\partial y^i} \qquad \text{for } \vec{u} = u^a \frac{\partial}{\partial x^a}}$$

Here $F^i_a = \frac{\partial y^i}{\partial x^a}$ is the Jacobian matrix.

③ Pullback of covector

$$\underset{\text{covec}}{\phi^*} : T^*_{\phi(p)}N \xrightarrow{\text{lin}} T^*_pM \quad \boxed{\langle \underset{\substack{\text{covec} \\ \cap \\ T^*N}}{\phi^*} \alpha \mid \underset{\substack{\cap \\ TM}}{\vec{u}} \rangle := \langle \alpha \mid \underset{\text{vec}}{\phi_*} \vec{u} \rangle}$$

$$\langle \alpha \mid \underset{\text{vec}}{\phi_*} \vec{u} \rangle = \alpha_i \left(\frac{\partial y^i}{\partial x^a} u^a \right) = \left(\alpha_i \frac{\partial y^i}{\partial x^a} \right) u^a$$

$$\Rightarrow \boxed{\underset{\text{covec}}{\phi^*} \alpha = \left(\frac{\partial y^i}{\partial x^a} \alpha_i \right) dx^a \qquad \text{for } \alpha = \alpha_i \, dy^i}$$

Chain Rule

$\boxed{(\phi \circ \psi)_* = \phi_* \psi_*}$ Pushforward is covariant

$\boxed{(\phi \circ \psi)^* = \psi^* \phi^*}$ Pullbacks are contravariant

$\boxed{d \underset{\text{fcn}}{\phi^*} g = \underset{\text{covec}}{\phi^*} dg}$ d & pullback commute.

(pf of the 3rd one) $\langle d\,\phi^* g \,|\, \vec{u} \rangle = \vec{u} \triangleright \phi^* g = (\phi_* \vec{u}) \triangleright g = \langle dg \,|\, \phi_* \vec{u} \rangle = \langle \phi^* dg \,|\, \vec{u} \rangle$. $\square$

Inner product structure

An inner product structure $\langle \cdot , \cdot \rangle$ for a vector space V is a bilinear form $\langle \cdot , \cdot \rangle : V \times V \to \mathbb{R}$ so that

- $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$ (symmetric)
- $\underbrace{\langle \vec{u}, \vec{u} \rangle}_{``|\vec{u}|^2} \geq 0$ and $|\vec{u}|^2 = 0 \Leftrightarrow \vec{u} = 0$ (positive definite.)

Table the inner product of basis vectors: $g_{ij} := \langle \vec{e}_i, \vec{e}_j \rangle$

<u>Inner product matrix</u> or <u>metric tensor</u> $\quad g_{ij} = g_{ji}$.

$\langle u^i \vec{e}_i, v^j \vec{e}_j \rangle = u^i v^j g_{ij}$.

$g_{ij} = \delta_{ij} \Leftrightarrow$ The basis is orthonormal.

Lowering and Raising indices

We can convert a vector to a covector by

$$\flat : V \to V^* \,, \qquad \underbrace{\langle \flat \vec{u} \,|\, \vec{v} \rangle}_{\text{dual pairing}} := \underbrace{\langle \vec{u}, \vec{v} \rangle}_{\text{inner product}}$$

Let $\sharp : V^* \to V$ be its inverse.

In basis $\langle \vec{u}, \vec{v} \rangle = g_{ij} u^i v^j \qquad \vec{u} = u^i \vec{e}_i, \ \vec{v} = v^j \vec{e}_j$

$$= \underbrace{(g_{ij} u^i) v^j}_{\text{dual pairing}}$$

$$\Rightarrow \boxed{(\flat \vec{u})_i = g_{ij} u^j}$$

and

$$\boxed{(\sharp \alpha)^i = g^{ij} \alpha_j}$$

where g^{ij} is the <u>inverse matrix</u> of g_{ij}.

<u>Define inner products for covectors</u> by $\langle \alpha, \beta \rangle := \langle \alpha^\sharp, \beta^\sharp \rangle$

$$= g^{ij} \alpha_j \underbrace{g_{ik} g^{kl}}_{\delta_i^l} \beta_l$$

$$= g^{jl} \alpha_j \beta_l.$$

$\boxed{\text{Gradient vector}}$ $\quad \mathrm{grad} f := \# df \qquad (\mathrm{grad} f)^i = g^{ij} \dfrac{\partial f}{\partial x^j}$.

Example What is the gradient vector in polar coord?

$$x = r\cos\theta \qquad y = r\sin\theta \qquad \left(r = \sqrt{x^2+y^2}, \quad \theta = \tan^{-1} y/x \right)$$

orthonormal

Let $\vec{e}_x = \frac{\partial}{\partial x}$, $\vec{e}_y = \frac{\partial}{\partial y}$ be dual basis to $dx\ dy$

$\vec{e}_r, \vec{e}_\theta$ be dual basis to $dr\ d\theta$

" ∂_r " ∂_θ

gradient in x,y-coord is

$$(\mathrm{grad} f) = [\vec{e}_x \ \ \vec{e}_y] \begin{bmatrix} \partial f/\partial x \\ \partial f/\partial y \end{bmatrix} = [\vec{e}_r \ \ \vec{e}_\theta] \begin{bmatrix} ? \\ ? \end{bmatrix}$$

$$\begin{bmatrix} dx \\ dy \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}}_{\frac{\partial(x,y)}{\partial(r,\theta)}} \begin{bmatrix} dr \\ d\theta \end{bmatrix}$$

inverse

$$[\vec{e}_x \ \ \vec{e}_y] = [\vec{e}_r \ \ \vec{e}_\theta] \underbrace{\begin{bmatrix} \cos\theta & \sin\theta \\ -\frac{1}{r}\sin\theta & \frac{1}{r}\cos\theta \end{bmatrix}}_{\frac{\partial(r,\theta)}{\partial(x,y)}}$$

$$df = \left[\frac{\partial f}{\partial x} \ \ \frac{\partial f}{\partial y}\right] \underbrace{\begin{bmatrix} dx \\ dy \end{bmatrix}}_{\begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} \begin{bmatrix} dr \\ d\theta \end{bmatrix}} = \left[\frac{\partial f}{\partial r} \ \ \frac{\partial f}{\partial \theta}\right] \begin{bmatrix} dr \\ d\theta \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \partial f/\partial x \\ \partial f/\partial y \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\frac{1}{r}\sin\theta & \frac{1}{r}\cos\theta \end{bmatrix}^T \begin{bmatrix} \frac{\partial f}{\partial r} \\ \frac{\partial f}{\partial \theta} \end{bmatrix}$$

$$\mathrm{grad} f = [\vec{e}_r \ \ \vec{e}_\theta] \begin{bmatrix} \cos\theta & \sin\theta \\ -\frac{1}{r}\sin\theta & \frac{1}{r}\cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\frac{1}{r}\sin\theta & \frac{1}{r}\cos\theta \end{bmatrix}^T \begin{bmatrix} \partial f/\partial r \\ \partial f/\partial \theta \end{bmatrix}$$

$$= [\vec{e}_r \ \ \vec{e}_\theta] \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{r^2} \end{bmatrix}}_{} \begin{bmatrix} \frac{\partial f}{\partial r} \\ \frac{\partial f}{\partial \theta} \end{bmatrix} \qquad \square$$

This is the $\#$ in the $\vec{e}_r\ \vec{e}_\theta$ basis

Summary

Vector	$\vec{v} \in V$	$\vec{v} = v^i \vec{e}_i = v^i \frac{\partial}{\partial x^i}$
Covector	$\alpha \in V^*$	$\alpha = \alpha_i dx^i$
vec-covec pairing	$\langle \alpha \| \vec{v} \rangle = \alpha[\![\vec{v}]\!]$	$= \alpha_i v^i$
Differential	$df\|_p \in T_p^*M$, $df[\![\vec{v}]\!] = \vec{v} \triangleright f$ directional derivative	$df = \frac{\partial f}{\partial x^i} dx^i$
Change of coord	$M \xrightarrow{\phi} N$	$\phi = \begin{bmatrix} y^1(x^1,\dots,x^m) \\ \vdots \\ y^n(x^1,\dots,x^m) \end{bmatrix}$
Pullback fcn	$\underset{\text{fcn}}{\phi^*} : (N \to \mathbb{R}) \to (M \to \mathbb{R})$ $\underset{\text{fcn}}{\phi^*} g := g \circ \phi$	$(\underset{\text{fcn}}{\phi^*} g)(x^1,\dots,x^m) = g\big(y^1(x^1,\dots,x^m),\dots,y^n(x^1,\dots,x^m)\big)$
Pushforward of vector	$\underset{\text{vec}}{\phi_*} : TM \to TN$ $(\underset{\text{vec}}{\phi_*} \vec{u}) \triangleright g := \vec{u} \triangleright (\underset{\text{fcn}}{\phi^*} g)$	$F^i_a = \frac{\partial y^i}{\partial x^a}$ $(\phi_* \vec{u}) = (F^i_a u^a) \frac{\partial}{\partial y^i}$
Pullback of covector	$\underset{\text{covec}}{\phi^*} : T^*N \to T^*M$ $\langle \underset{\text{covec}}{\phi^*} \alpha \| \vec{u} \rangle := \langle \alpha \| \underset{\text{vec}}{\phi_*} \vec{u} \rangle$	$(\phi^* \alpha) = F^i_a \alpha_i dx^a$
Metric	$\langle\ ,\ \rangle : V \times V \to \mathbb{R}$ $\flat : V \to V^*$ $\langle \flat\vec{u} \| \vec{v} \rangle := \langle \vec{u}, \vec{v} \rangle$ $\sharp : V^* \to V$ $\sharp = \flat^{-1}$	$\langle \vec{u}, \vec{v} \rangle = g_{ij} u^i v^j$ $g_{ij} := \langle \vec{e}_i, \vec{e}_j \rangle$ $(\flat \vec{u})_i = g_{ij} u^j$ $(\sharp \alpha)^i = g^{ij} \alpha_j$ $\underbrace{g^{ij}\, g_{jk}}_{\text{inverse matrix}} = \delta^i_k$
gradient	$\text{grad} f = \sharp\, df$	$(\text{grad} f)^i = g^{ij} \frac{\partial f}{\partial x^j}$