CSE 291 (SP24)
Topics in CSE:
Dimensional Analysis

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What is Dimensional Analysis?

- It is a powerful tool for modeling and analysis.
- It allows us to
  - find the dimensionless quantities
  - reduce the number of parameters
  - perform equivalent system in a scaled model

Dimensions and Units

- Dimensions and units
- Dimension homogeneity
- Dimensionless equations
- Buckingham Pi Theorem

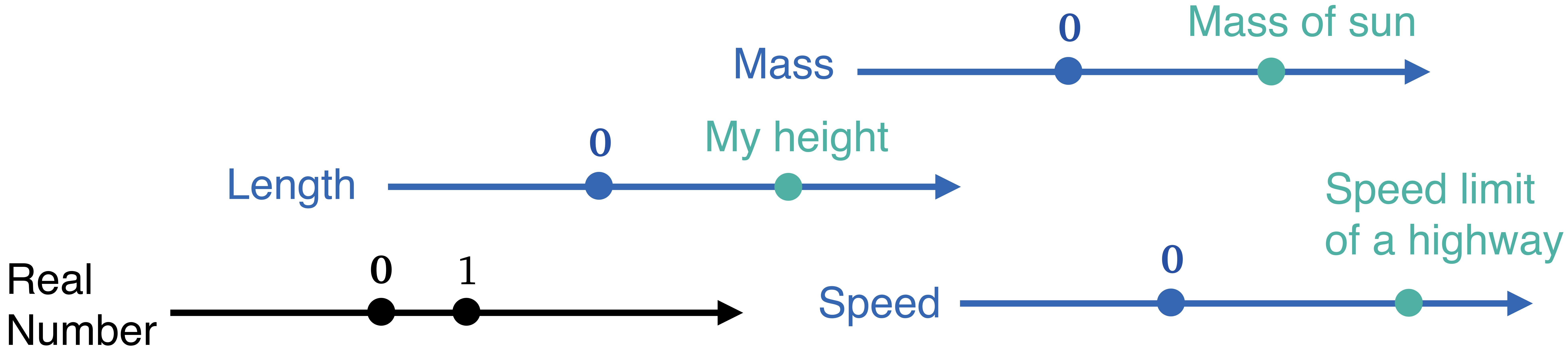
Dimension

- A (physical) **dimension** is

  a collection of measurements of one type of physical quantity.

- Example of dimension
  - Length, mass, speed, acceleration, force, energy,…

- Mathematically, a dimension is a one-dimensional vector space.
  - Every element of each 1D vector space is a physical measurement



Unit

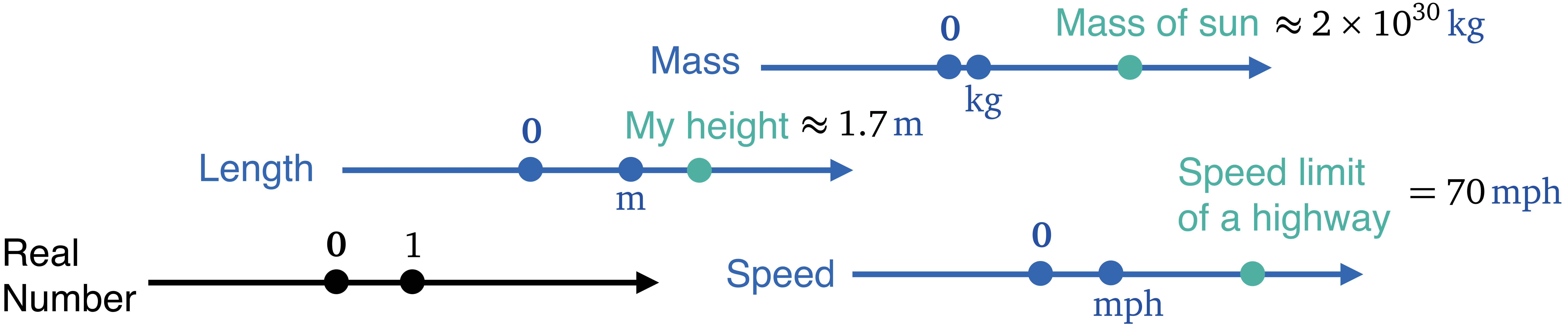
- A (physical) **unit** for a dimension is

  a basis for the 1D vector space representing the dimension

- A unit is a way to assign number for each measurement
- Given any two element of the same dimension

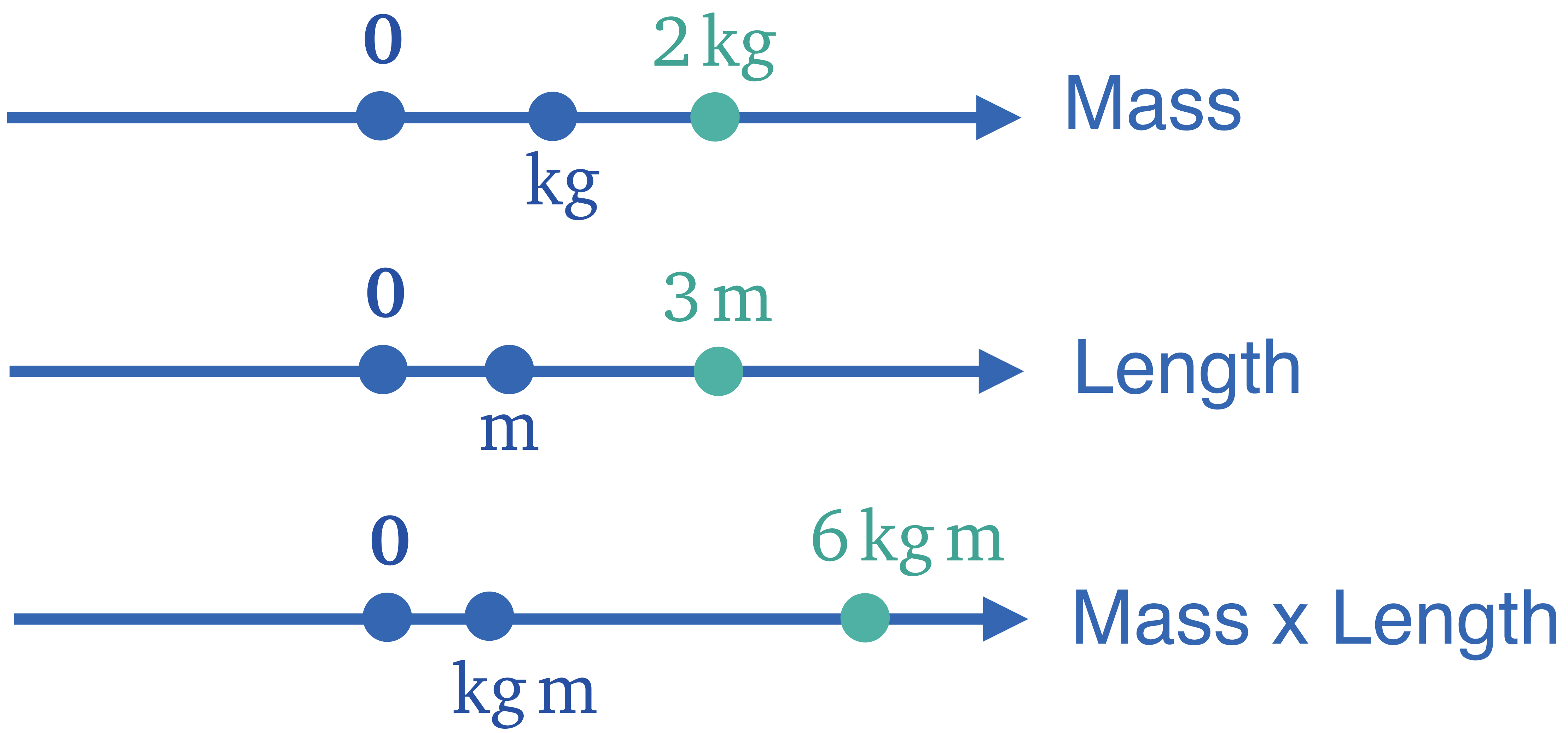
$$x, y \in \mathrm{D} \qquad y \neq 0$$

there exists a unique scalar $s \in \mathbb{R}$ so that $x = s\,y$



Multiplication between dimensions

- We don't add/subtract measurements of different dimensions
- We can multiply/divide two measurements of different dimensions



$$(2\,\mathrm{kg})(3\,\mathrm{m}) = 6\,\mathrm{kg\,m}$$

- The collection of all possible dimensions involved in a context would look like:

$$\{\mathbb{R}, \text{Mass}, \text{Length}, \text{Mass Length}, \text{Mass}^2\,\text{Length}, \text{Mass Length}^{-1}, \ldots\}$$

Notation

- A dimension is denoted by sans serif font D.
- Each element $x \in \mathsf{D}$ is a measurement.
- The dimensionless dimension is $\mathbb{R}$ which has a standard unit 1.
- The bracket of a measurement is the dimension it is in:

$$\mathsf{D} = [x]$$

$$[\mathrm{kg}] = [5\,\mathrm{kg}] = [\mathrm{lb}] = \mathsf{M}$$

- Products of measurements produce products in dimensions

$$[x][y] = [xy]$$

- We can consider a collection of all dimensions that can be involved

$$\mathscr{D} = \{\mathbb{R}, \mathsf{D}_1, \mathsf{D}_2, \ldots\}$$

which should be closed under multiplication, forming an abelian group

Physical dimension

- In physics, the space of dimensions is a finitely generated free Abelian group by 7 standard base dimensions

Primary dimension	Symbol	SI Unit
Mass	M	kg (kilogram)
Length	L	m (meter)
Time	T	s (second)
Temperature	Θ	K (Kelvin)
Electric current	I	A (Ampere)
Amount of light	C	cd (candela)
Amount of matter	N	mol (mole)

- Every physical dimension takes the form of

$$\mathsf{M}^{n_1}\mathsf{L}^{n_2}\mathsf{T}^{n_3}\Theta^{n_4}\mathsf{I}^{n_5}\mathsf{C}^{n_6}\mathsf{N}^{n_7}$$

- Represent it as a 7D vector $(n_1, \ldots, n_7)^\intercal \in \mathbb{R}^7$
  - Multiplications between dimensions become additions between 7D vectors

Physical dimension

- What is the 7D vector representation of the "force" dimension?

hint: force is mass times acceleration

$$[\text{force}] = \mathsf{MLT}^{-2} \qquad (1, 1, -2, 0, 0, 0, 0)^{\mathsf{T}}$$

$$\mathsf{M}^{n_1}\mathsf{L}^{n_2}\mathsf{T}^{n_3}\Theta^{n_4}\mathsf{I}^{n_5}\mathsf{C}^{n_6}\mathsf{N}^{n_7}$$

- What is the 7D vector representation of the dimension of a dimensionless number?

$$(0, 0, 0, 0, 0, 0, 0)^{\mathsf{T}}$$

Dimension Homogeneity

- Dimensions and units
- Dimension homogeneity
- Dimensionless equations
- Buckingham Pi Theorem

Dimension homogeneity

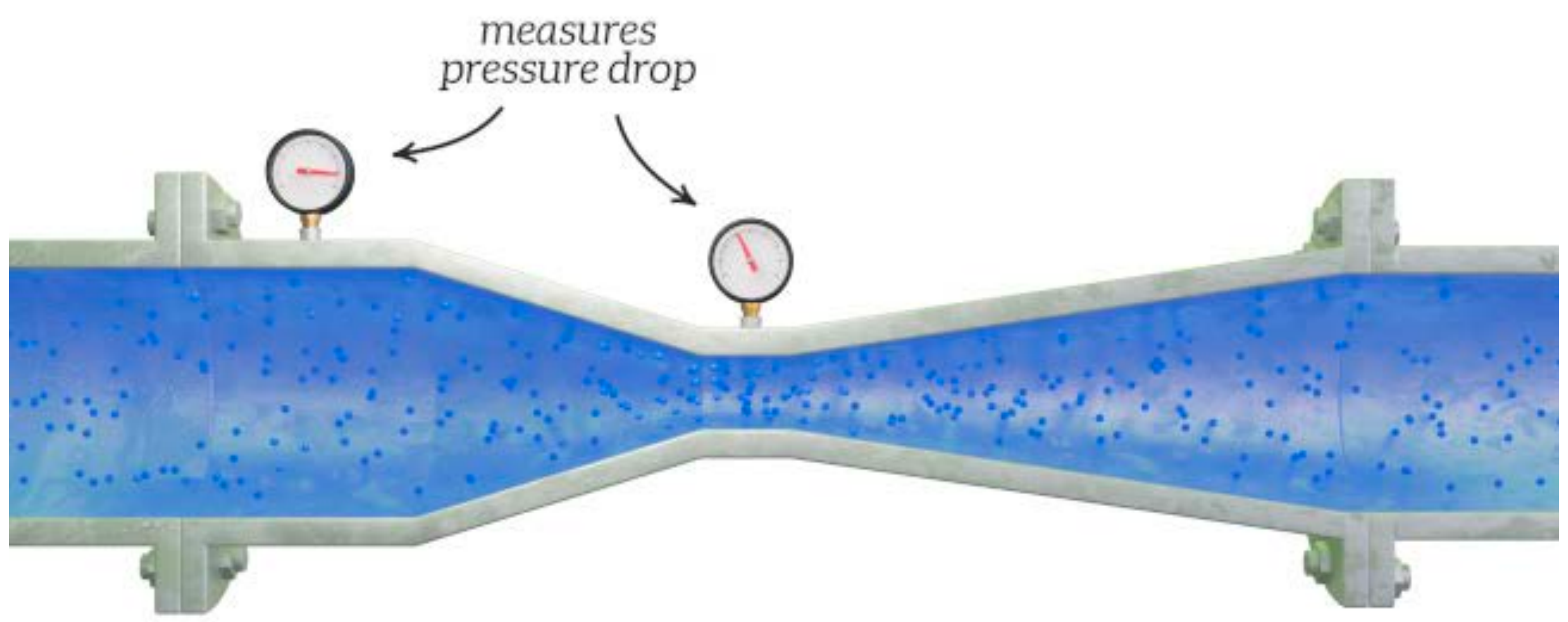
In an equation or inequality, every additive term must have the same dimension.

Dimension homogeneity: Example 1

- Example: Bernoulli equation

In a steady incompressible fluid flow,

the measurements at each point



ρ : density
g : gravity
v : velocity
z : height
p : pressure

satisfy

$$\frac{1}{2}\rho v^2 + p + \rho g z = \text{Constant}$$

Dimension homogeneity: Example 2

- Example: Sobolev embedding

Sobolev space of functions is defined by

$$W^{k,p}(\mathbb{R}^n) = \left\{ f : \mathbb{R}^n \to \mathbb{R} \,\middle|\, \int_{\mathbb{R}^n} |D^k f|^p \, dx_1 \dots dx_n < \infty \right\}$$

- Can you get the relationship between p, q given k, l, n?

Sobolev inequality

3 languages

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From Wikipedia, the free encyclopedia

In mathematics, there is in mathematical analysis a class of **Sobolev inequalities**, relating norms including those of Sobolev spaces. These are used to prove the **Sobolev embedding theorem**, giving inclusions between certain Sobolev spaces, and the Rellich–Kondrachov theorem showing that under slightly stronger conditions some Sobolev spaces are compactly embedded in others. They are named after Sergei Lvovich Sobolev.

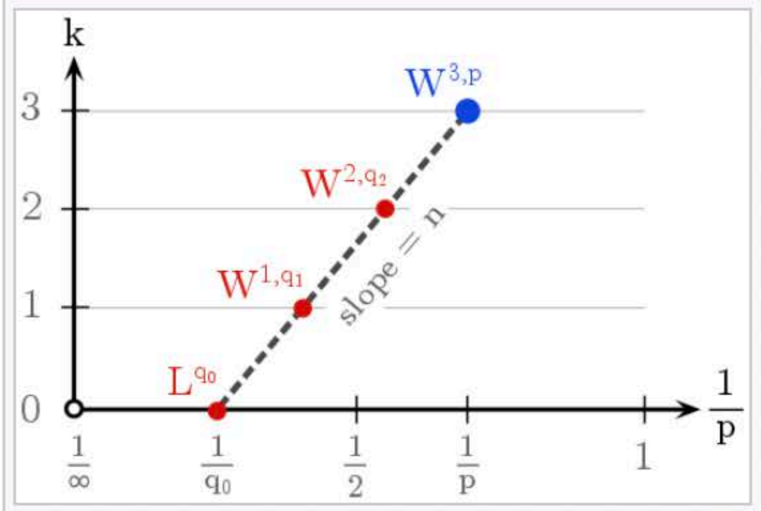
Sobolev embedding theorem [edit]

Let $W^{k,p}(\mathbf{R}^n)$ denote the Sobolev space consisting of all real-valued functions on $\mathbf{R}^n$ whose first k weak derivatives are functions in L^p. Here k is a non-negative integer and $1 \le p < \infty$. The first part of the Sobolev embedding theorem states that if $k > \ell$, $p < n$ and $1 \le p < q < \infty$ are two real numbers such that

$$\frac{1}{p} - \frac{k}{n} = \frac{1}{q} - \frac{\ell}{n},$$

then

$$W^{k,p}(\mathbf{R}^n) \subseteq W^{\ell,q}(\mathbf{R}^n)$$



Graphical representation of the

Dimensionless Equations

- Dimensions and units
- Dimension homogeneity
- Dimensionless equations
- Buckingham Pi Theorem

Dimensionless equations

- Rescale the physical variables so that they are all dimensionless. Rescale the equation so that it becomes a ***relationship between dimensionless variables***.

- The process will leave us ***as few parameters as possible***.

- We obtain the ***same result*** when the dimensionless parameters matches, even if the original problems are ***at vastly different scale***.

Parabolic free fall

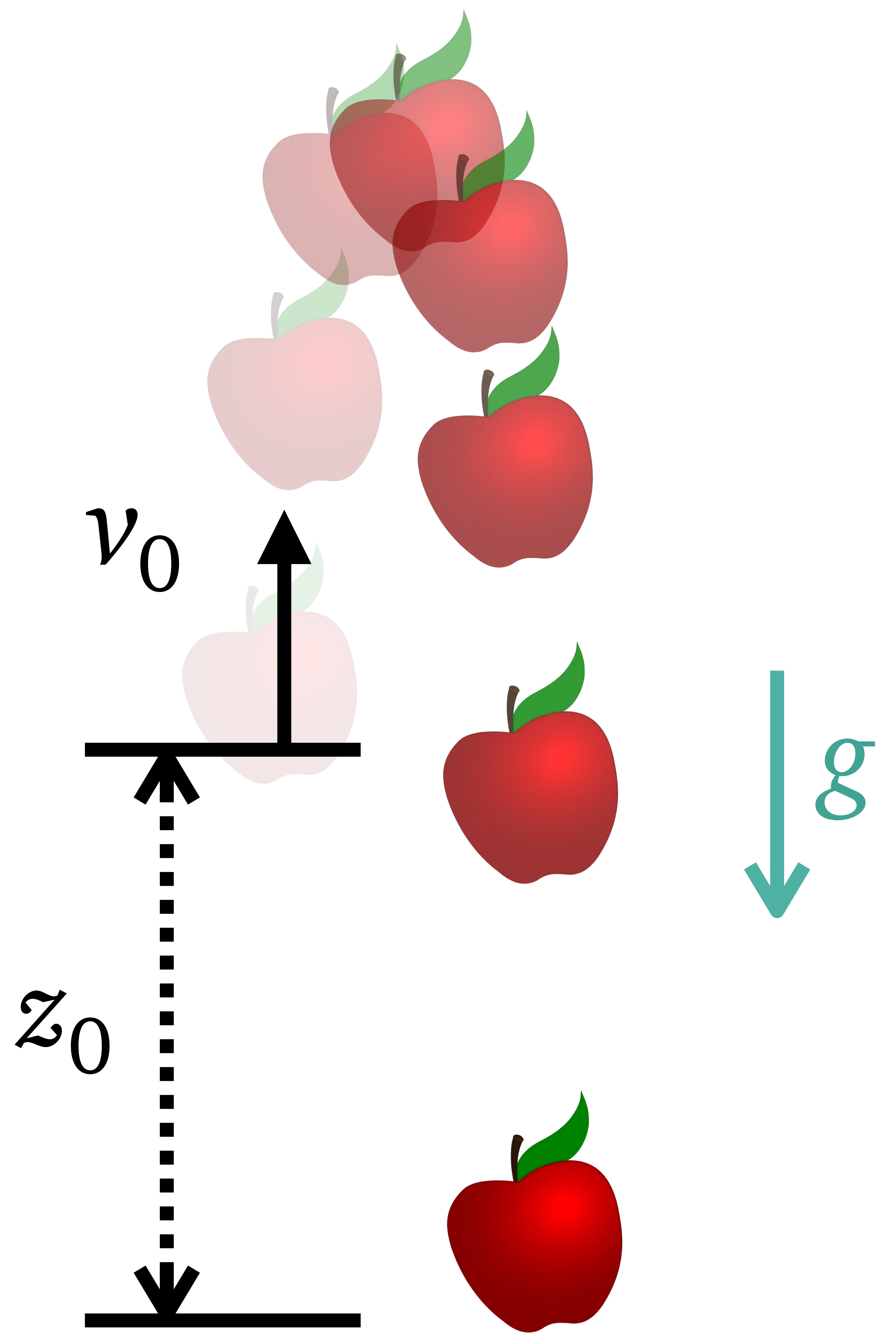
- Height as a function of time, initial velocity, initial height, and gravity:

$$z(t) = z_0 + v_0 t - \frac{g}{2} t^2$$

- It's the solution to

$$z''(t) = -g, \quad z(0) = z_0, \quad z'(0) = v_0$$

- There are 3 parameters, but we can reduce them into 1 parameter!



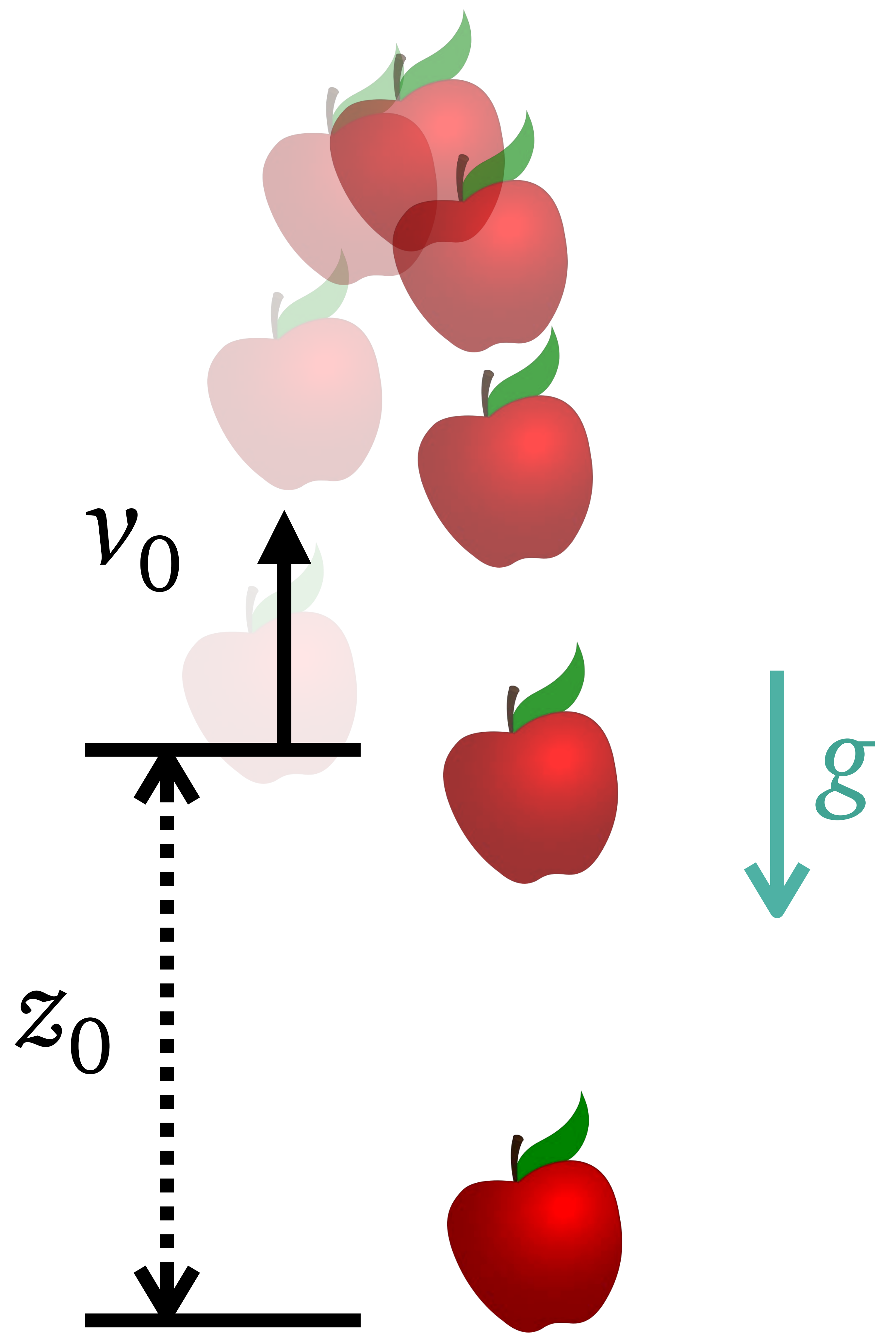
Parabolic free fall

$$z^* = \frac{z}{z_0} \qquad t^* = \frac{t v_0}{z_0}$$

- Froude number $$\mathrm{Fr} = \frac{v_0}{\sqrt{g z_0}}$$

$$z^* = 1 + t^* - \frac{1}{2\mathrm{Fr}^2} t^{*2}$$

$$z^*_{\mathrm{max}} = 1 + \frac{\mathrm{Fr}^2}{2}$$



Navier–Stokes equation

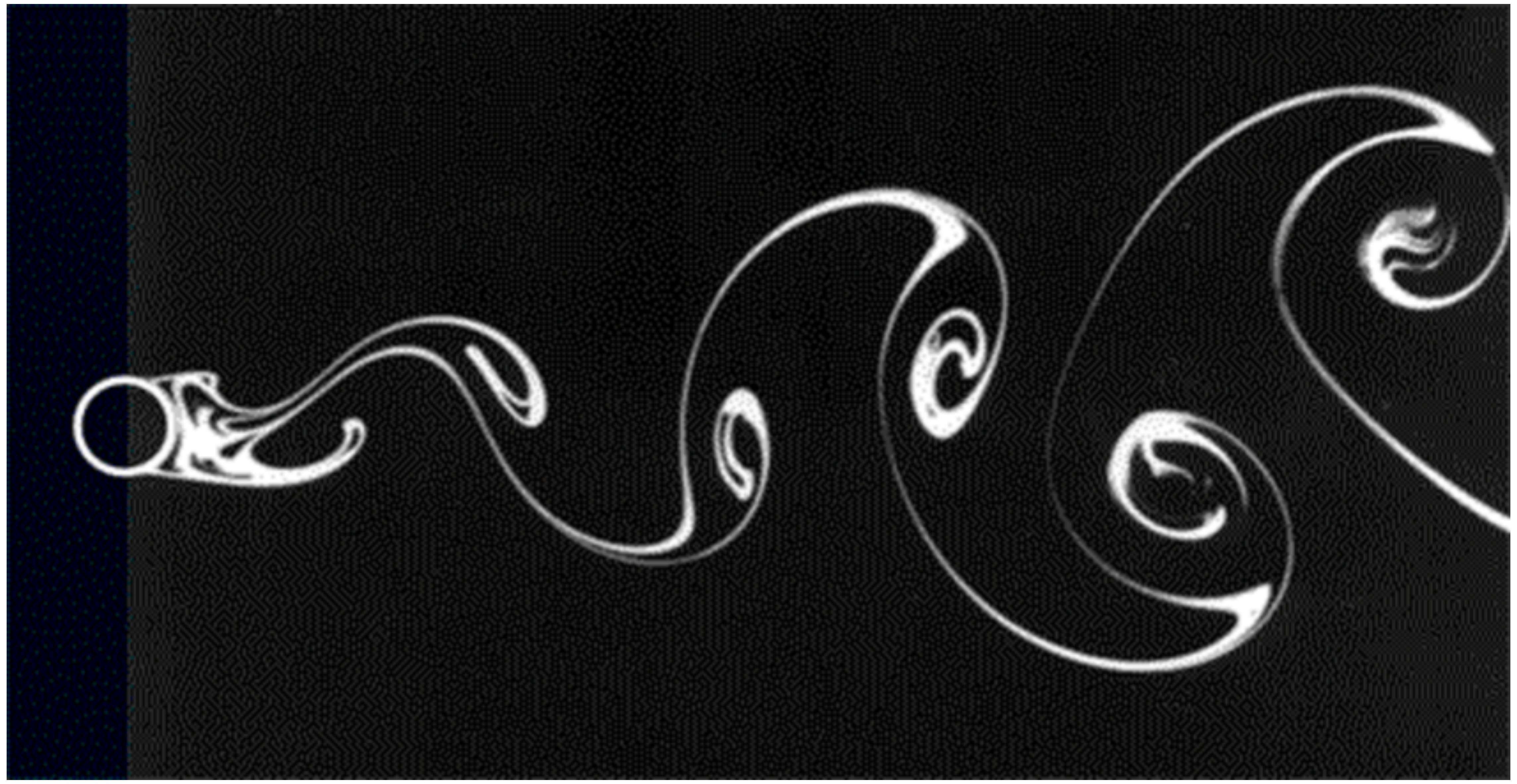
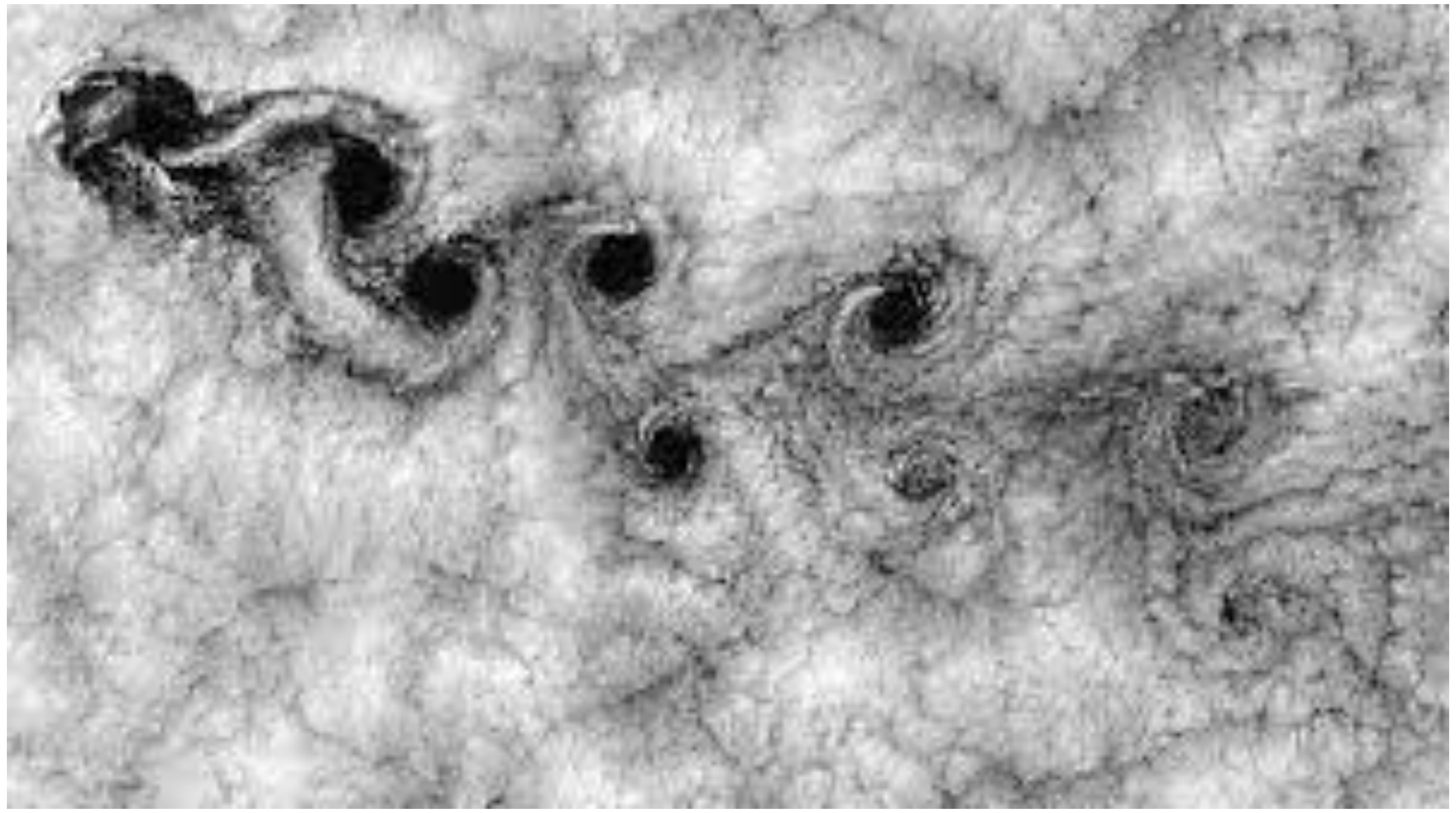
- Navier–Stokes equation

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{\nabla p}{\rho} + \frac{\mu}{\rho} \nabla \cdot \nabla \mathbf{u}$$

  - Time and time derivative: $[t] = \mathsf{T}$ $\quad [\frac{\partial}{\partial t}] = \mathsf{T}^{-1}$
  - Space and spatial derivatives: $[\mathbf{x}] = \mathsf{L}$ $\quad [\nabla] = \mathsf{L}^{-1}$ $\quad [\nabla \cdot \nabla] = \mathsf{L}^{-2}$
  - Velocity: $[\mathbf{u}] = \mathsf{LT}^{-1}$
  - Density and pressure: $[\rho] = \mathsf{ML}^{-3}$ $\quad [p] = \mathsf{ML}^{-1}\mathsf{T}^{-2}$
  - Viscosity (stress per speed gradient) $[\mu] = \mathsf{ML}^{-1}\mathsf{T}^{-1}$
- (Assume density and viscosity are constant)

Navier–Stokes equation

- Characteristic length L and characteristic speed U
- They can be the diameter of obstacle and the background speed

Von Karman vortex street

Navier–Stokes equation

$$\mathbf{x}^* := {}^{\mathbf{x}}/_{L}$$

$$t^* := ({}^{U}/_{L})t$$

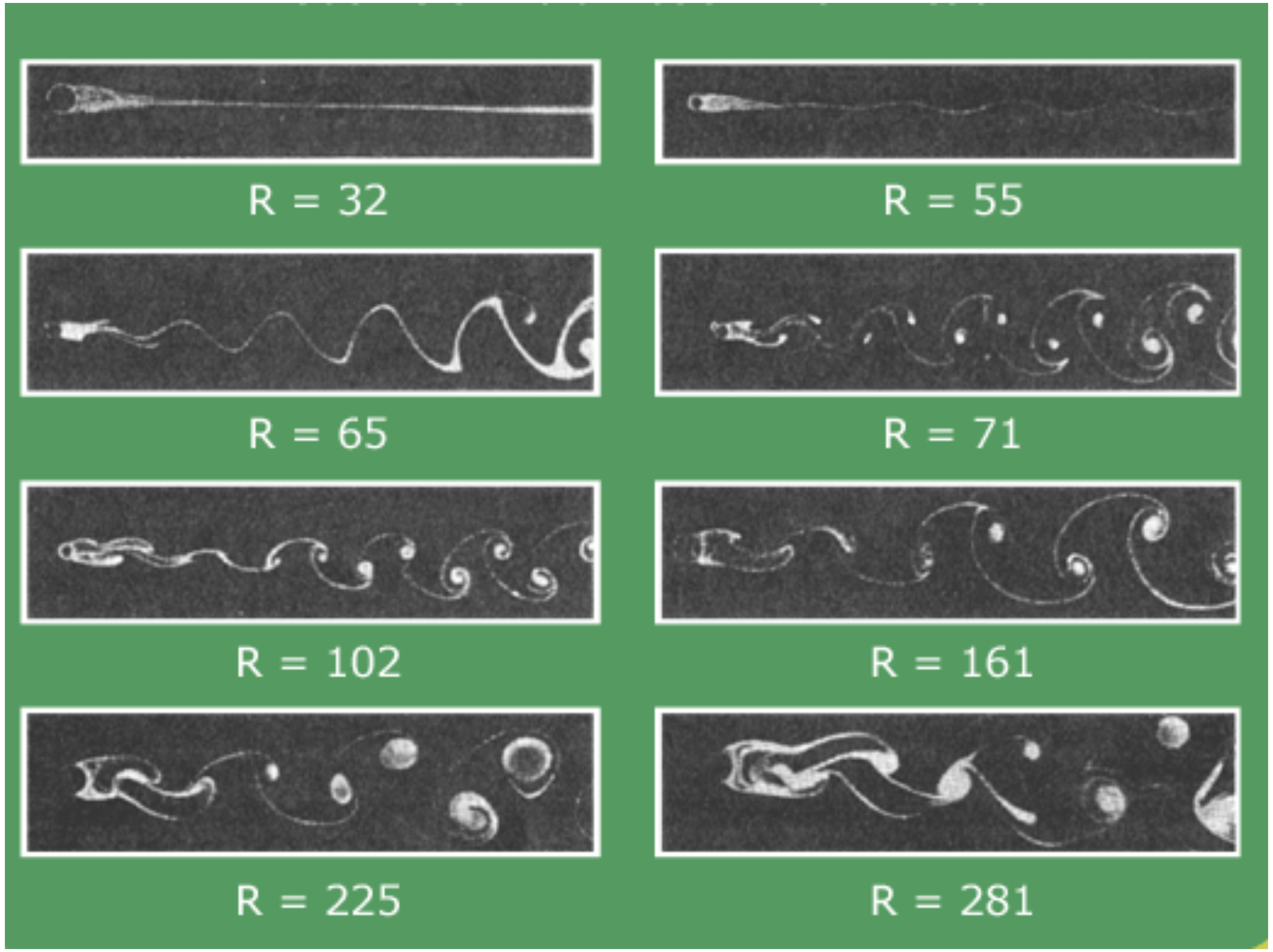
$$\mathbf{u}^* := {}^{\mathbf{u}}/_{U}$$

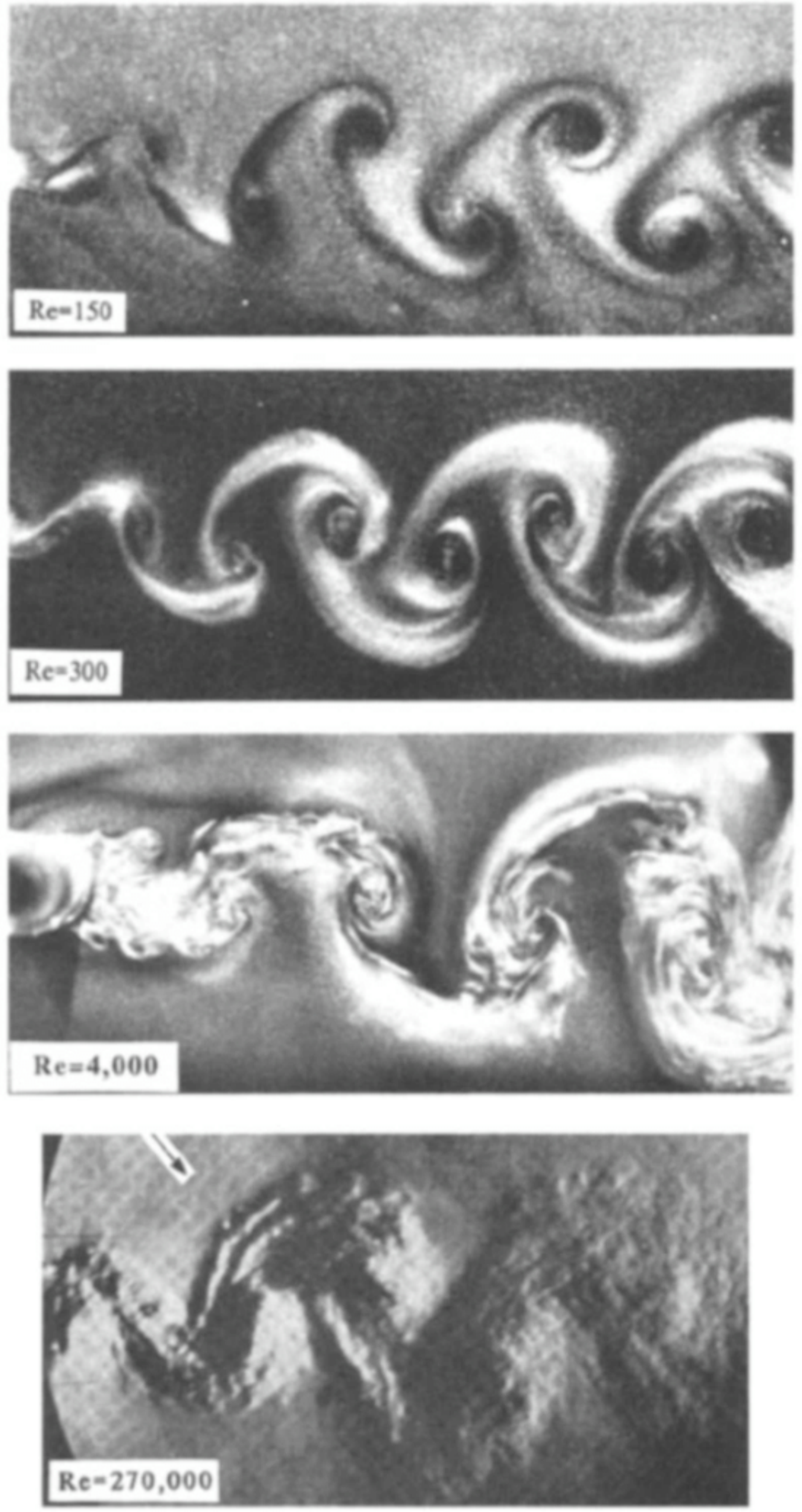
$$p^* := \frac{p}{\rho U^2}$$

$$\frac{\partial \mathbf{u}^*}{\partial t^*} + \mathbf{u}^* \cdot \nabla^* \mathbf{u}^* = -\nabla^* p^* + \frac{1}{\mathrm{Re}} \nabla^* \cdot \nabla^* \mathbf{u}^*$$

- Reynolds number: $\mathrm{Re} = \dfrac{\rho U L}{\mu}$

Navier–Stokes equation



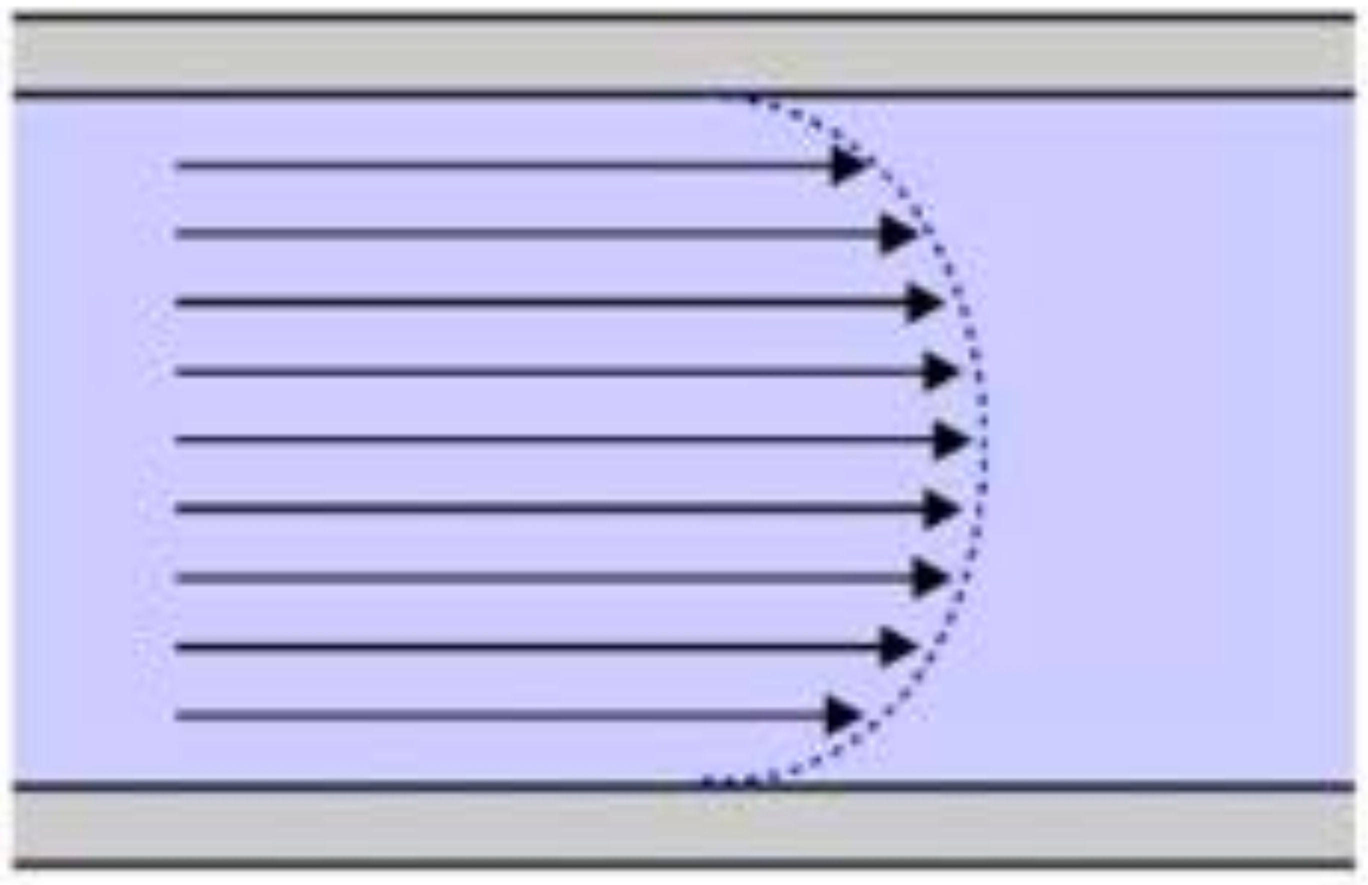


Navier–Stokes equation

$$\mathrm{Re} = \frac{\rho U L}{\mu}$$

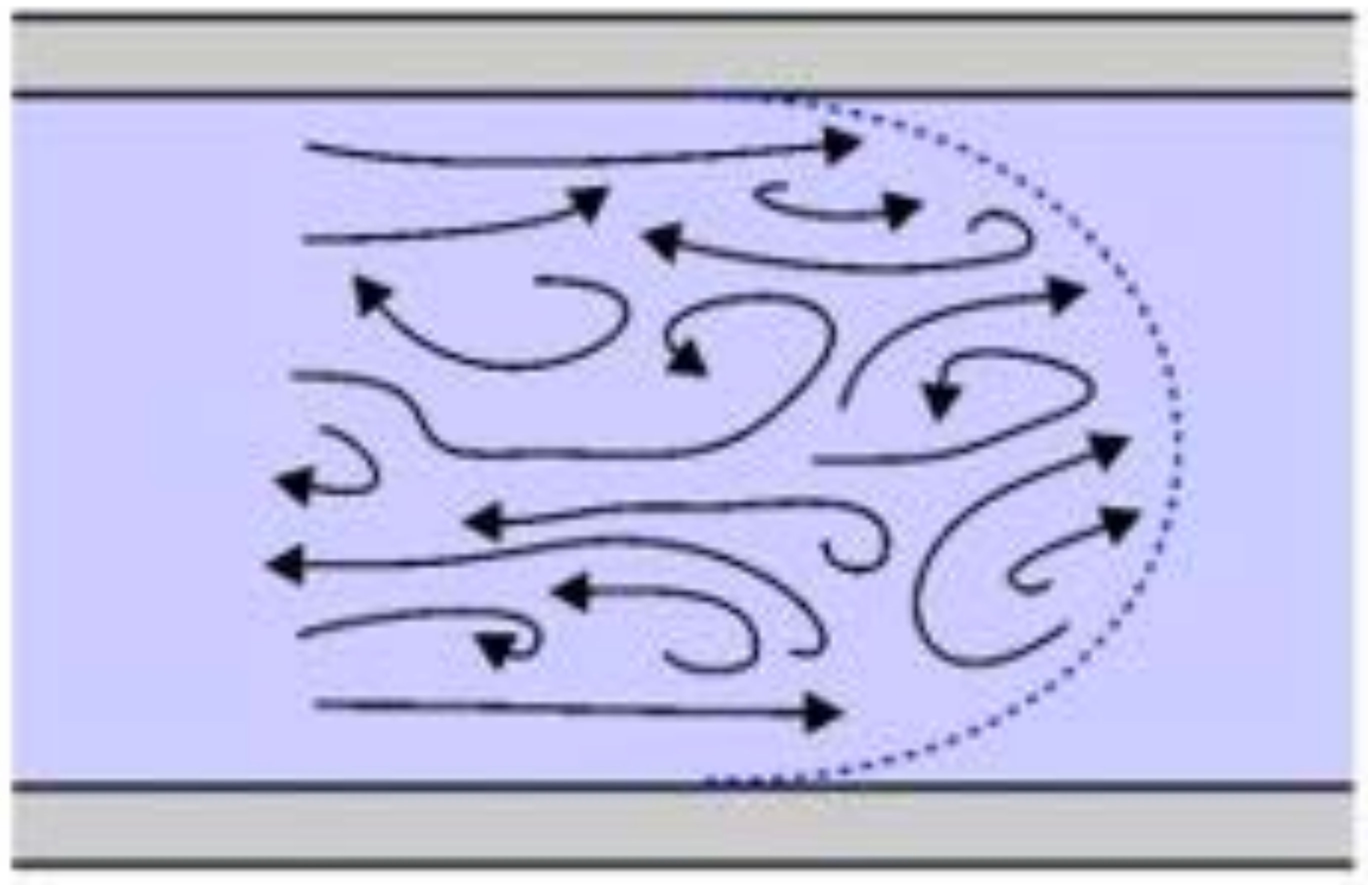
Navier–Stokes equation

Laminar flow
(low Reynolds)



Efficient fluid transport

Turbulent flow
(high Reynolds)

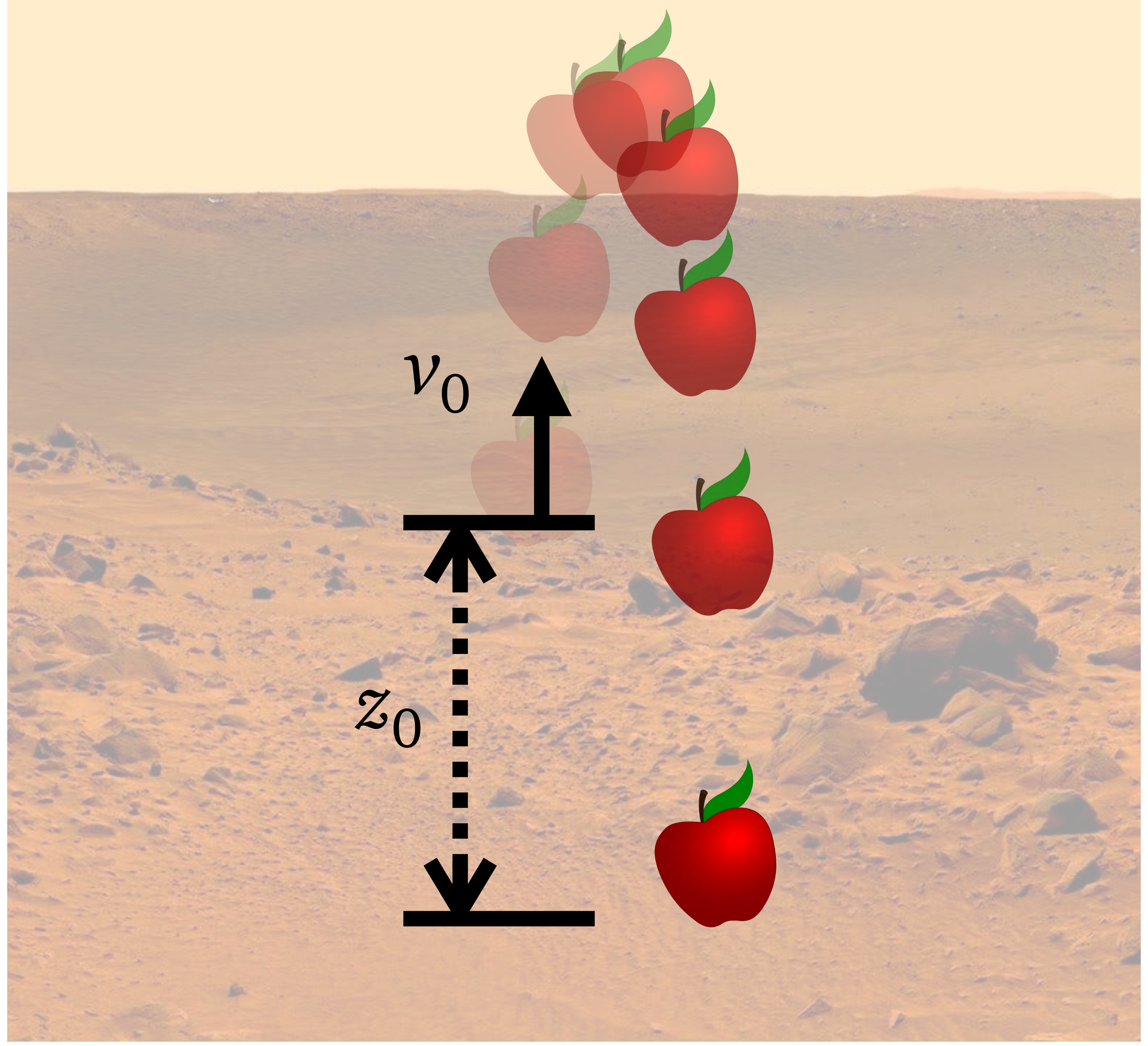


Inefficient transport
(causing Asthma)

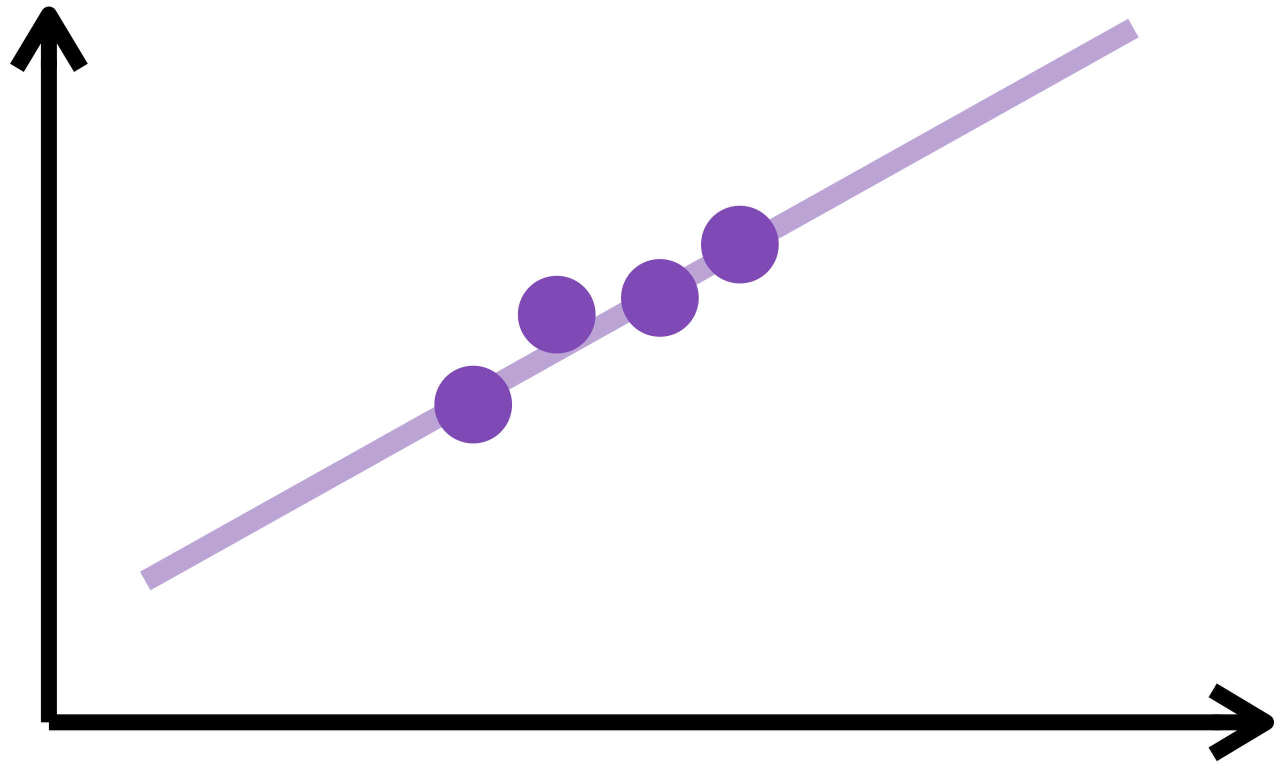
$$\mathrm{Re} = \frac{\rho U L}{\mu}$$

Dimensionless equations: summary

- Reduction of parameters
- Similarity between systems



- Extrapolation of data



Buckingham Pi Theorem

- Dimensions and units
- Dimension homogeneity
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Buckingham Pi Theorem

Buckingham Π-theorem

- Suppose we have a physical equation relating *n* quantities

$$f(q_1, \ldots, q_n) = 0$$

- Suppose the *n* quantities only involve *k* independent physical dimensions.
- Then the equation can be restated as

$$F(\Pi_1, \ldots, \Pi_p) = 0$$

for some $p = n - k$ dimensionless variables/parameters

$$\Pi_1, \ldots, \Pi_p$$

Buckingham Pi Theorem

$$f(q_1, \ldots, q_n) = 0$$

- Consider base dimension (e.g. the 7 base dimension)

$$\mathrm{B}_1, \ldots, \mathrm{B}_\ell$$

- Expand the dimensions of the quantities in this base

$$[q_i] = \mathrm{B}_1{}^{m_{1i}} \ldots \mathrm{B}_\ell{}^{m_{\ell i}}$$

Buckingham Pi Theorem

- Expand the dimensions of the quantities in this base

$$[q_i] = \mathrm{B}_1^{\;m_{1i}} \ldots \mathrm{B}_\ell^{\;m_{\ell i}}$$

- Take log: $\log[q_i] = m_{1i} \log \mathrm{B}_1 + \ldots + m_{\ell i} \log \mathrm{B}_\ell$
- This is a change of basis formula

$$\begin{pmatrix} \log[q_1] & \cdots & \log[q_n] \end{pmatrix} = \begin{pmatrix} \log \mathrm{B}_1 & \cdots & \log \mathrm{B}_\ell \end{pmatrix} \begin{pmatrix} m_{11} & \cdots & m_{1n} \\ \vdots & \ddots & \vdots \\ m_{\ell 1} & \cdots & m_{\ell n} \end{pmatrix}$$

$\mathbf{M}$ dimension matrix

Buckingham Pi Theorem

$$\begin{pmatrix}\log[q_1] & \cdots & \log[q_n]\end{pmatrix} = \begin{pmatrix}\log \mathrm{B}_1 & \cdots & \log \mathrm{B}_\ell\end{pmatrix}\begin{pmatrix} m_{11} & \cdots & m_{1n} \\ \vdots & \ddots & \vdots \\ m_{\ell 1} & \cdots & m_{\ell n}\end{pmatrix}$$

$\mathbf{M}$ dimension matrix

- The physical quantities involve only k independent dimension $\iff$ $\mathbf{M}$ has rank k (dimension of the image)
- Rank–Nullity Thm: The null space of $\mathbf{M}$ has dimension $p = n - k$
- Find a basis for the null space $\mathbf{a}_1 = \begin{pmatrix} a_{11} \\ \vdots \\ a_{n1}\end{pmatrix}, \ldots, \mathbf{a}_p = \begin{pmatrix} a_{1p} \\ \vdots \\ a_{np}\end{pmatrix}$

$$\mathbf{M}\mathbf{a}_j = \mathbf{0}$$

Buckingham Pi Theorem

$$\begin{pmatrix}\log[q_1] & \cdots & \log[q_n]\end{pmatrix} = \begin{pmatrix}\log \mathrm{B}_1 & \cdots & \log \mathrm{B}_\ell\end{pmatrix}\begin{pmatrix} m_{11} & \cdots & m_{1n} \\ \vdots & \ddots & \vdots \\ m_{\ell 1} & \cdots & m_{\ell n}\end{pmatrix}$$

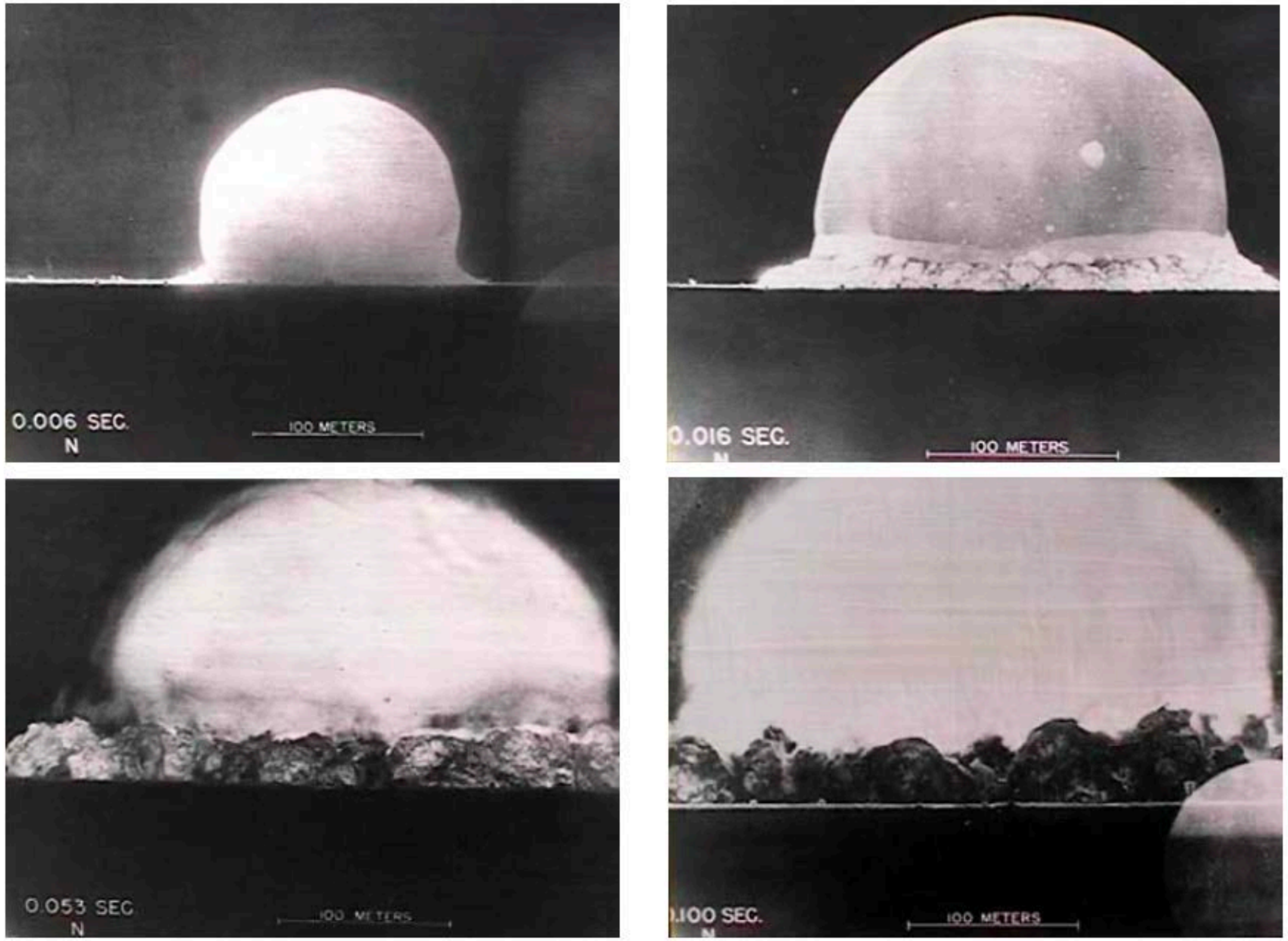
$\mathbf{M}$ dimension matrix

- Find a basis for the null space $\mathbf{a}_1 = \begin{pmatrix} a_{11} \\ \vdots \\ a_{n1}\end{pmatrix}, \ldots, \mathbf{a}_p = \begin{pmatrix} a_{1p} \\ \vdots \\ a_{np}\end{pmatrix}$

$$\mathbf{M}\mathbf{a}_j = \mathbf{0}$$

- Then $\begin{pmatrix}\log[q_1] & \cdots & \log[q_n]\end{pmatrix}\mathbf{a}_j = \mathbf{0}$
- That is, $\Pi_j := C_j {q_1}^{a_{1j}} \cdots {q_n}^{a_{nj}}$ are dimensionless

Example 1: Atomic bomb



Trinity test (1945)

From the released photograph
G.I. Taylor estimated the energy (which was confidential)

Example 1: Atomic bomb

- Assume that the relevant variables are
  - Radius r of the fireball $[r] = \mathsf{M}^0\mathsf{L}^1\mathsf{T}^0$
  - Density ρ of surrounding air $[\rho] = \mathsf{M}^1\mathsf{L}^{-3}\mathsf{T}^0$
  - Energy E released by the bomb $[E] = \mathsf{M}^1\mathsf{L}^2\mathsf{T}^{-2}$
  - Time t since the ignition $[t] = \mathsf{M}^0\mathsf{L}^0\mathsf{T}^1$
- Dimension matrix

$$\mathbf{M} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & -3 & 2 & 0 \\ 0 & 0 & -2 & 1 \end{pmatrix}$$

- One dimensional null space spanned by $\mathbf{a} = \begin{pmatrix} -5 \\ -1 \\ 1 \\ 2 \end{pmatrix}$

Example 1: Atomic bomb

- Assume that the relevant variables are
  - Radius r of the fireball
  - Density ρ of surrounding air
  - Energy E released by the bomb
  - Time t since the ignition
- One dimensional null space spanned by $\mathbf{a} = \begin{pmatrix} -5 \\ -1 \\ 1 \\ 2 \end{pmatrix}$
- Found a dimensionless quantity $\Pi = r^{-5}\rho^{-1}E^{1}t^{2} = \dfrac{Et^2}{r^5\rho}$
- Physical law takes the form $F(\Pi) = 0$
- This mathematically implies $\Pi = \text{Constant}$

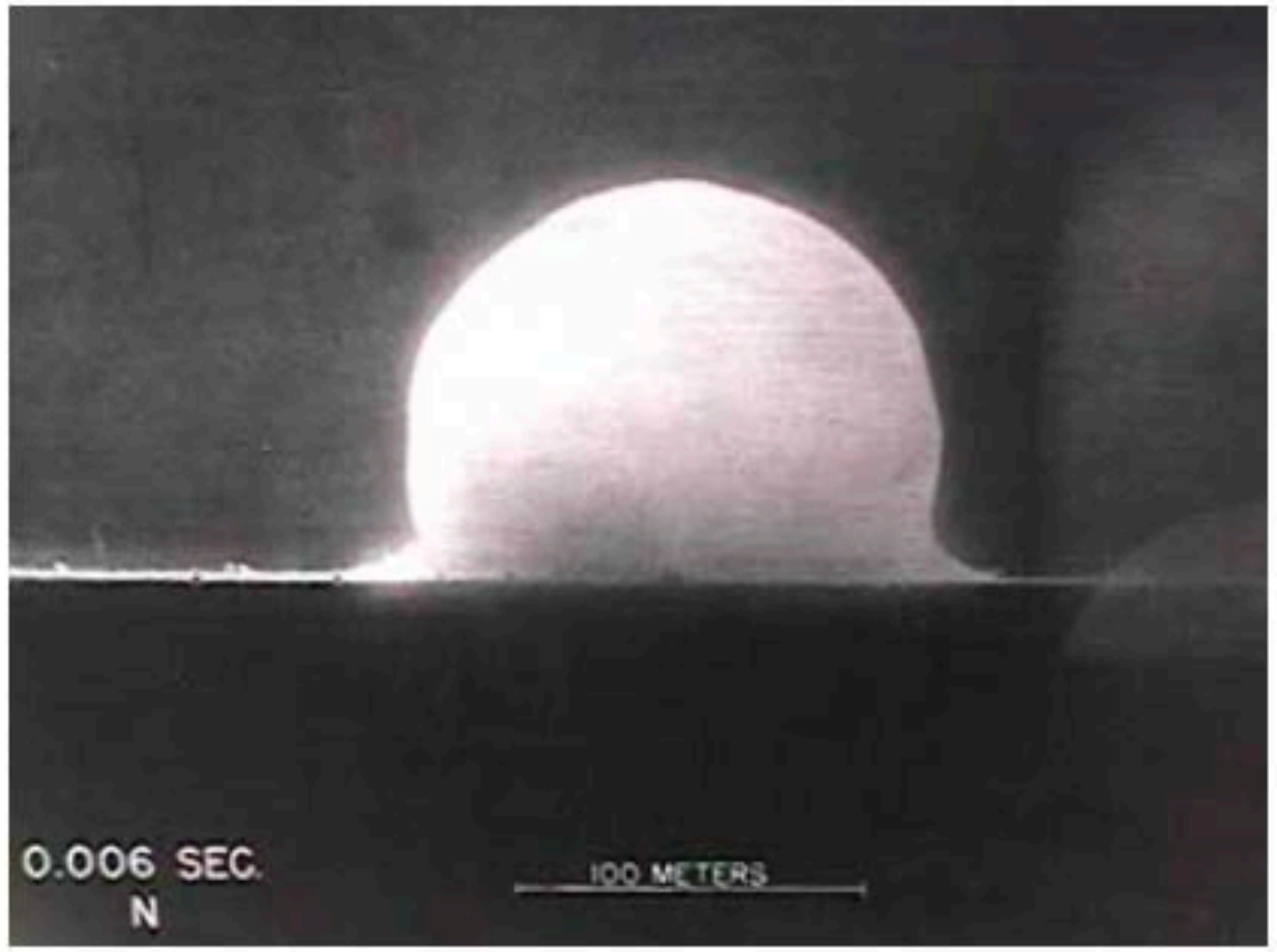
Example 1: Atomic bomb

- Any explosion satisfies $\frac{Et^2}{r^5\rho} = \mathrm{C}$
- Some smaller experiments suggest $\mathrm{C} \approx 1.033$
- Given the photograph indicating

$$t = 0.006\,\mathrm{s} \qquad r = 80\,\mathrm{m} \qquad \rho_{\mathrm{TestSite}} = 1.1\,\mathrm{kg/m^3}$$

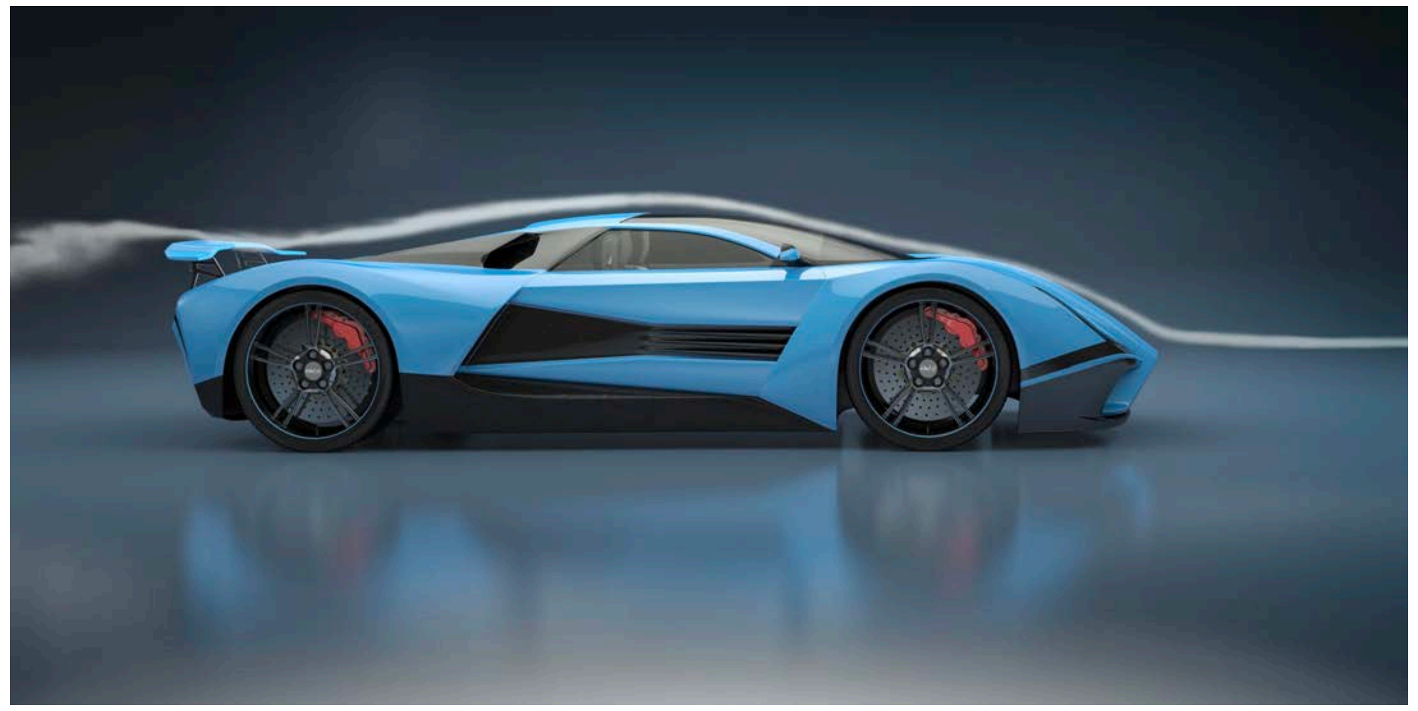
$$E \approx \frac{Cr^5\rho}{t^2} \approx 10^{14}\,\mathrm{J}$$

≈ 24 kilotons of TNT



- Taylor got 22 kilotons TNT using more frames
- Ground truth is 20 kilotons TNT

Example 2: Drag of a car



- A moving car will experience aerodynamic drag
- Reasonable postulate: there exists a function relating the following 5 quantities
  - Car's length scale L $\quad [L] = \mathrm{M}^0\mathrm{L}^1\mathrm{T}^0$
  - Car speed V $\quad [V] = \mathrm{M}^0\mathrm{L}^1\mathrm{T}^{-1}$
  - Air density ρ $\quad [\rho] = \mathrm{M}^1\mathrm{L}^{-3}\mathrm{T}^0$
  - Air viscosity μ $\quad [\mu] = \mathrm{M}^1\mathrm{L}^{-1}\mathrm{T}^{-1}$
  - Drag force F $\quad [F] = \mathrm{M}^1\mathrm{L}^1\mathrm{T}^{-2}$
- Find a set of dimensionless parameters (how many?)
- Given a scaled wind tunnel experiment, deduce the drag force in real-life.