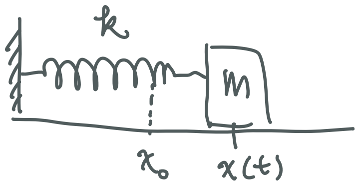
ODE time evolution eq.

Example Mass spring system



$$\begin{cases} \dot{x} = v \\ \dot{v} = -\frac{k}{m} x \end{cases} \qquad y = \begin{bmatrix} x \\ v \end{bmatrix}$$

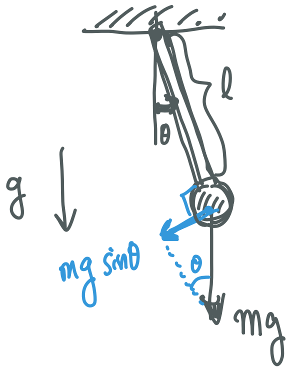
$$[\dot{y}] = \begin{bmatrix} y_2 \\ -\frac{k}{m} y_1 \end{bmatrix}$$

$$m\ddot{x}(t) = -k(x(t) - x_0)$$

$$\tilde{x}(t) := x(t) - x_0$$
$$\ddot{\tilde{x}}(t) = \ddot{x}(t)$$

$$\boxed{m\ddot{\tilde{x}} = -k\tilde{x}}$$

Example Pendulum



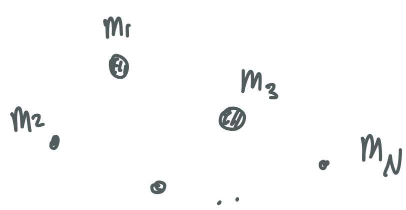
$\theta(t)$: "position"

$$\cancel{m} l \ddot{\theta} = -\cancel{m} g \sin\theta$$

$$\Rightarrow \boxed{\ddot{\theta} = -\frac{g}{l}\sin\theta}$$

$$\frac{1}{s^2} = \frac{\cancel{m}/s^2}{\cancel{m}}$$

Example n-body problem



$$m_i \ddot{\vec{x}}_i = \sum_{j \neq i} \frac{G m_i m_j (\vec{x}_i - \vec{x}_j)}{|x_i - x_j|^n} \qquad i = 1, \cdots, N$$

$$\vec{x}_i(t) \in \mathbb{R}^n$$

Example Flocking of birds

speed fixed

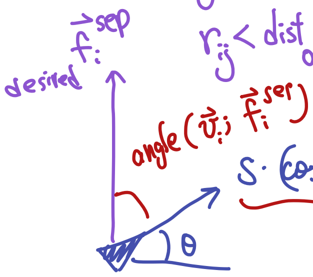
$$\vec{v}_i(t) = s\,(\cos(\theta_i(t)), \sin(\theta_i(t)))$$

$$\begin{cases} \dot{\vec{x}}_i(t) = s\,(\cos(\theta_i(t)), \sin(\theta_i(t))) \\ \dot{\theta}_i(t) = \underline{\text{steering}} \end{cases}$$

Boid's algorithm

① Separation

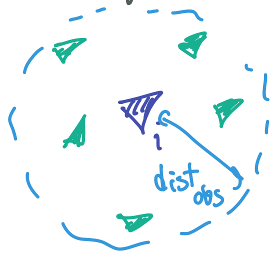
$$\vec{f}_i^{sep} = \sum_{\substack{j \neq i \\ r_{ij} < dist_{ok}}} \frac{\vec{x}_i - \vec{x}_j}{r_j^2}$$



$$\underbrace{S \cdot (\cos\theta, \sin\theta)}_{v_i}$$

$$c_{sep} \underbrace{angle(v_i ; f_i^{sep})}_{Steer_{sep}}$$

② Alignment



$$\overrightarrow{desired}_i = \left(\frac{1}{\#}\right) \sum_{\substack{j \neq i \\ r_{ij} < dist_{obs}}} \vec{v}_j$$

$$Steer_{align} = angle(\vec{v}_i, \overrightarrow{desired}_i)$$

③ Cohesion

$$\overrightarrow{avepos} = \left(\frac{1}{\#}\right) \sum_{\substack{j \neq i \\ r_{ij} < dist_{obs}}} \vec{x}_j$$

$$Steer_{coh} = angle(\vec{v}_i, \overrightarrow{avepos} - \vec{x}_i)$$

$$\underline{\dot{\theta}_i = C_{sep} Steer_{sep} + C_{align} steer_{align} + C_{coh} Steer_{coh}.}$$

Example $\quad u_1(t), \dots, u_N(t) \xrightarrow{N \to \infty} u(t, \underline{x, y})$

e.g. u temperature at pos x, y at time t

$$\underline{\frac{\partial u}{\partial t}} = k \underline{\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)}$$

Classification of ODE

$\vec{y}(t) \in \mathbb{R}^m \qquad \vec{y}(t) = \begin{bmatrix} y_1(t) \\ \vdots \\ y_m(t) \end{bmatrix}$

Order of ODE

$$\mathcal{F}_t\left(\vec{y}(t), \dot{\vec{y}}(t), \ddot{\vec{y}}(t), \cdots, \vec{y}^{\boxed{(k)}}(t) \right) = 0$$

$\ddot{y} = -\sin y$

$\mathcal{F}(u_1, u_2, u_3) = u_3 + \sin u_1$

$\mathcal{F}(y, \dot{y}, \ddot{y}) = 0$

- If $m=1$ then the ODE is a scalar ODE.

  $m > 1$: ODE system

- If $\mathcal{F}$ is linear in all of its argment, then we say the ODE is linear

  e.g. $\ddot{x}(t) = -\frac{k}{m}(x(t) - x_0)$

  $\to a(t)\ddot{x} + b(t)\dot{x} + c(t)x = d(t)$ inhomogeneous

  If $a(t)\ddot{x} + b(t)\dot{x} + c(t)x = 0$ then we say the ODE is linear & homogeneous.

  If $x_1(t)$ & $x_2(t)$ are 2 sols of hom. linear ODE then $c_1 x_1(t) + c_2 x_2(t)$ will also be a sol.

  For inhom linear ODE

  x_1 is a sol, and x_2 is a sol to its hom ODE

  $x_1 + x_2$ will still be a sol to inhom ODE.

- If F_t is time indep. Then we say ODE is autonomous.

First order or high order	Scalar or system	Linear or nonlinear	time indep or dep	
1st	scalar	lin	indep	$\dot{y} = ay \Rightarrow y = ce^{at}$
			dep	sep var
		nonlinear	indep	sep var
			dep	??
	system	lin	indep	$\dot{\vec{y}} = A\vec{y} \Rightarrow \vec{y}(t) = e^{At}\vec{c}$
			dep	$\dot{\vec{y}}(t) = A(t)\,\vec{y}(t)$?? $e^{A}e^{B} \neq e^{A+B}$, $AB \neq BA$
		nonlinear		???
high	scalar or system	lin nonlinear		

$$\dot y = ay \Rightarrow \frac{dy}{dt} = a\,y \overset{\text{sep of var}}{\Rightarrow} \frac{dy}{y} = a\,dt$$

$$\Rightarrow \int \frac{dy}{y} = \int a\,dt \Rightarrow \ln(y) + c = at$$

$$\Rightarrow y\,e^{c} = e^{at}$$

$$\Rightarrow y(t) = \tilde{c} \cdot e^{at}.$$

Inhom case

$$\dot y = a(t)\,y + b(t)$$

Let $h(t)$ solve $\dot h = a(t)\,h$

to solve

Then we can solve $y(t)$ by $y(t) = h(t)\,v(t)$

$$(hv)^{\cdot} = a(hv) + b$$

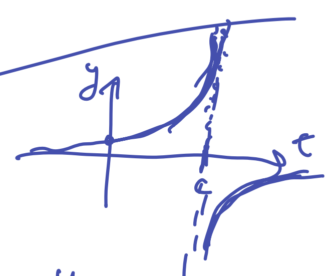
$$\cancel{\dot h v} + h\dot v = \cancel{ah}\,v + b \qquad \dot h + ah = 0$$

$$\dot v = \frac{b}{h} \qquad v = \int \frac{b}{h}\,dt$$

$$y = h(t)\int \frac{b}{h}\,dt.$$

$$\dot y = a(t)\,y \qquad \frac{dy}{y} = a(t) \Rightarrow \ln y = \int^t a(t) + \tilde{C}$$

$$y = C e^{\int^t a(t)}$$



$$\dot y(t) = y^2(t) \Rightarrow \frac{dy}{y^2} = dt \Rightarrow -\frac{1}{y} = t - c \Rightarrow y = \frac{1}{c-t}$$

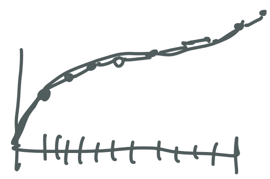
$$\frac{dy(t)}{dt} = y^2(t) + t \quad ??$$

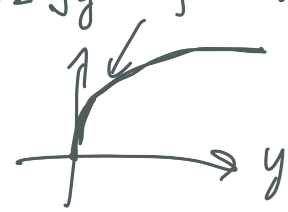
All ODEs can be written as 1st order (nonlinear) system

$$\begin{cases} \dot{\vec{y}} = f(t, \vec{y}) \\ \vec{y}(0) = \vec{y}_0 \end{cases} \qquad f : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$$

Thm (Peano existence thm)

If f is continuous, then sol exists.



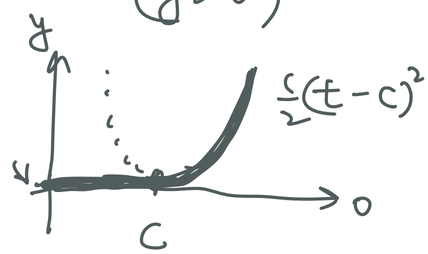


$\dot{y} = \sqrt{y}$ $\quad (y \geq 0)$

$\frac{dy}{\sqrt{y}} = dt$

$\Rightarrow 2\sqrt{y} = t - c$

$\Rightarrow y = \frac{(t-c)^2}{2}$



Sol not unique

Thm (Picard existence & uniqueness Thm)

If f is Lipschitz continous "f' is bounded"

then sol exists and is unique.