1.1 Dimensional Analysis

Exercise 1.1 — 10 pts. Consider a liquid in a cylindrical container that is rotating at a certain frequency $\Omega$ (e.g. counted as rounds per minute). At equilibrium, the liquid surface will have an elevation difference $h$ between the center and the rim. We postulate that this height $h$ is a function of the rotating frequency $\Omega$, the radius $R$ of the container, the liquid’s density $\rho$, and the gravitational acceleration $g$. Use dimensional analysis to answer the following questions.

(a) We already know that the aspect ratio $\Pi_1 = \frac{h}{R}$ is one of the dimensionless variables. Using the Buckingham $\Pi$ Theorem, this aspect ratio is determined by some other dimensionless variables $\Pi_1 = f(\Pi_2, \ldots)$. How many and what are these other dimensionless variables? (Choose $\Pi_2, \ldots$ so that they do not involve $h$ by a suitable linear combination of the null vectors of the dimension matrix.)

(b) Does the aspect ratio $\Pi_1 = \frac{h}{R}$ depend on the density $\rho$?

(c) Perform the rotating bucket experiment on both Earth and Mars using the same bucket size $R$. To achieve the same aspect ratio ($\frac{h}{R}$) for the liquid surfaces on both planets, how much faster or slower should the rotation frequency ($\Omega$) on Mars be compared to the rotation frequency on Earth? (Earth’s gravity is 9.806 m/s$^2$, while Mars’ gravity is 3.721 m/s$^2$.)
1.2 Vectors, Covectors, Differentials, and Adjoint

Exercise 1.2 — 20 pts. Consider a Cartesian–Euclidean $xy$-plane and an alternative coordinate system in the first quadrant defined by the following two coordinate functions:

$$
\theta_1 = \sqrt{xy}, \quad \theta_2 = \sqrt{\frac{x}{y}}. \tag{1}
$$

This can be expressed equivalently as:

$$
x = \theta_1 \theta_2, \quad y = \frac{\theta_1}{\theta_2} \tag{2}
$$

assuming that $\theta_1, \theta_2, x$, and $y$ are all positive. The differentials of the coordinate functions yield two covector bases, $(dx, dy)$ and $(d\theta_1, d\theta_2)$. A coordinate vector basis is defined as the dual basis of a covector basis. Specifically, $(\vec{e}_x, \vec{e}_y)$ is the dual basis to $(dx, dy)$, and $(\vec{e}_{\theta_1}, \vec{e}_{\theta_2})$ is the dual basis to $(d\theta_1, d\theta_2)$.

(a) Establish the transformation matrices for the change of covector bases. That is, what are the coefficients $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ in the following relationships?

$$
\begin{align*}
&dx = a_{11} d\theta_1 + a_{12} d\theta_2, & dy = a_{21} d\theta_1 + a_{22} d\theta_2 \tag{3} \\
&d\theta_1 = b_{11} dx + b_{12} dy, & d\theta_2 = b_{21} dx + b_{22} dy. \tag{4}
\end{align*}
$$

More concisely,

$$
\begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} d\theta_1 \\ d\theta_2 \end{bmatrix}, \quad \begin{bmatrix} d\theta_1 \\ d\theta_2 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}. \tag{5}
$$
Please express both \( a_{ij} \)’s and \( b_{ij} \)’s in terms of \( \theta_1 \) and \( \theta_2 \).

**Hint** Matrix inversion.

(b) What are the coefficients \( \mathbf{F} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \) and \( \mathbf{G} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \) as expressions of \( \theta_1, \theta_2 \) that relate the coordinate vectors?

\[
\begin{bmatrix} \vec{e}_{\theta_1} & \vec{e}_{\theta_2} \end{bmatrix} = \begin{bmatrix} \vec{e}_x & \vec{e}_y \end{bmatrix} \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}, \quad \begin{bmatrix} \vec{e}_{\theta_1} & \vec{e}_{\theta_2} \end{bmatrix} = \begin{bmatrix} \vec{e}_x & \vec{e}_y \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}.
\]

(6)

**Hint** Use the results from (a).

(c) Consider the function \( U = x^2 + y^2 = \theta_1^2 \theta_2^2 + \theta_1^2 \). Given that \( dU = 2xdx + 2ydy = 2\theta_1 \theta_2 dx + 2\theta_1 \theta_2 dy \) (since \( \frac{\partial U}{\partial x} = 2x \) and \( \frac{\partial U}{\partial y} = 2y \)), compute the partial derivatives \( \frac{\partial U}{\partial \theta_1} \) and \( \frac{\partial U}{\partial \theta_2} \) of \( U \) with respect to the \( (\theta_1, \theta_2) \) coordinate system.

**Hint** By definition \( dU = \frac{\partial U}{\partial \theta_1} d\theta_1 + \frac{\partial U}{\partial \theta_2} d\theta_2 \). Apply one of the transformation matrices from (a), (b).

(d) Consider again \( U = x^2 + y^2 = \theta_1^2 \theta_2^2 + \theta_1^2 \). What is the gradient vector \( \vec{v} = \nabla U = -\vec{e}_{\theta_1} + \vec{e}_{\theta_2} \) written in the \( (\theta_1, \theta_2) \) coordinate system? Note that \( \vec{e}_x, \vec{e}_y \) are orthonormal, so \( \vec{v} = \nabla U = 2x \vec{e}_x + 2y \vec{e}_y \).

\( \blacksquare \)

**Exercise 1.3 — 10 pts.** Let \( U, V \) be two vector spaces of the same dimension. The space \( \text{Hom}(U, V) \) of all linear maps, \( \text{Hom}(U, V) = \{ \mathbf{X} : U \overset{\text{linear}}{\rightarrow} V \} \), from \( U \) to \( V \) forms a vector space itself. The dual space \( \text{Hom}(U, V)^* \) is indeed \( \text{Hom}(U, V)^* = \text{Hom}(V, U) = \{ \mathbf{Y} : V \overset{\text{linear}}{\rightarrow} U \} \), where the dual pairing \( \langle \mathbf{Y} | \mathbf{X} \rangle \) for \( \mathbf{Y} \in \text{Hom}(V, U) \) and \( \mathbf{X} \in \text{Hom}(U, V) \) is given by

\[
\langle \mathbf{Y} | \mathbf{X} \rangle = \text{tr}(\mathbf{Y} \mathbf{X}) = \text{tr}(\mathbf{X} \mathbf{Y}).
\]

Let \( \mathbf{B} \in \text{Hom}(V, V^*) \) be a given symmetric bilinear form, with \( \mathbf{B}^* = \mathbf{B} \). Consider the following (nonlinear) function:

\[
f : \text{Hom}(U, V) \rightarrow \text{Hom}(U, U^*), \quad f(\mathbf{X}) := \mathbf{X}^* \mathbf{B} \mathbf{X}.
\]

(8)

(a) What is the differential \( df|_\mathbf{X} : \text{Hom}(U, V) \overset{\text{linear}}{\rightarrow} \text{Hom}(U, U^*) \) of \( f \) at \( \mathbf{X} \)? That is, for a given perturbation direction \( \mathbf{E} \in \text{Hom}(U, V) \), compute \( df|_\mathbf{X}[\mathbf{E}] = \frac{d}{d\epsilon} f(\mathbf{X} + \epsilon \mathbf{E}) \bigg|_{\epsilon=0} \).

(b) What is the adjoint \( (df|_\mathbf{X})^* : \text{Hom}(U^*, U) \overset{\text{linear}}{\rightarrow} \text{Hom}(V, U) \) of the linear map \( df|_\mathbf{X} \) from part (a)? That is, for a given \( \mathbf{S} \in \text{Hom}(U^*, U) \), compute \( (df|_\mathbf{X})^*[\mathbf{S}] \), which should satisfy \( \langle (df|_\mathbf{X})^*[\mathbf{S}] | \mathbf{E} \rangle = \langle \mathbf{S} | df|_\mathbf{X}[\mathbf{E}] \rangle \) for all \( \mathbf{E} \in \text{Hom}(U, V) \).

**Hint** The trace operator satisfies linearity \( \text{tr}(\mathbf{A}_1 + \mathbf{A}_2) = \text{tr}(\mathbf{A}_1) + \text{tr}(\mathbf{A}_2) \), cyclic permutability \( \text{tr}(\mathbf{A}_1 \mathbf{A}_2 \cdots \mathbf{A}_n) = \text{tr}(\mathbf{A}_2 \cdots \mathbf{A}_n \mathbf{A}_1) \), and \( \text{tr}(\mathbf{A}) = \text{tr}(\mathbf{A}^*) \).

\( \blacksquare \)