Recall that the least action principle:

A physical path $q:[0,T]\to\mathbb{R}^m$ is a critical path of the action $\int_0^T L(q(t),\dot{q}(t))\,dt$

The Euler–Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i} \qquad i=1,\cdots,m.$$

Define $p_i = \dfrac{\partial L(\vec{q},\vec{v})}{\partial v_i}$ called momentum.

Fix q. Then $p(q,v)=F_q(v)$ is a function of $\vec{v}$.

$$F_q : T_qM \longrightarrow T_q^*M$$
$$\vec{v} \longmapsto \vec{p}$$

Let G_q be the inverse function of F_q

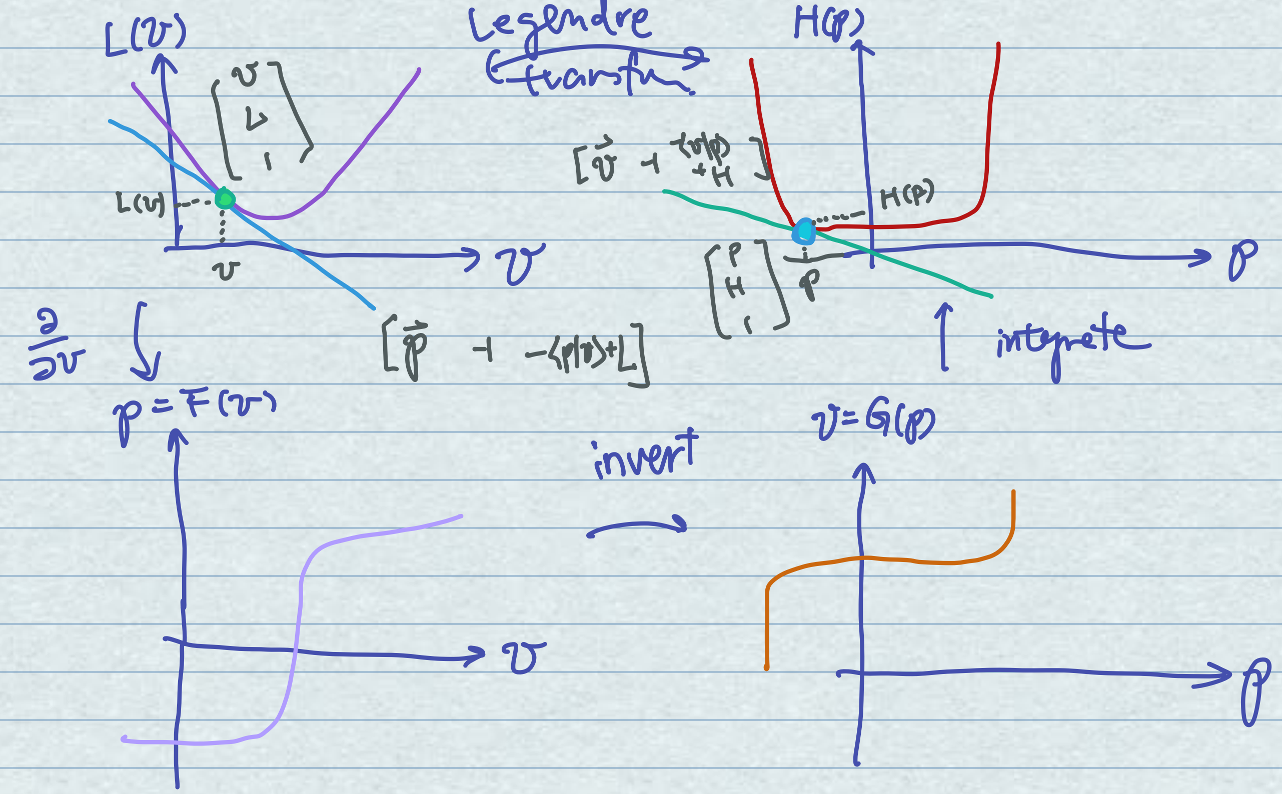
$$G_q : T_q^*M \longrightarrow T_qM. \qquad G_q\circ F_q = id$$
$$\vec{p} \longmapsto \vec{v}$$

<u>Theorem</u> If $F_q = \dfrac{\partial L(q,v)}{\partial v}$ is a pure gradient

then so is its inverse fun. i.e.

there is a function $H(q\ p)$ so that

$$G_q = \frac{\partial H(q,p)}{\partial p}.$$



Explicitly $$H(q,p) = \sup_v \langle p|v\rangle - L(q,v)$$

Solve this maximization : $v^* = \operatorname{argmax} \langle p|v\rangle - L(q,v)$

which should satisfy $$\left.\frac{\partial(\langle p|v\rangle - L(q,v))}{\partial v}\right|_{v=v^*} = 0$$

$$p - \left.\frac{\partial L}{\partial v}\right|_{v^*} = 0$$

$$H(q,p) = \langle p|v^*\rangle - L(q,v^*)$$

Check $\frac{\partial H}{\partial p}$: $$\frac{\partial H}{\partial p} = |v^*\rangle + \langle p|\frac{\partial v^*}{\partial p}\rangle - \langle \underbrace{\frac{\partial L}{\partial v}(q,v^*)}_{p} | \frac{\partial v^*}{\partial p}\rangle$$

$$= |v^*\rangle.$$

Check $\frac{\partial H}{\partial q} = -\frac{\partial L}{\partial}$

Hamilton's equation

EL: $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i} \Rightarrow \begin{cases} \frac{d}{dt} p_i = \frac{\partial L}{\partial q_i} = -\frac{\partial H}{\partial q_i} \\ \frac{d}{dt} q_i = \frac{\partial H}{\partial p_i} \end{cases}$

We call $\vec{q}-\vec{p}$ space the <u>phase space</u> $\mathcal{P}$

Hamiltonian H is a function defined on the phase space.

<u>Example</u>

$$L = \frac{m}{2}|\dot{q}|^2 - U(q)$$

$$H(q,p) = \sup_v \underbrace{\langle p|v\rangle - \frac{m}{2}|v|^2}_{mv^* = p \quad v^* = \frac{p}{m}} + U(q)$$

$$\downarrow$$

$$H(q,p) = \langle p | \tfrac{p}{m}\rangle - \frac{m}{2}\left|\frac{p}{m}\right|^2 + U(q)$$

$$= \frac{|p|^2}{2m} + U(q).$$

<u>Poisson system.</u>

Let f be some function on the phase space $\mathcal{P}$

Let $\gamma(t) \in \mathcal{P}$ follow $\begin{cases} \dot{q} = \frac{\partial H}{\partial p} \\ \dot{p} = -\frac{\partial H}{\partial q} \end{cases}$

What is $\frac{d}{dt}(f\circ\gamma)$?

$$\frac{d}{dt}(f\circ\gamma) = \left\langle \frac{\partial f}{\partial q} \middle| \dot{q} \right\rangle + \left\langle \frac{\partial f}{\partial p} \middle| \dot{p} \right\rangle = \left\langle \frac{\partial f}{\partial q} \middle| \frac{\partial H}{\partial p} \right\rangle - \left\langle \frac{\partial f}{\partial p} \middle| \frac{\partial H}{\partial q} \right\rangle$$

$$=: \{f, H\}.$$

Def The <u>Poisson bracket</u> $\{\cdot,\cdot\} : C^\infty(P) \times C^\infty(P) \xrightarrow{\text{bilinear}} C^\infty(P)$ defined by $\{f, g\} = \left\langle \frac{\partial f}{\partial q} \middle| \frac{\partial g}{\partial p} \right\rangle - \left\langle \frac{\partial f}{\partial p} \middle| \frac{\partial g}{\partial q} \right\rangle$.

($\{f, g\} = -\{g, f\}$ and
$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$
$\{f, \cdot\} : C^\infty \to C^\infty$ is a derivation)

Geometry of phase space

Def A symplectic form is 2-form $\sigma \in \Omega^2(M)$
such that
- $d\sigma = 0$ (closed)
- Non-degenerate $i_X \sigma \neq 0$ $\forall X \in T_pM$.

[For example $M = \mathbb{R}^n \times \mathbb{R}^n$ with coordinate $q_1, \dots, q_n$, $p_1, \dots, p_n$

$$\sigma = dp_1 \wedge dq_1 + dp_2 \wedge dq_2 + \dots + dp_n \wedge dq_n$$

Let f be any function $f: M \to \mathbb{R}$
$\Rightarrow df \in \Omega^1(M)$.

Define its "symplectic gradient"

sgrad $f \in \Gamma(TM)$

by

$\Longleftrightarrow \quad i_{\text{sgrad} f}\,\sigma = df$

$$\sigma(\text{sgrad} f, X) = df[X] \quad \forall X \in \Gamma(TM)$$

(This is similar to how gradient is defined

$$\langle \text{grad} f, X \rangle = df$$

↖ non-degenerate symmetric bilinear form)

Similar to gradient descent, consider

$$\dot{x} = -\,\text{sgrad}\, f\big|_x .$$

<u>Example</u> $M = \mathbb{R}^n \times \mathbb{R}^n$, coord: $(\vec{q}, \vec{p})$, $\sigma = \sum_i dp_i \wedge dq_i$.

cord vectors $(\vec{e}_{q_1}, \cdots, \vec{e}_{q_n}, \vec{e}_{p_1}, \cdots, \vec{e}_{p_n})$

Let H be any function $H(q, p)$

$$dH = \frac{\partial H}{\partial q_1} dq_1 + \cdots + \frac{\partial H}{\partial q_n} dq_n + \frac{\partial H}{\partial p_1} dp_1 + \cdots + \frac{\partial H}{\partial p_n} dp_n .$$

$$Y = \text{sgrad}\, H = Y_{q_1} \vec{e}_{q_1} + \cdots + Y_{q_n} \vec{e}_{q_n} + \cdots + Y_{p_n} \vec{e}_{p_n} .$$

$$i_Y \sigma = \sum_{i=1}^{n} i_Y\, dp_i \wedge dq_i = \sum_{i=1}^{n} Y_{p_i} dq_i - Y_{q_i} dp_i$$

Comparing $i_Y \sigma = dH$ we get

$$Y_{p_i} = \frac{\partial H}{\partial q_i} \qquad Y_{q_i} = -\frac{\partial H}{\partial p_i}$$

$\Rightarrow$ The sgrad descent is

$$\frac{d}{dt}\begin{bmatrix} q \\ p \end{bmatrix} = \begin{bmatrix} \frac{\partial H}{\partial p} \\ -\frac{\partial H}{\partial q} \end{bmatrix}.$$

Liouville's thm

Under the ODE system $\dot{y} = \text{sgrad}\, H$,

$$\mathcal{L}_{-\text{sgrad} H}\, \sigma = 0 \quad \text{i.e. } \sigma \text{ is preserved}$$

If $\sigma = \sum_i dp_i \wedge dq_i = d\left(\sum_i p_i\, dq_i\right)$

i.e. $\int_{S_t} \sigma = \int_{S_0} \sigma$ (S_t ← 2D surface flowing with the flow)

$\sum_i \oint_{C_t} p_i\, dq_i$ is time-indep.

(pf) $\mathcal{L}_{\text{sgrad} H}\, \sigma = i_{\text{sgrad} H} \underbrace{d\sigma}_{0} + d\, \underbrace{i_{\text{sgrad} H}\, \sigma}_{dH}$

$= d(dH) = 0.$

The $2n$-D volume given by $\sigma \wedge \cdots \wedge \sigma$ is also conserved.

$$\left(\mathcal{L}_Y(\sigma \wedge \cdots \wedge \sigma) = (\mathcal{L}_Y \sigma) \wedge \sigma \cdots \wedge \sigma + \cdots + \sigma \wedge \cdots \wedge \sigma \wedge (\mathcal{L}_Y \sigma)\right)$$

$$= 0$$

So our ODE flow is divergence-free on M.

Noether's thm

Back to the Poisson bracket.

An observable f is conserved under the Hamiltonian system generated by H if $\{H, f\} = 0$

Or $df[\text{sgrad}\, H] = 0$

$$= \sigma[\text{sgrad}\, f, \text{sgrad}\, H] = 0$$

So we have a nice expression

$$\{f, g\} = \sigma(\text{sgrad}\, f, \text{sgrad}\, g).$$

<u>Thm</u> H is conserved along the flow.

<u>Thm</u> If H is conserved along the flow generated by G, then G is conserved along the flow generated by H.

<u>Ex</u> $p_1 : M \to \mathbb{R}$ (coord function reading off the momentum of the 1st component)

- $\text{sgrad}\, p_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ \hline 0 \\ \vdots \\ 0 \end{bmatrix}$ translation in the 1st direction

$\Rightarrow$ If H is translationally invariant, the dynamical system preserves momentum.

Kirchhoff point vortex dynamics.

Vortex locations $(\underbrace{x_1, y_1}_{\vec{x}_1}, \cdots, x_n, y_n)$ Vortex strength $\kappa_1, \cdots, \kappa_n$.

$$\frac{d}{dt}\begin{bmatrix} x_i \\ y_i \end{bmatrix} = \sum_{j \neq i} \frac{\kappa_j}{2\pi} \frac{\begin{pmatrix} -y_i + y_j \\ x_i - x_j \end{pmatrix}}{\left| \begin{pmatrix} x_i \\ y_i \end{pmatrix} - \begin{pmatrix} x_j \\ y_j \end{pmatrix} \right|^2}.$$

This is actually a Hamiltonian system.

$$M = \{(\vec{x}_1, \cdots, \vec{x}_n)\} = \mathbb{R}^{2n}$$

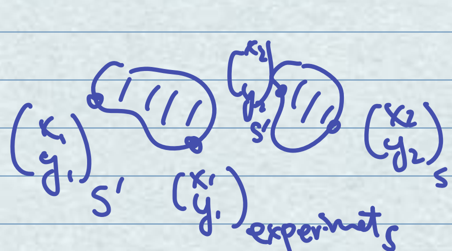
$$\sigma\left[\!\left[\begin{pmatrix} \vec{u}_1 \\ \vdots \\ \vec{u} \end{pmatrix}, \begin{pmatrix} \vec{v}_1 \\ \vdots \\ \vec{v}_n \end{pmatrix}\right]\!\right] := \sum_i \frac{\kappa_i}{2} \underbrace{\left(R^{90^\circ}\vec{u}_i \cdot \vec{v}_i\right)}_{\det\begin{bmatrix} | & | \\ \vec{u}_i & \vec{v}_i \\ | & | \end{bmatrix} = \vec{u} \times \vec{v}_i}$$

$$H = \sum_{i \neq j} \frac{-\kappa_i \kappa_j}{4\pi} \ln |\vec{x}_i - \vec{x}_j| \quad \text{(Kirchhoff–Routh function)}$$

$$dH\left[\!\left[\begin{bmatrix} 0 \\ \vdots \\ \vec{v}_i \\ \vdots \\ 0 \end{bmatrix}\right]\!\right] = \sum_{i \neq j} -\frac{\kappa_i \kappa_j}{4\pi} \frac{\langle \vec{x}_i - \vec{x}_j, \vec{v}_i \rangle}{|\vec{x}_i - \vec{x}_j|^2}$$

$$\Rightarrow (\operatorname{sgrad} H)_i = \sum_i \frac{\kappa_j}{2\pi} R^{90^\circ} \frac{(\vec{x}_i - \vec{x}_j)}{|\vec{x}_i - \vec{x}_j|^2}$$

Liouville thm $\Rightarrow$ Consider a loop of experiments



The κ-weighted area will be conserved.

Noether's thm

- H is conserved.

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$\oplus$

- $A_x := \sum_i \kappa_i x_i$ $\qquad dA_x\begin{bmatrix} \vdots \\ v_i \\ \vdots \end{bmatrix} = \kappa_i v_i$

  $A_y := \sum_i \kappa_i y_i$ $\qquad \Rightarrow$ sgrad A_x = translation in y_i direction

  H is translationally invariant

  $\Rightarrow A_x, A_y$ are conserved.

- $B = \sum_i \kappa_i (x_i^2 + y_i^2)$ is also conserved

  (rotation invariance of H)