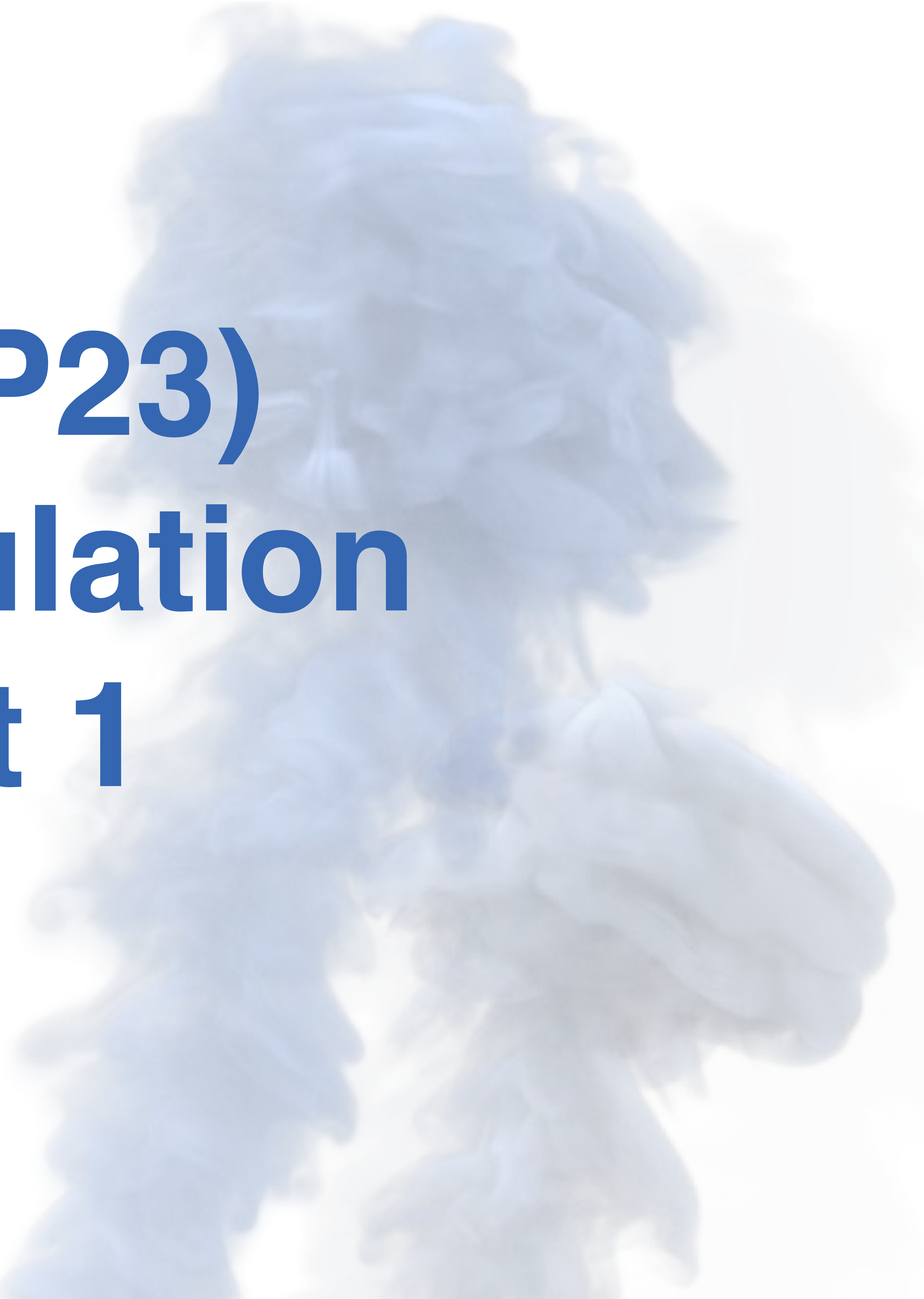
CSE 291 (SP23)
Physical Simulation
Fluid: Part 1

Albert Chern



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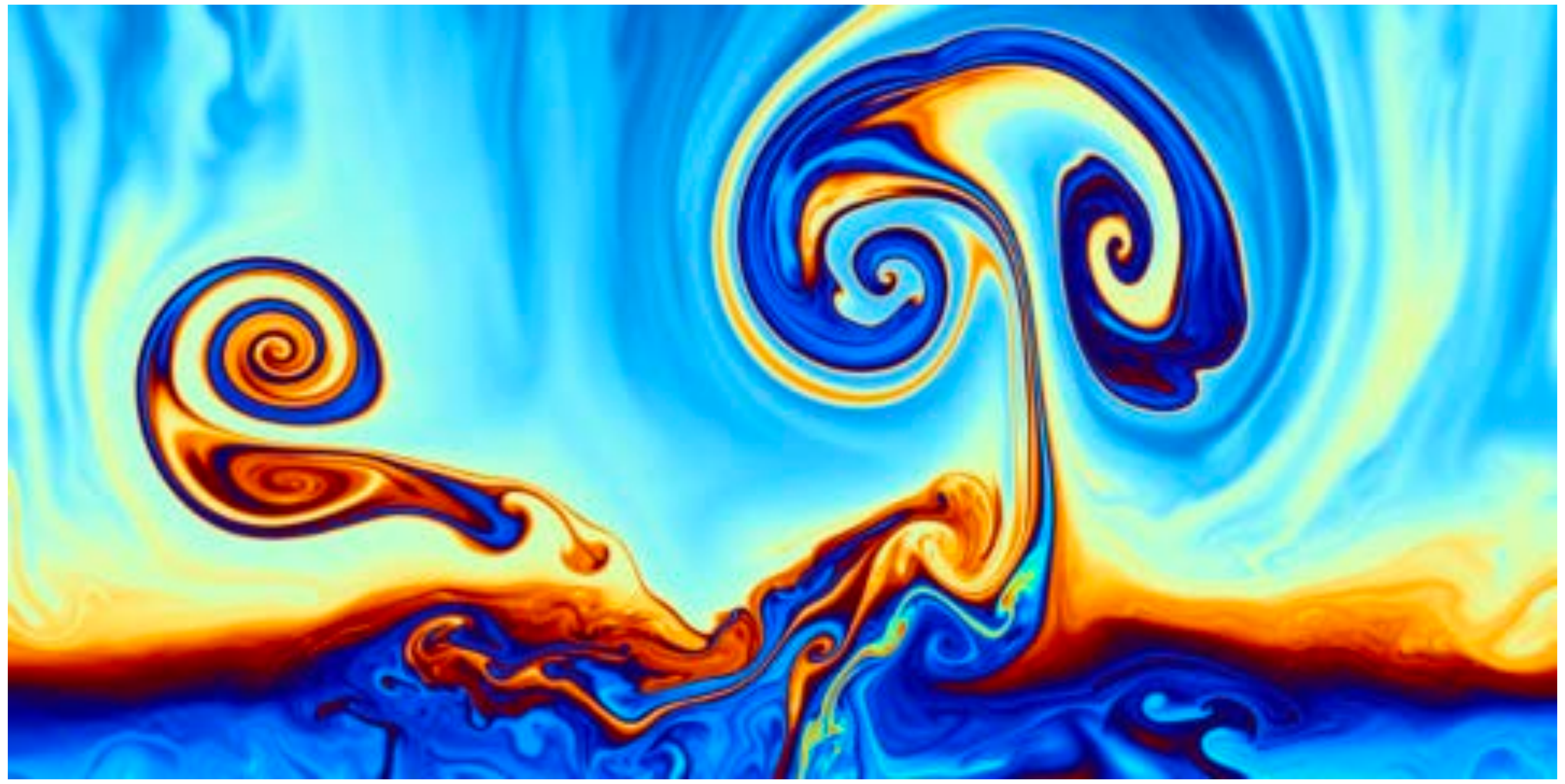
Fluid as a special type of elastic body

- Fluid as elastic body
- Velocity and acceleration
- Navier–Stokes equations
- Lie derivative
  - Cartan's formula
  - Lie material derivative
- Vorticity

Fluid as an elastic body

- A fluid is an elastic body whose potential only depends on the change of volume.
- There is no resistance in volume-preserving shearing.
- One may model dissipation function as a function of rate of change of deformation, resulting in viscosity



Flow map

- Recall the setup for deformable body
  - Manifold M : **Material** coordinate, **Lagrangian** coordinate
  - Manifold W : **World** coordinate, **Eulerian** coordinate
  - A state of a deformable body is a map (called **flow map**)

$$\phi_t : M \to W$$

- We also have time-independent fluid mass density $\rho_M \in \Omega^n(M)$ and time-independent world coordinate metric $\flat_W \in \Gamma(T^*W \odot T^*W)$

- For fluid $M = W$, and Lagrangian coordinate represent state at *time = 0*, and Eulerian coordinate represent *time = t*.

Fluid as an elastic body

- Deformation gradient $\mathbf{F} = d\phi$
- Cauchy–Green $\sharp_M \mathbf{C} = \sharp_M \mathbf{F}^\intercal \flat_W \mathbf{F}$
- Potential energy $U(\mathbf{C}) = w(\det(\mathbf{C}))$ $\qquad \mathbf{C} = \mathbf{F}^\intercal \mathbf{F}$
- 2nd Piola $\mathbf{S} = 2\frac{\partial U}{\partial \mathbf{C}} = 2w'_{(\det(\mathbf{C}))} \det(\mathbf{C})\mathbf{C}^{-\intercal}$
  $= 2w'_{(\det(\mathbf{C}))} \det(\mathbf{C})\mathbf{F}^{-1}\mathbf{F}^{-\intercal}$
- 1st Piola $\mathbf{P} = \mathbf{F}\mathbf{S} = 2w'_{(\det(\mathbf{C}))} \det(\mathbf{C})\mathbf{F}^{-\intercal}$
  $= 2w'_{J^2} J^2 \mathbf{F}^{-\intercal}$ $\qquad J = \det(\mathbf{F})$
- Cauchy stress $\boldsymbol{\sigma} = (2w'_{J^2} J)\mathbf{I} = -p\mathbf{I}$ $\qquad$ p is called **pressure**

Fluid as an elastic body

- Equation of motion

$$\rho_M \overset{\nabla}{\ddot{\phi}} = -J(\mathrm{grad}\, p) \circ \phi$$

- Let μ_M, μ_W be volume forms on Lagrangian and Eulerian space.
- The determinant $J = \frac{\phi^* \mu_W}{\mu_M}$
- Let $\rho_t \in \Omega^n(W)$ be mass density so that $\rho_M = \phi_t{}^* \rho_t$
- Let $q_t = \frac{\rho_t}{\mu_W}$ $\qquad q \circ \phi = \frac{1}{J} \frac{\rho_M}{\mu_M}$
- The pressure p is only a function of scalar mass density q

Velocity and Acceleration Fields

- Fluid as elastic body
- Velocity and acceleration
- Navier–Stokes equations
- Lie derivative
  - Cartan's formula
  - Lie material derivative
- Vorticity

Flow map

- Let's look at the time-derivative of flow map

$$\dot{\phi}_{(t,\mathbf{x})} = \frac{\partial}{\partial t}\phi_{(t,\mathbf{x})}$$

  - Since the domain of ϕ is the Lagrangian coordinate, the partial time derivative means fixing the Lagrangian spatial variable.
  - This time derivative measures the velocity of a particle.
  - Type $\dot{\phi}_{(t,\mathbf{x})} \in T_{\phi(\mathbf{x})}W$

$$\dot{\phi}_t \in \Gamma(T_\phi W)$$

Velocity field

- The **velocity field** is defined on the Eulerian (world) coordinate

$$\vec{u}_{(t,\mathbf{x})} \in \Gamma(T_{\mathbf{x}}W) \qquad \vec{u}_t \in \Gamma(TW)$$

  - It is the change of coordinate of

$$\dot{\phi}_{(t,\mathbf{X})} \in T_{\phi(\mathbf{X})}W \qquad \dot{\phi}_t \in \Gamma(T_\phi W)$$

i.e.

$$\dot{\phi}_{(t,\mathbf{X})} = \vec{u}_{(t,\phi(\mathbf{X}))} \qquad \dot{\phi} = \vec{u} \circ \phi$$

$$\dot{\phi} = \underbrace{\phi^* \ \vec{u}}_{\text{0-form}}$$

Acceleration of particle

- What is acceleration of particle?
  - Is it time derivative of velocity $\dot{\phi}$ or velocity $\vec{u}$?
  - Since the spatial domain for $\vec{u}$ is W, its time derivative $\dot{\vec{u}} = \frac{\partial}{\partial t}\vec{u}$ fixes a world location instead of following a fixed particle.
  - The acceleration of a particle is

$$\overset{\triangledown}{\ddot{\phi}} = \frac{\triangledown}{\partial t}\frac{\partial}{\partial t}\phi$$

Acceleration of particle

- The acceleration of a particle is

$$\overset{\triangledown}{\ddot{\phi}} = \frac{\triangledown}{\partial t} \frac{\partial}{\partial t} \phi$$

- In terms of world-space velocity field

$$\begin{aligned} \frac{\triangledown}{\partial t} \dot{\phi} = \frac{\triangledown}{\partial t} (\vec{u} \circ \phi) &= (\tfrac{\partial}{\partial t} \vec{u}) \circ \phi + (\nabla \vec{u})_{\phi} [\![\dot{\phi}]\!] \\ &= (\tfrac{\partial}{\partial t} \vec{u}) \circ \phi + (\nabla \vec{u})_{\phi} [\![\vec{u} \circ \phi]\!] \\ &= (\tfrac{\partial}{\partial t} \vec{u}) \circ \phi + (\nabla_{\vec{u}} \vec{u}) \circ \phi \\ &= (\tfrac{\partial}{\partial t} \vec{u} + \nabla_{\vec{u}} \vec{u}) \circ \phi \end{aligned}$$

Summary

- Velocity field as Eulerian-valued function on Lagrangian coordinate

$$\dot{\phi}_t \in \Gamma(T_\phi W)$$

- Acceleration field as Eulerian-valued function on Lagrangian coord.

$$\overset{\nabla}{\ddot{\phi}} \in \Gamma(T_\phi W)$$

- Velocity field in Eulerian coordinate

$$\vec{u} \in \Gamma(TW) \qquad \dot{\phi} = \underset{\text{0-form}}{\phi^*} \vec{u}$$

- Acceleration field in Eulerian coordinate

$$\frac{\partial}{\partial t}\vec{u} + \nabla_{\vec{u}}\vec{u} \in \Gamma(TW) \qquad \overset{\nabla}{\ddot{\phi}} = \underset{\text{0-form}}{\phi^*} \left(\frac{\partial}{\partial t}\vec{u} + \nabla_{\vec{u}}\vec{u}\right)$$

Material derivative

$$\frac{\partial}{\partial t}\vec{u} + \nabla_{\vec{u}}\vec{u} \in \Gamma(TW)$$

- The derivative on Eulerian coordinate

$$\frac{D}{Dt} = \left(\frac{\partial}{\partial t} + \nabla_{\vec{u}}\right)$$

is called **Material derivative**.
It corresponds to the time derivative when fixing the material coord.

$$\frac{\partial}{\partial t}\Big(\underbrace{\phi^* \ f}_{\text{0-form}}\Big) = \underbrace{\phi^*}_{\text{0-form}}\left(\frac{D}{Dt}f\right)$$

- In general (we'll see later on Lie derivatives)

$$\frac{\partial}{\partial t}(\phi^* f) = \phi^*\left[\left(\frac{\partial}{\partial t} + \mathscr{L}_{\vec{u}}\right)f\right]$$

Navier–Stokes Equations

- Fluid as elastic body
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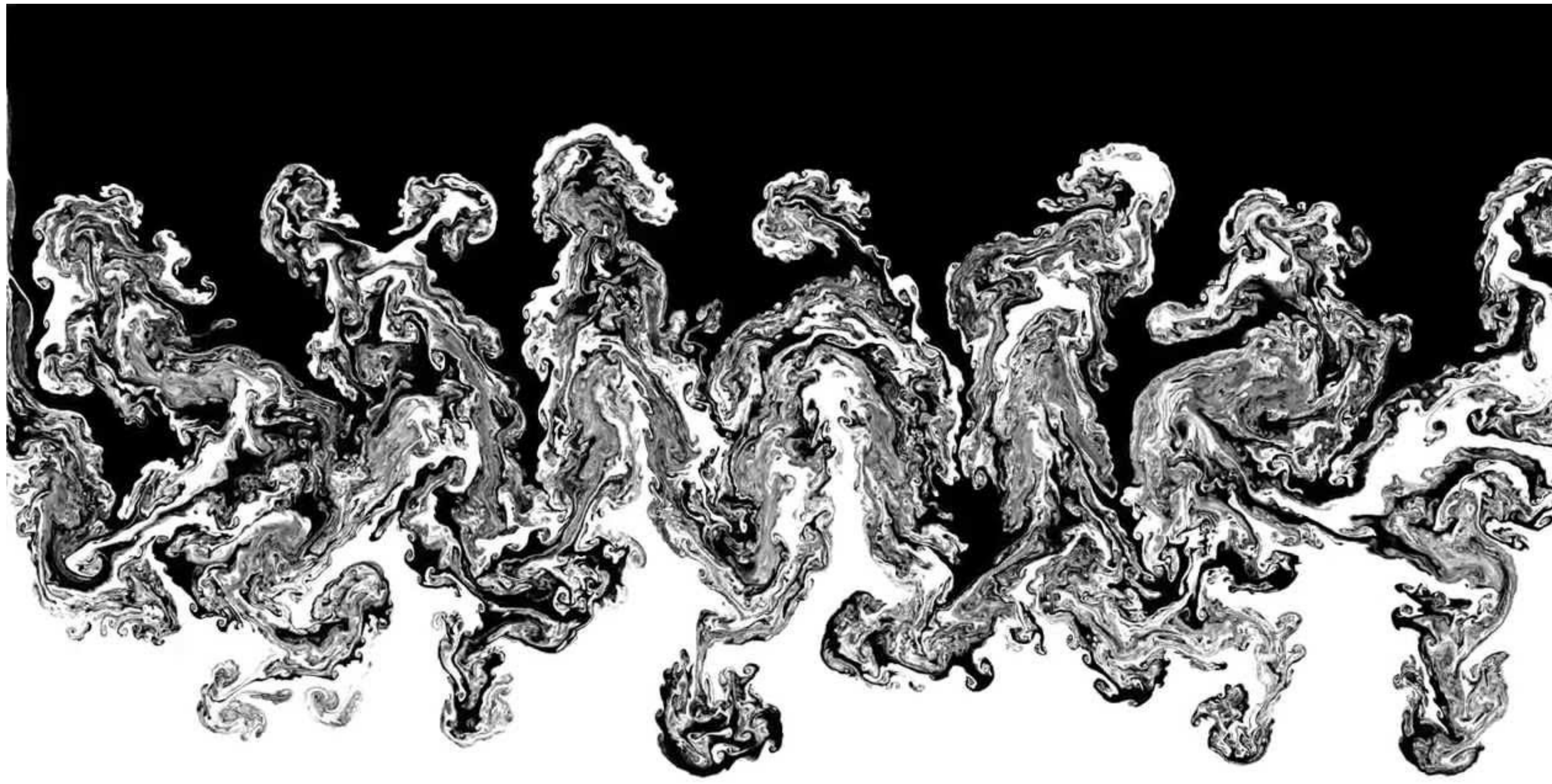
Equation of motion in Eulerian coord.

- Unlike standard elastic solid, flow map becomes “bad” function very quickly.
- It is better to use Eulerian coordinate and describe how velocity field evolve.



- Since the energy never depends on the deformation other than J (encoded in density), the fluid just keeps forgetting the initial Lagrangian coordinate.
- Constantly resetting Lagrangian space = pure Eulerian description.

Equation of motion in Eulerian coord.

- Equation of motion $\rho_M \overset{\triangledown}{\ddot{\phi}} = -J(\operatorname{grad} p) \circ \phi$

$$J = \frac{\phi^* \mu_W}{\mu_M} \qquad q_t = \frac{\rho_t}{\mu_W} \qquad \overset{\triangledown}{\ddot{\phi}} = \underset{\text{0-form}}{\phi^*} \left(\frac{\partial}{\partial t}\vec{u} + \nabla_{\vec{u}}\vec{u}\right)$$

- Write in Eulerian coordinate

$$\left(\frac{\partial}{\partial t}\vec{u} + \nabla_{\vec{u}}\vec{u}\right) = -\frac{1}{q}\operatorname{grad} p$$

Momentum equation

- The scalar density field satisfies

$$\frac{\partial}{\partial t} q + \operatorname{div}(q\vec{u}) = 0$$

Continuity equation
(derived later)

- Some density–pressure relation

$$p = \pi(q)$$

Equation of state

Euler equation

$$\left(\frac{\partial}{\partial t}\vec{u} + \nabla_{\vec{u}}\vec{u}\right) = -\frac{1}{q}\operatorname{grad} p$$

Momentum equation

$$\frac{\partial}{\partial t}q + \operatorname{div}(q\vec{u}) = 0$$

Continuity equation
(derived later)

$$p = \pi(q)$$

Equation of state

- Together, these are called (compressible) **barotropic Euler equation**
- Suppose the density–pressure relation is very stiff, to the point that fluid's volume is rigid, then we get an **incompressible fluid**.

$$J = 1 \qquad q = \text{constant} \quad (q = q_0) \qquad \operatorname{div}\vec{u} = 0$$

Euler equation

- Incompressible Euler equation

$$\frac{\partial}{\partial t}\vec{u} + \nabla_{\vec{u}}\vec{u} = -\frac{1}{q_0}\operatorname{grad} p$$
$$\operatorname{div}\vec{u} = 0$$

- Navier–Stokes equation (in Euclidean space)

$$\frac{\partial}{\partial t}\vec{u} + \nabla_{\vec{u}}\vec{u} = -\frac{1}{q_0}\operatorname{grad} p + \frac{\mu}{q_0}\nabla\cdot\nabla\vec{u}$$
$$\operatorname{div}\vec{u} = 0$$

  - Rayleigh dissipation function $\mathcal{R}(\dot{\phi}) = \int_W R(\mathbf{L})\mu_W$

deformation rate $\dot{\mathbf{F}} = (\nabla\mathbf{u})\mathbf{F} =: \mathbf{L}\mathbf{F}$ strain rate $\mathbf{E} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^{\intercal})$

$$R(\mathbf{L}) = \left(\frac{\lambda}{2}\operatorname{tr}(\mathbf{E})^2 + \mu\operatorname{tr}(\mathbf{E}^2)\right)$$

Lie derivative

- Fluid as elastic body
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- Vorticity

Lie derivative

- Let $\vec{u} \in \Gamma(TW)$ be a world-coord vector field, being the velocity field of a flow map $\phi : M \to W$

$$\dot{\phi} = \vec{u} \circ \phi$$

- Let $T \in \Gamma(\otimes^k TW \otimes^\ell T^*W)$ be some time-independent tensor field on the world coordinate.

- Define Lie derivative

$$\mathscr{L}_{\vec{u}} : \Gamma(\otimes^k TW \otimes^\ell T^*W) \to \Gamma(\otimes^k TW \otimes^\ell T^*W)$$

so that

$$\frac{\partial}{\partial t}(\phi^* T) = \phi^*(\mathscr{L}_{\vec{u}} T)$$

Lie derivative for functions

- Let $f \in C^{\infty}(W)$

$$\underset{\text{0-form}}{\phi^* f} = f \circ \phi$$

- Time derivative

$$\frac{\partial}{\partial t}\underset{\text{0-form}}{(\phi^* f)} = \frac{\partial}{\partial t}(f \circ \phi) = df[\![\vec{u}]\!] \circ \phi = \underset{\text{0-form}}{\phi^*(df[\![\vec{u}]\!])}$$

- Therefore, the Lie derivative is the directional derivative

$$\underset{\text{0-form}}{\mathscr{L}_{\vec{u}} f} = df[\![\vec{u}]\!] = \vec{u} f$$

Lie derivative for vectors

- Let $\vec{v} \in \Gamma(TW)$

$$\underbrace{\phi^* \vec{v}}_{\text{vec}} = \phi_*^{-1} \vec{v}$$

- If we have function $f \in C^\infty(W)$, then we have

$$\underbrace{\phi^*}_{\text{0-form}} (\vec{v} f) = \underbrace{(\phi^* \vec{v})}_{\text{vec}} (\underbrace{\phi^*}_{\text{0-form}} f)$$

- Take derivative of this formula

$$\frac{\partial}{\partial t} \underbrace{\phi^*}_{\text{0-form}} (\vec{v} f) = \underbrace{(\frac{\partial}{\partial t} \phi^* \vec{v})}_{\text{vec}} (\underbrace{\phi^*}_{\text{0-form}} f) + \underbrace{(\phi^* \vec{v})}_{\text{vec}} (\frac{\partial}{\partial t} \underbrace{\phi^*}_{\text{0-form}} f)$$

Lie derivative for vectors

$$\underbrace{\frac{\partial}{\partial t}\phi^*(\vec{v}f)}_{\text{0-form}} = \underbrace{(\frac{\partial}{\partial t}\phi^*\vec{v})}_{\text{vec}}\underbrace{(\phi^* f)}_{\text{0-form}} + \underbrace{(\phi^*\vec{v})}_{\text{vec}}\underbrace{(\frac{\partial}{\partial t}\phi^* f)}_{\text{0-form}}$$

$$\underbrace{\phi^*(\vec{u}\vec{v}f)}_{\text{0-form}} = \underbrace{(\frac{\partial}{\partial t}\phi^*\vec{v})}_{\text{vec}}\underbrace{(\phi^* f)}_{\text{0-form}} + \underbrace{(\phi^*\vec{v})}_{\text{vec}}\underbrace{(\phi^* \vec{u}f)}_{\text{0-form}}$$

$$\underbrace{\phi^*(\vec{u}\vec{v}f)}_{\text{0-form}} = \underbrace{(\frac{\partial}{\partial t}\phi^*\vec{v})}_{\text{vec}}\underbrace{(\phi^* f)}_{\text{0-form}} + \underbrace{\phi^*(\vec{v}\vec{u}f)}_{\text{0-form}}$$

$$\underbrace{(\frac{\partial}{\partial t}\phi^*\vec{v})}_{\text{vec}}\underbrace{(\phi^* f)}_{\text{0-form}} = \underbrace{\phi^*([\vec{u},\vec{v}]f)}_{\text{0-form}} = \underbrace{(\phi^*[\vec{u},\vec{v}])}_{\text{vec}}\underbrace{(\phi^* f)}_{\text{0-form}}$$

- Therefore $\underbrace{\frac{\partial}{\partial t}\phi^*\vec{v}}_{\text{vec}} = \underbrace{(\phi^*[\vec{u},\vec{v}])}_{\text{vec}}$

Lie derivative for vectors

- Therefore $\underbrace{\frac{\partial}{\partial t} \phi^* \vec{v}}_{\text{vec}} = (\underbrace{\phi^*[\vec{u}, \vec{v}]}_{\text{vec}})$

$$\underbrace{\mathcal{L}_{\vec{u}} \vec{v}}_{\text{vec}} = [\vec{u}, \vec{v}] = \nabla_{\vec{u}} \vec{v} - \nabla_{\vec{v}} \vec{u}$$

Lie derivative for differential forms

- Let $\alpha \in \Omega^k(W)$
- Its Lie derivative is given by **Cartan's magic formula**

$$\underset{\text{form}}{\mathcal{L}_{\vec{u}}}\alpha = i_{\vec{u}}(d\alpha) + d(i_{\vec{u}}\alpha) \qquad \underset{\text{form}}{\mathcal{L}_{\vec{u}}} = i_{\vec{u}}d + di_{\vec{u}}$$

- Proof
  - From $\underset{\text{form}}{\phi^*}(d\alpha) = d(\underset{\text{form}}{\phi^*}\alpha)$ and $\underset{\text{form}}{\phi^*}(\alpha\wedge\beta) = (\underset{\text{form}}{\phi^*}\alpha)\wedge(\underset{\text{form}}{\phi^*}\beta)$ we have

$$d\underset{\text{form}}{\mathcal{L}_{\vec{u}}} = \underset{\text{form}}{\mathcal{L}_{\vec{u}}}d \quad \text{and} \quad \underset{\text{form}}{\mathcal{L}_{\vec{u}}}(\alpha\wedge\beta) = (\underset{\text{form}}{\mathcal{L}_{\vec{u}}}\alpha)\wedge\beta + \alpha\wedge(\underset{\text{form}}{\mathcal{L}_{\vec{u}}}\beta)$$

  - There can only be one operator that satisfies these rules and agrees on the behavior when applied on 0-forms.
  - Check that Cartan's magic formula obeys these rules.

Lie derivative for differential forms

- Under integral evaluation

$$\int_{\phi_t(S)} \alpha = \int_S \underbrace{\phi_t{}^* \alpha}_{\text{form}}$$

$$\frac{d}{dt} \int_{\phi_t(S)} \alpha = \frac{d}{dt} \int_S \underbrace{\phi_t{}^* \alpha}_{\text{form}} = \int_S \underbrace{\phi_t{}^*}_{\text{form}} (\underbrace{\mathscr{L}_{\vec{u}} \alpha}_{\text{form}})$$

$$= \int_{\phi_t(S)} \underbrace{\mathscr{L}_{\vec{u}} \alpha}_{\text{form}}$$

Lie derivative for differential forms

- Under integral evaluation

$$\frac{d}{dt}\int_{\phi_t(S)} \alpha = \int_{\phi_t(S)} \underset{\text{form}}{\mathcal{L}_{\vec{u}}}\,\alpha$$

- This is the derivative of integral with respect to integration domain
- In 1D, this is known as Leibniz's integral rule
- This fundamental calculus rule is not widely known; but its special cases are used independently in continuum mechanics, shape optimization, differentiable rendering.

On Cartan’s magic formula

- Fluid as elastic body
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Integral formulation

$$\int_A \alpha$$

k-form

(a test surface)
k-dim piece of geometry

Integral formulation

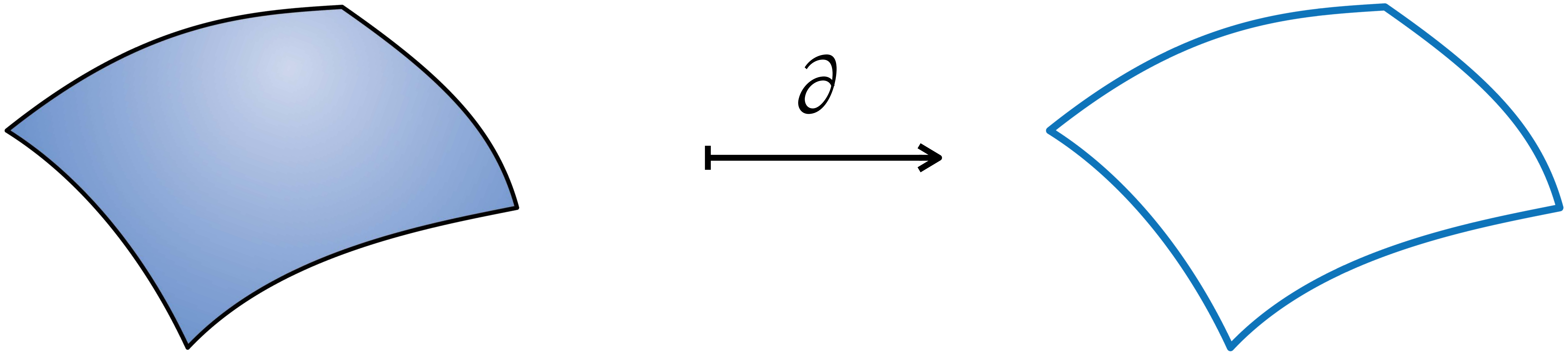
$$\int_{\partial A} \alpha = \int_A d\alpha$$

geometric operator

calculus operator

Operations

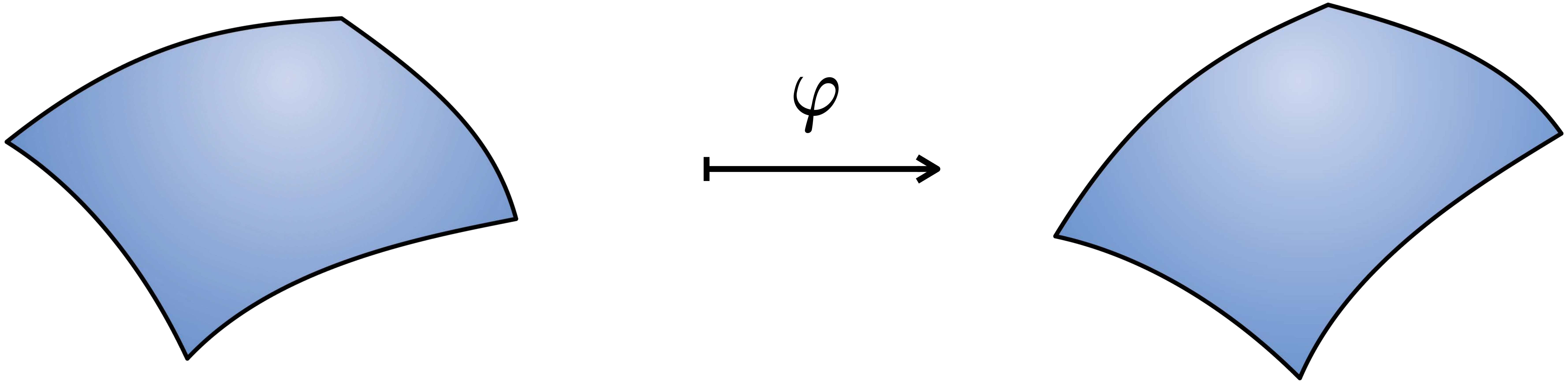
- Boundary



k-surface $\qquad$ $(k-1)$-surface

Operations

- Map

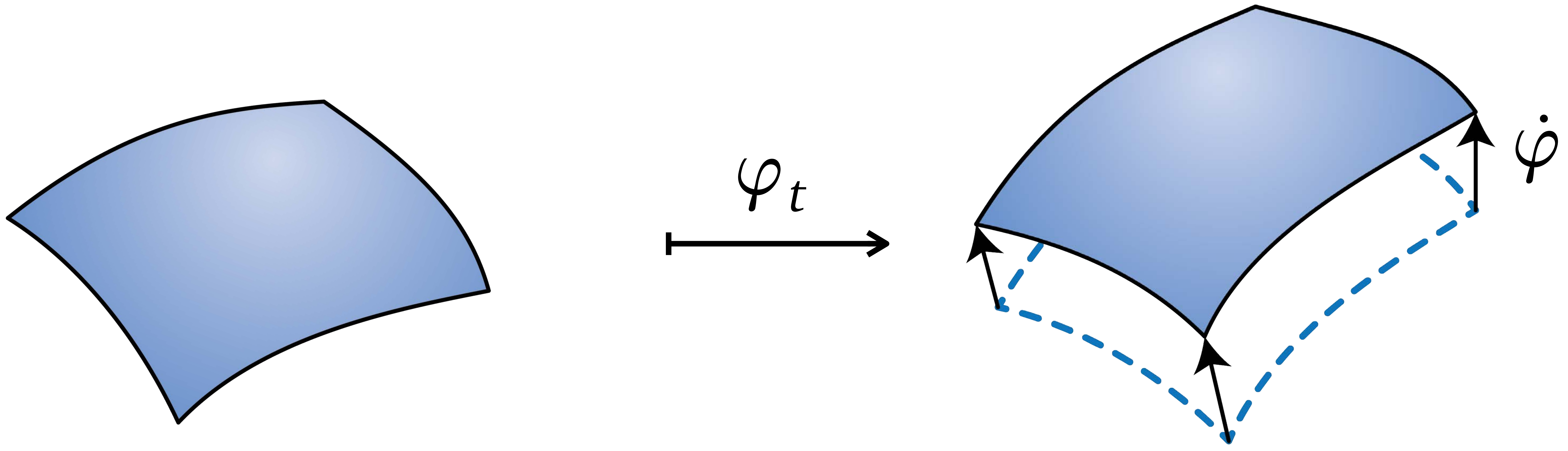


k-surface

k-surface possibly in another space

Operations

- Time-dependent map

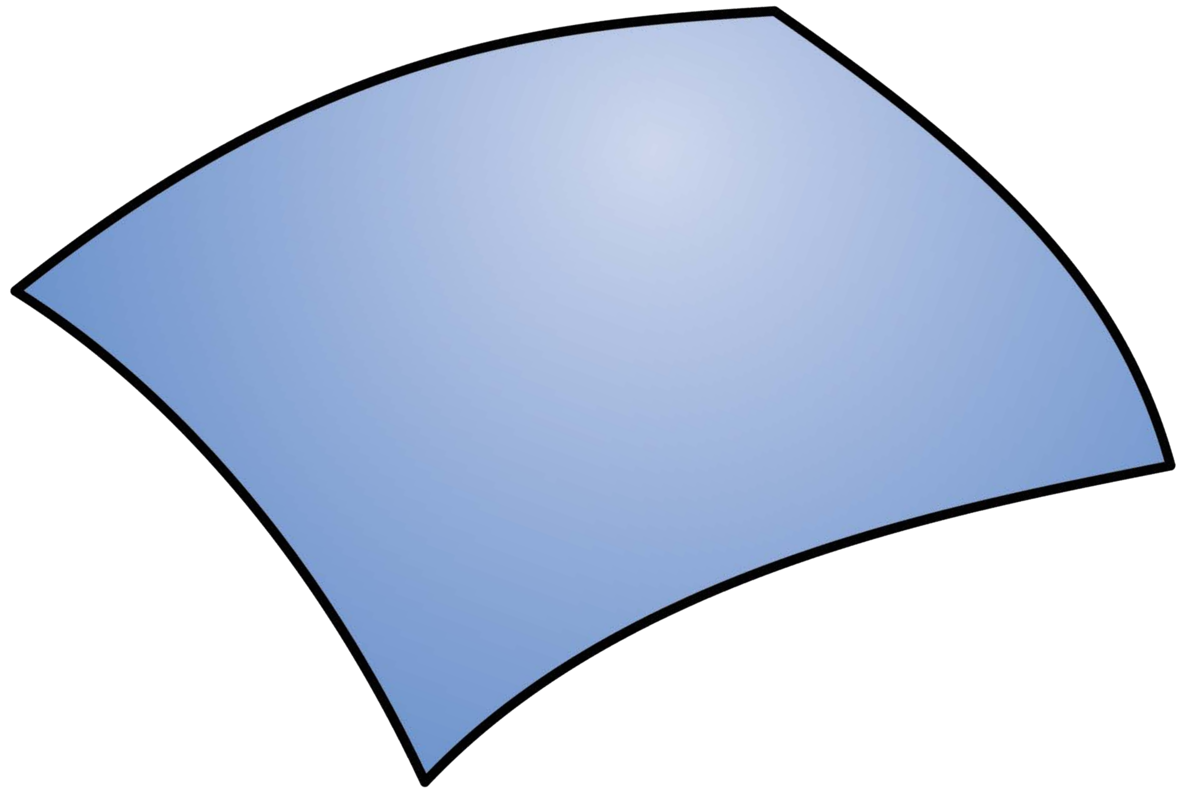


k-surface

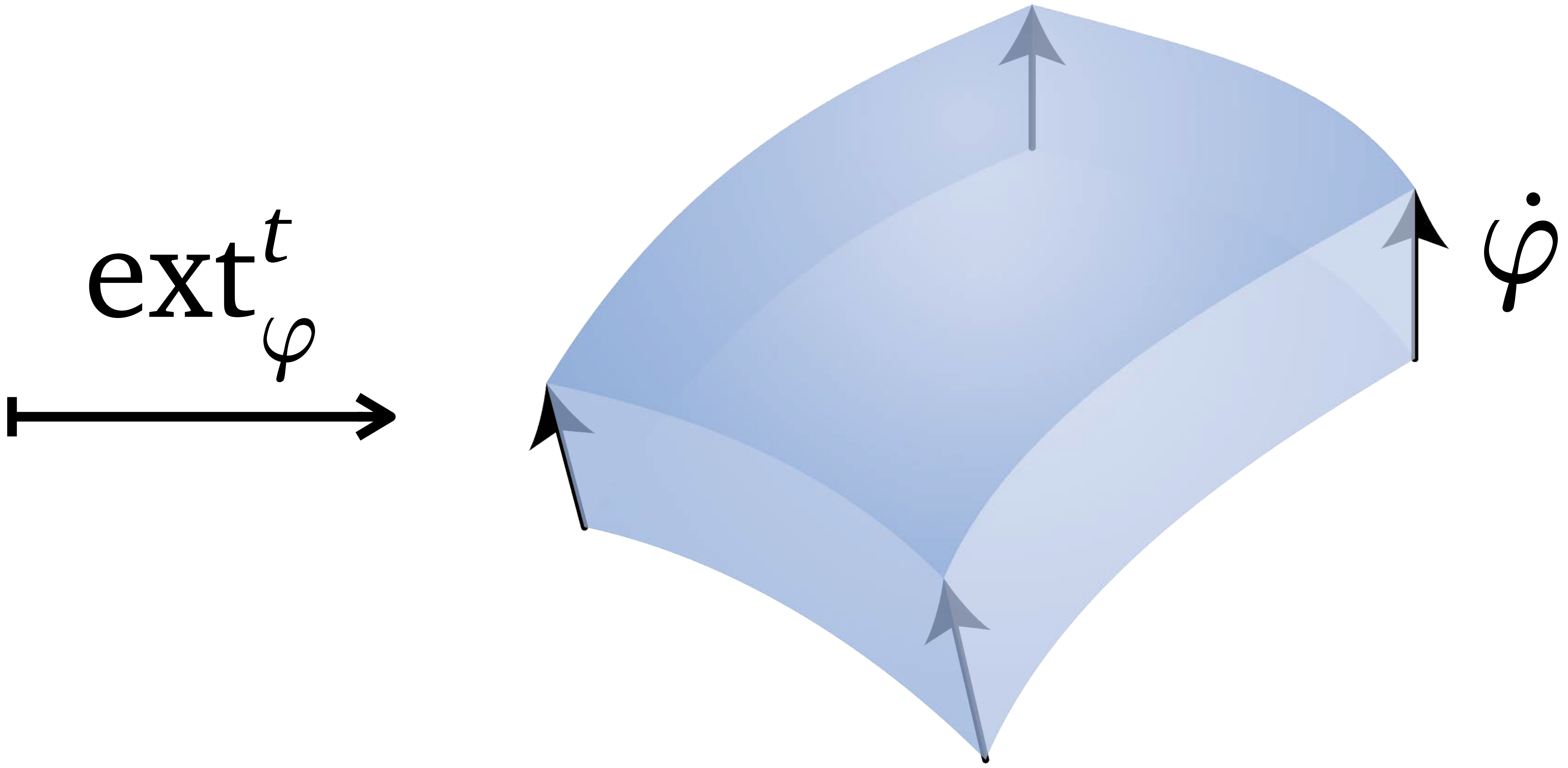
moving k-surface

Operations

- Extrusion

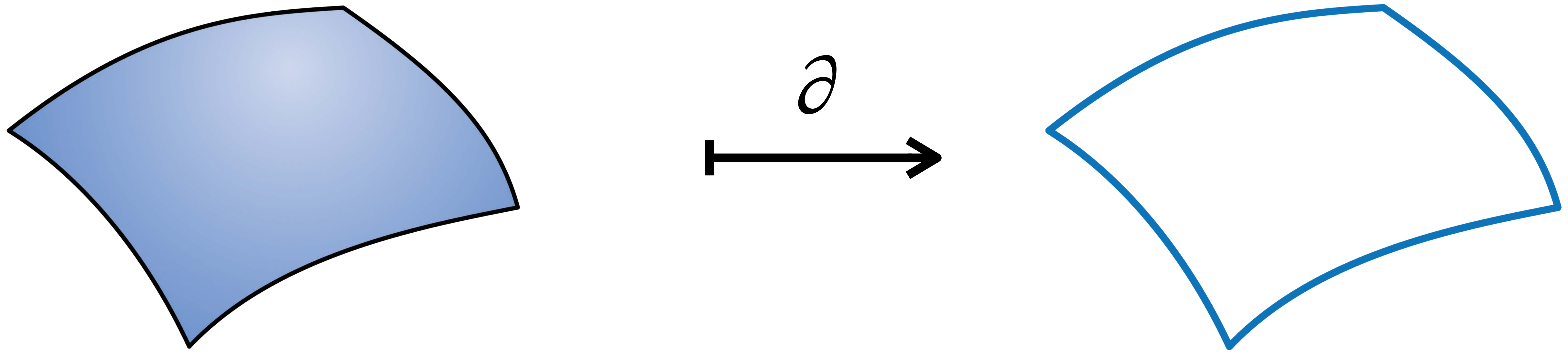


k-surface



$(k+1)$-surface

Stokes-like Theorems



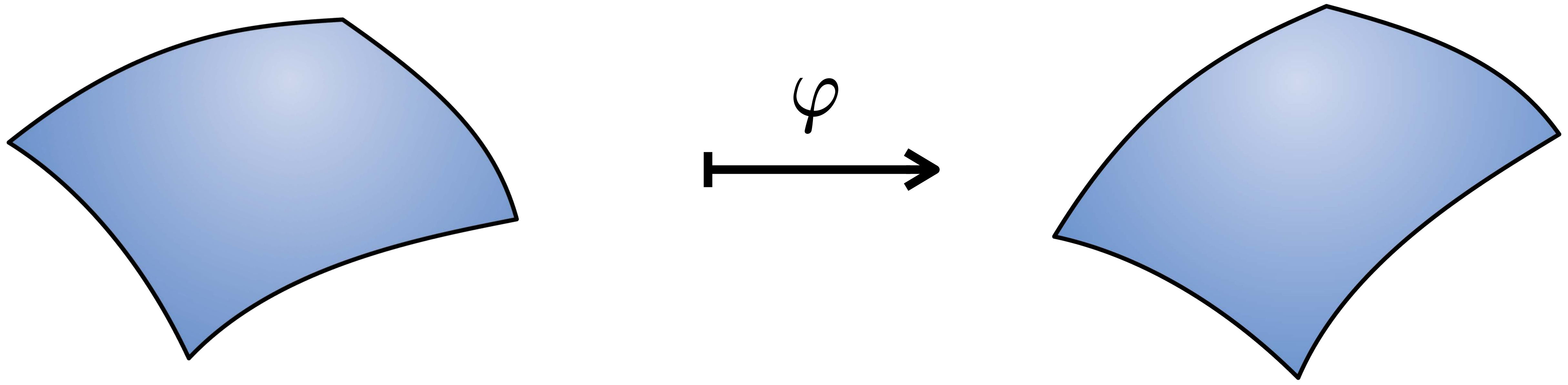
Boundary / Exterior derivative

$\partial : k$-surfaces $\longrightarrow (k-1)$-surfaces

$d : (k-1)$-forms $\longrightarrow k$-forms

$$\int_{\partial A} \alpha = \int_A d\alpha$$

Stokes-like Theorems



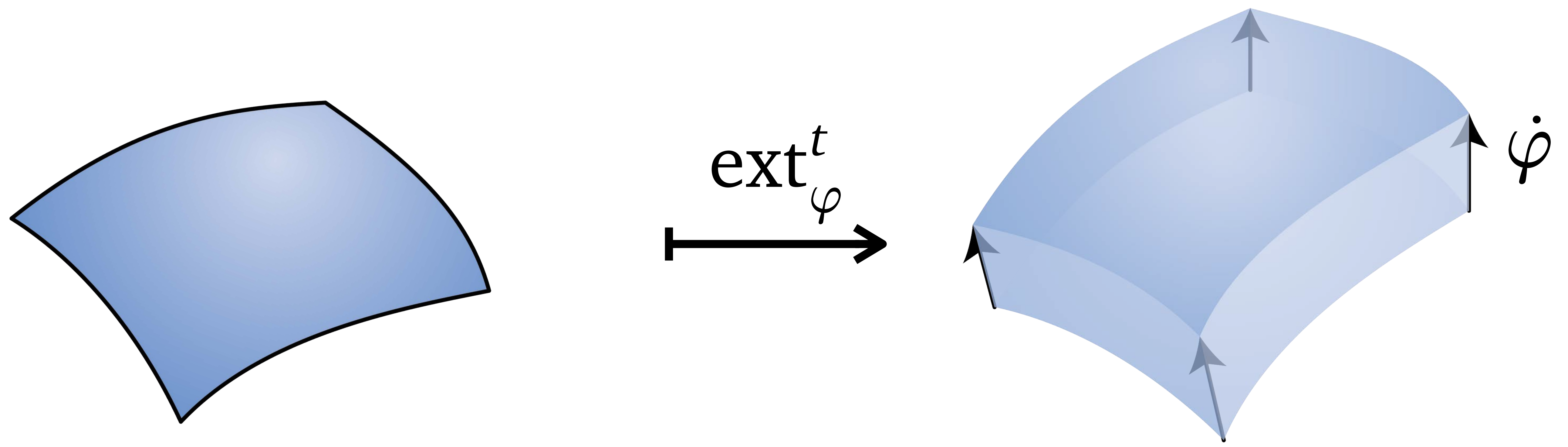
Map / Pullback operator

$\varphi : k$-surfaces $\longrightarrow k$-surfaces

$\varphi^* : k$-forms $\longrightarrow k$-forms

$$\int_{\varphi(A)} \alpha = \int_A \varphi^* \alpha$$

Stokes-like Theorems

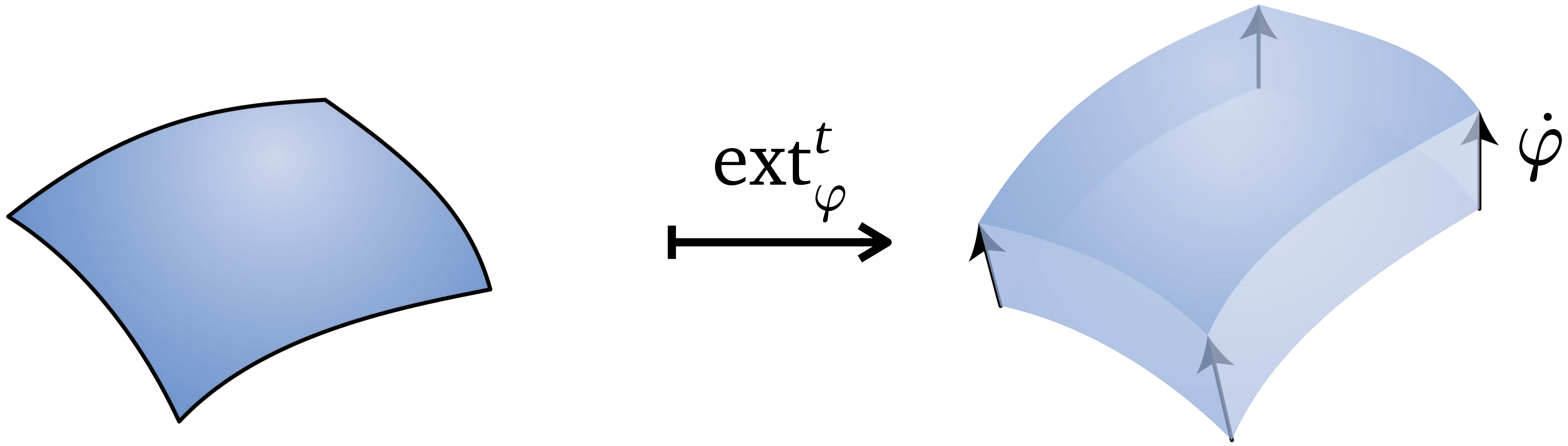


Extrusion /

$$\mathrm{ext}_\varphi^t : k\text{-surfaces} \longrightarrow (k+1)\text{-surfaces}$$
$$: (k+1)\text{-forms} \longrightarrow k\text{-forms}$$

$$\lim_{t\to 0} \frac{1}{t} \int_{\mathrm{ext}_\varphi^t(A)} \alpha =$$

Stokes-like Theorems

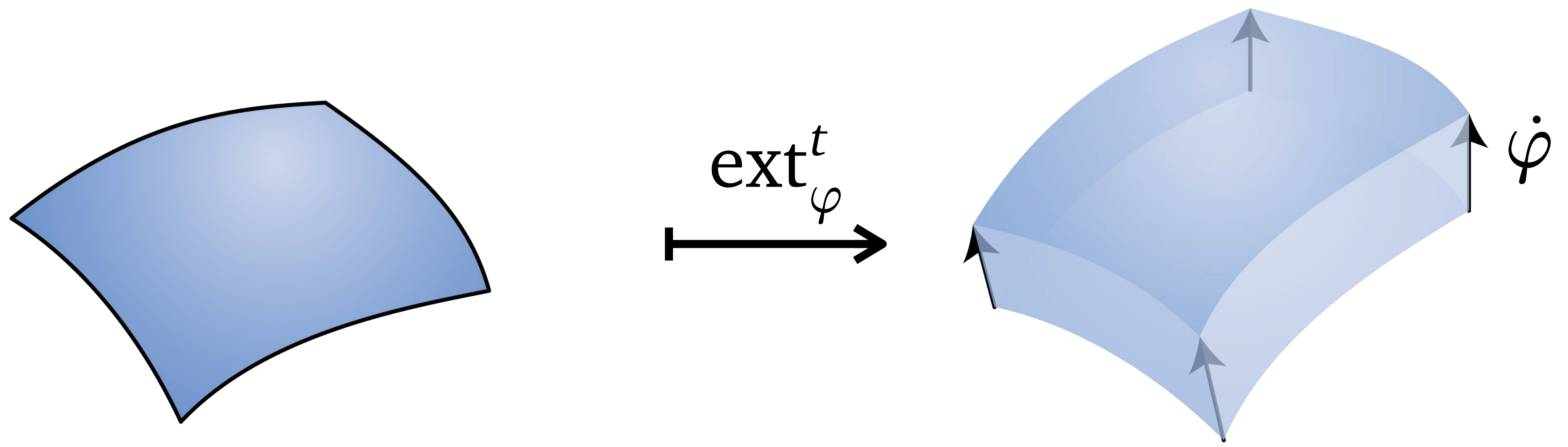


Extrusion /

$$\left.\frac{d}{dt}\right|_{t=0} \text{ext}_\varphi^t : k\text{-surfaces} \longrightarrow (k+1)\text{-surfaces}$$
$$: (k+1)\text{-forms} \longrightarrow k\text{-forms}$$

$$\left.\frac{d}{dt}\right|_{t=0} \int_{\text{ext}_\varphi^t(A)} \alpha =$$

Stokes-like Theorems



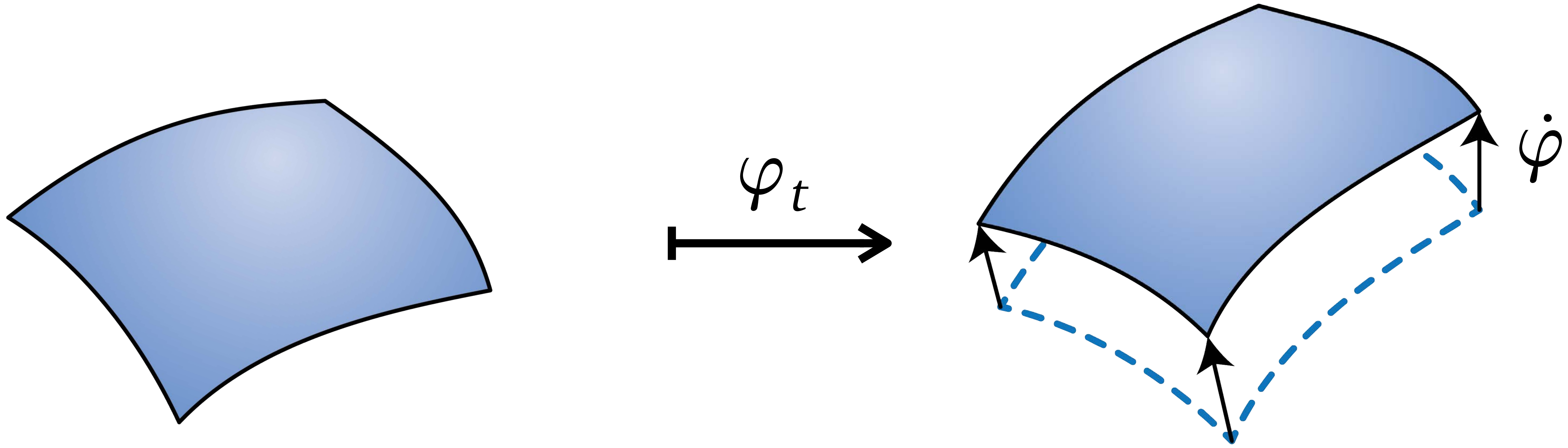
Extrusion / Interior product

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{ext}^t_\varphi : k\text{-surfaces} \to (k+1)\text{-surfaces}$$

$$i_{\dot\varphi} : (k+1)\text{-forms} \to k\text{-forms}$$

$$\left.\frac{d}{dt}\right|_{t=0} \int_{\mathrm{ext}^t_\varphi(A)} \alpha = \int_A i_{\dot\varphi}\,\alpha$$

Stokes-like Theorems



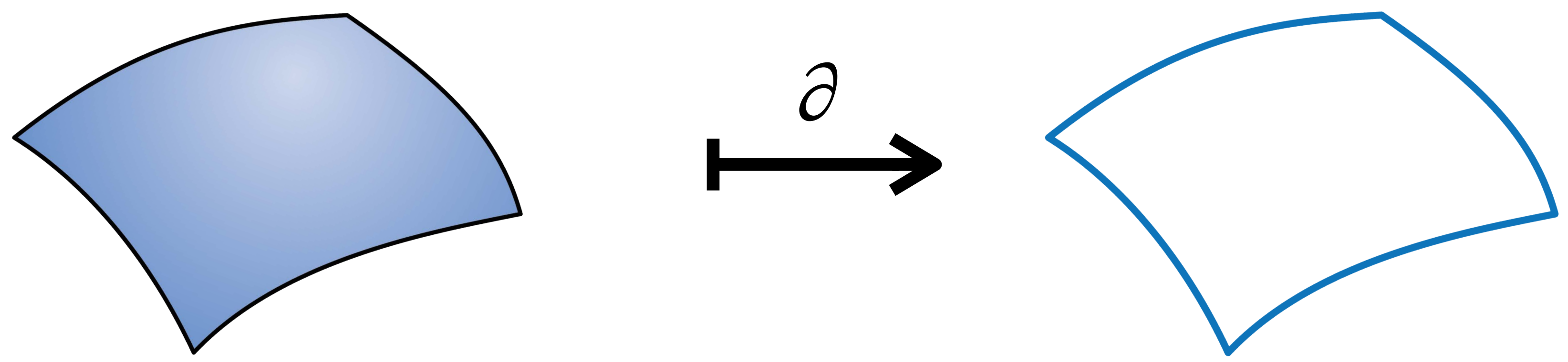
Time-dependent map / Lie derivative

$$\left.\frac{d}{dt}\right|_{t=0} \varphi_t : k\text{-surfaces} \to k\text{-surfaces}$$

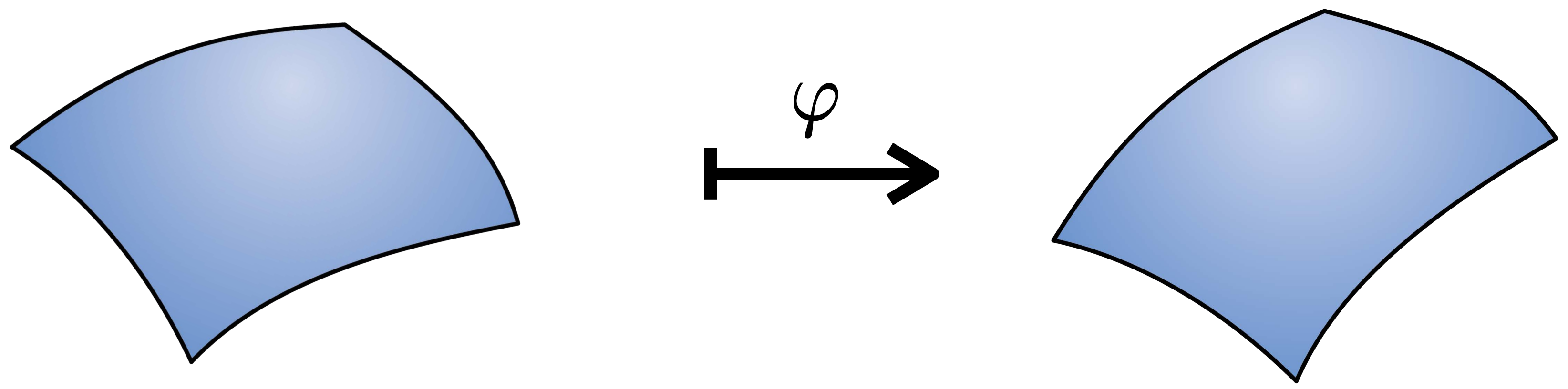
$$\mathcal{L}_{\dot{\varphi}} : k\text{-forms} \to k\text{-forms}$$

$$\left.\frac{d}{dt}\right|_{t=0} \int_{\varphi_t(A)} \alpha = \int_A \mathcal{L}_{\dot{\varphi}} \alpha$$

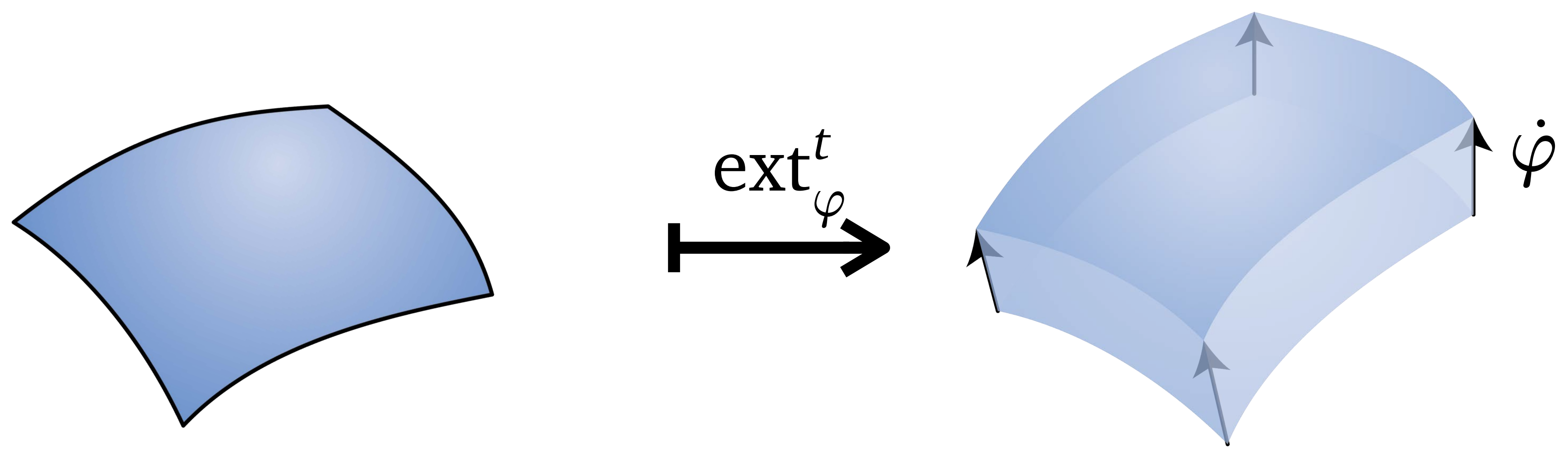
Stokes-like Theorems



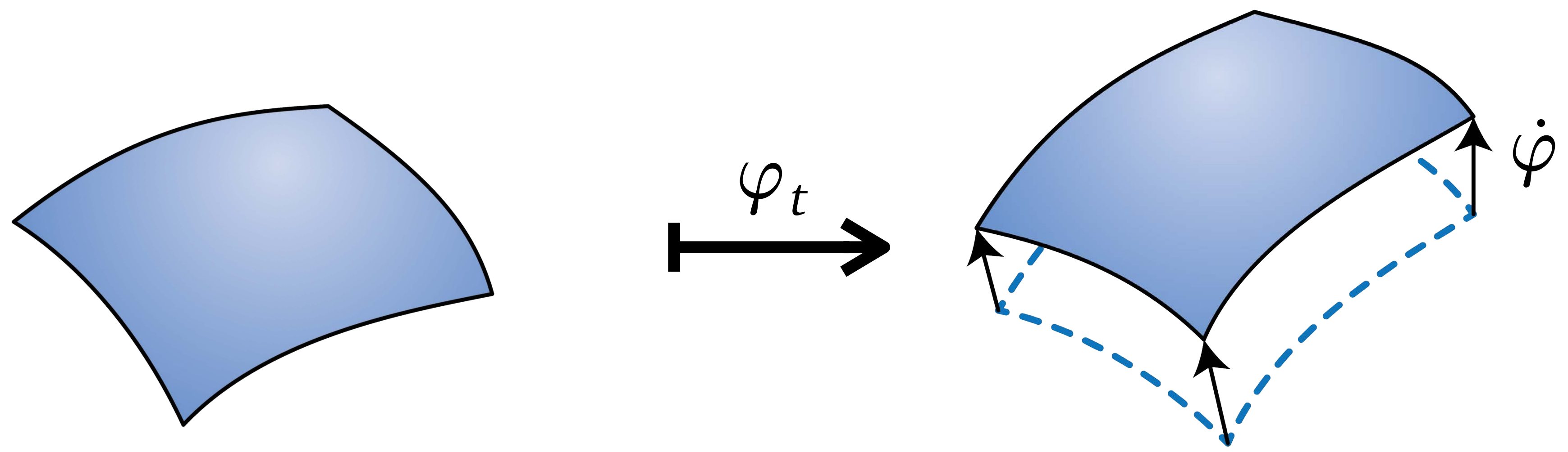
$$\int_{\partial A} \alpha = \int_A d\alpha$$



$$\int_{\varphi(A)} \alpha = \int_A \varphi^* \alpha$$



$$\left.\frac{d}{dt}\right|_{t=0} \int_{\mathrm{ext}^t_\varphi(A)} \alpha = \int_A i_{\dot{\varphi}} \alpha$$



$$\left.\frac{d}{dt}\right|_{t=0} \int_{\varphi_t(A)} \alpha = \int_A \mathscr{L}_{\dot{\varphi}} \alpha$$

Cartan Magic Formula

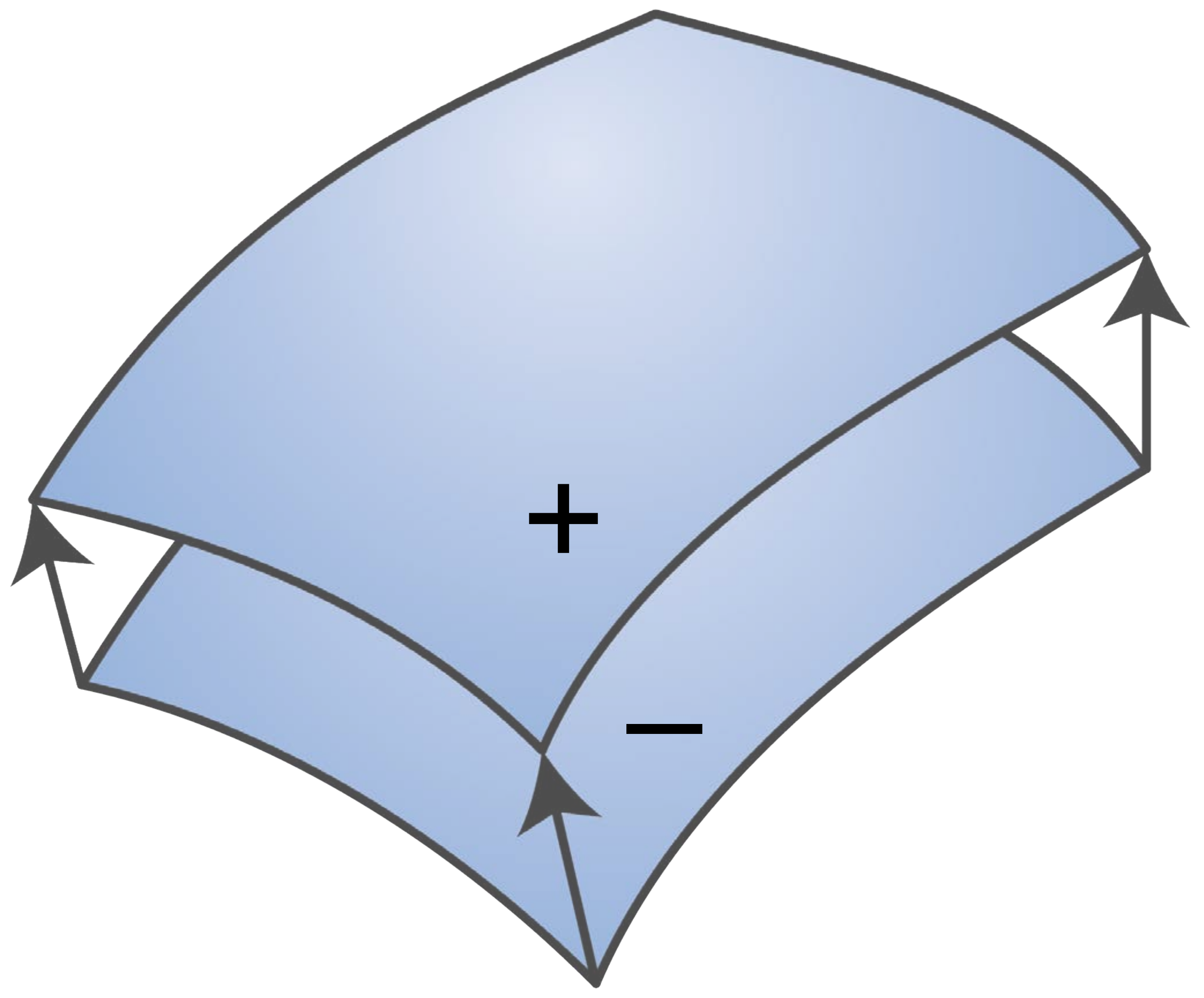
$$\frac{d}{dt}\int_{\varphi_t(A)} \alpha$$

Cartan Magic Formula

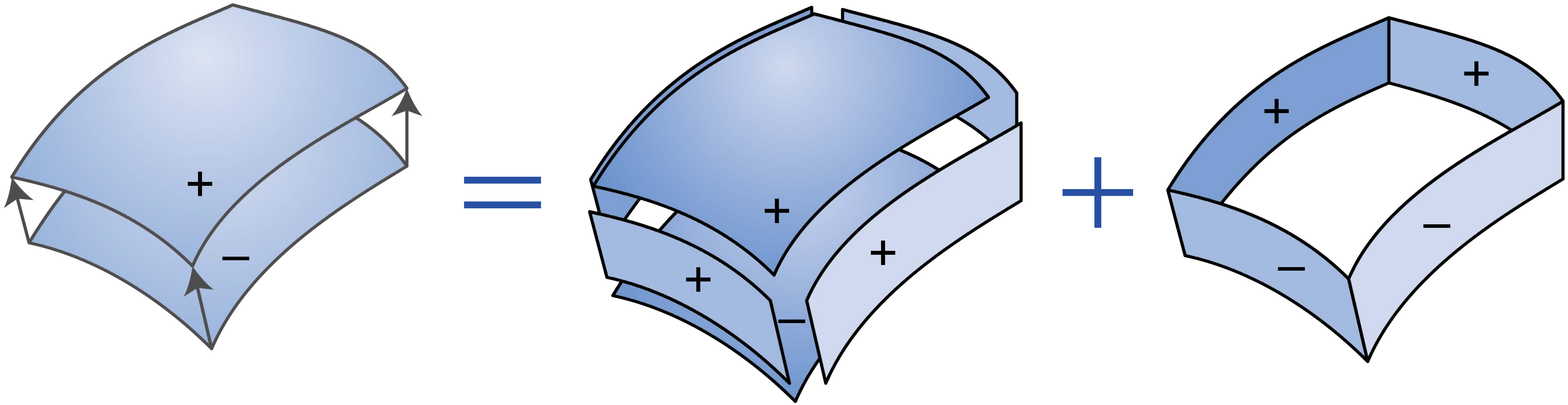
$$\frac{d}{dt}\varphi_t(A)$$

Cartan Magic Formula

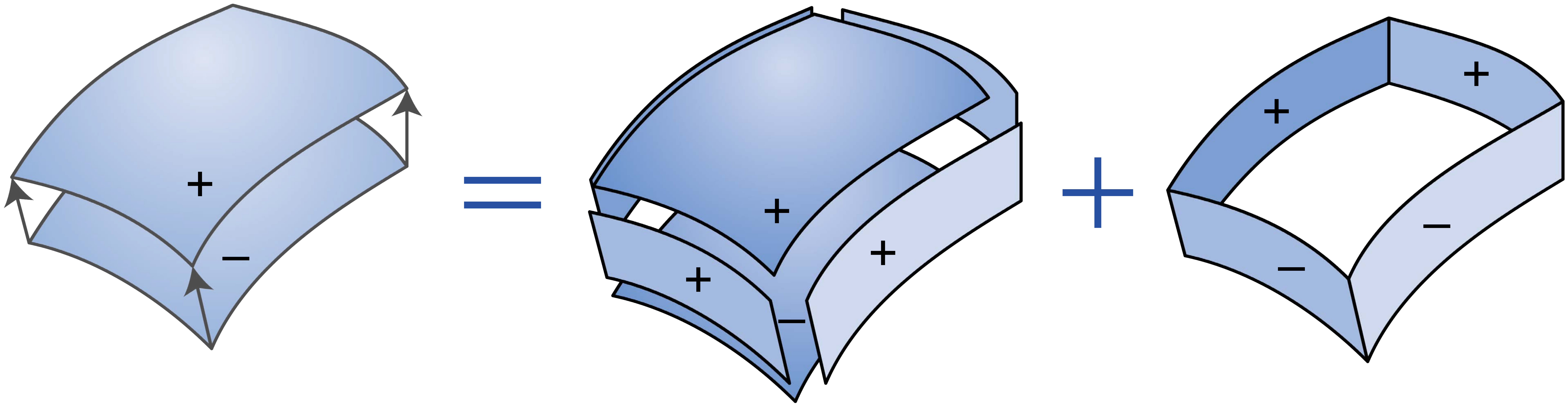
$$\frac{d}{dt}\varphi_t(A) =$$



Cartan Magic Formula



Cartan Magic Formula



$$\frac{d}{dt}\varphi_t(A) = \frac{d}{dt}\,\partial\left(\mathrm{ext}_{\varphi}^{t}(A)\right) + \frac{d}{dt}\,\mathrm{ext}_{\varphi}^{t}(\partial(A))$$

Cartan Magic Formula

Cartan Magic Formula

$$\mathscr{L}_{\dot{\varphi}}\alpha = i_{\dot{\varphi}}d\alpha + di_{\dot{\varphi}}\alpha$$

Cartan formula under integration

Reynolds Transport Theorem (Leibniz integral rule)

$$\frac{d}{dt}\int_{\varphi_t(A)} \alpha = \int_A i_{\dot{\varphi}}\, d\alpha + \oint_{\partial A} i_{\dot{\varphi}}\, \alpha$$

Cartan formula under integration

Reynolds Transport Theorem (Leibniz integral rule)

$$\frac{d}{dt}\int_{\varphi_t(A)} \alpha = \int_A i_{\dot\varphi} d\alpha + \oint_{\partial A} i_{\dot\varphi}\alpha$$

If α also depends on t then

$$\frac{\partial}{\partial t}\left(\varphi_t^* \alpha_t\right) = \frac{\partial \alpha}{\partial t} + \mathscr{L}_{\dot\varphi}\alpha = \frac{\partial \alpha}{\partial t} + i_{\dot\varphi} d\alpha + d i_{\dot\varphi}\alpha$$

$$\frac{d}{dt}\int_{\varphi_t(A)} \alpha_t = \int_A \frac{\partial \alpha}{\partial t} + \int_A i_{\dot\varphi} d\alpha + \oint_{\partial A} i_{\dot\varphi}\alpha$$

Cartan formula under integration

$$\frac{d}{dt}\int_{\varphi_t(A)} \alpha_t = \int_A \frac{\partial \alpha}{\partial t} + \int_A i_{\dot\varphi}\, d\alpha + \oint_{\partial A} i_{\dot\varphi}\, \alpha$$

density form

$$\frac{d}{dt}\iiint_{\varphi_t(V)} q\, dV = \iiint_V \frac{\partial q}{\partial t}\, dV + \oiint_{\partial V} q\dot\varphi \cdot \mathbf{n}\, dA$$

flux form

$$\frac{d}{dt}\iint_{\varphi_t(S)} \mathbf{v}\cdot\mathbf{n}\, dA = \iint_S \left(\frac{\partial \mathbf{v}}{\partial t} + (\nabla\cdot\mathbf{v})\dot\varphi\cdot\mathbf{n}\right) dA + \oint_{\partial S} (\dot\varphi\times\mathbf{v})\cdot d\mathbf{l}$$

Lie material derivative

- Fluid as elastic body
- Velocity and acceleration
- Navier–Stokes equations
- Lie derivative
  - Cartan’s formula
  - Lie material derivative
- Vorticity

Lie material derivative

- Let T_t be some tensor (function, vector field, or k-form)

$$\underset{\langle\text{type}\rangle}{\frac{\partial}{\partial t}\phi_t{}^* T_t} = \underset{\langle\text{type}\rangle}{\phi_t{}^*}\left(\frac{\partial}{\partial t} + \underset{\langle\text{type}\rangle}{\mathscr{L}_{\vec{u}}}\right)T_t$$

- Summary of Lie derivative
  - For functions $\underset{\text{0-form}}{\mathscr{L}_{\vec{u}}}\, f = (df)[\![\vec{u}]\!]$
  - For vector fields $\underset{\text{vec}}{\mathscr{L}_{\vec{u}}}\vec{v} = [\vec{u}, \vec{v}] = \nabla_{\vec{u}}\vec{v} - \nabla_{\vec{v}}\vec{u}$
  - For differential forms $\underset{\text{form}}{\mathscr{L}_{\vec{u}}}\alpha = i_{\vec{u}}(d\alpha) + d(i_{\vec{u}}\alpha)$

Lie material derivative

- Let T_t be some tensor (function, vector field, or k-form)

$$\frac{\partial}{\partial t} \underbrace{\phi_t^* T_t}_{\langle\text{type}\rangle} = \underbrace{\phi_t^*}_{\langle\text{type}\rangle} \Big(\frac{\partial}{\partial t} + \underbrace{\mathscr{L}_{\vec{u}}}_{\langle\text{type}\rangle}\Big) T_t$$

- We say a tensor field T_t is **purely advected** (as a specific type) if

$$\Big(\frac{\partial}{\partial t} + \underbrace{\mathscr{L}_{\vec{u}}}_{\langle\text{type}\rangle}\Big) T_t = 0$$

- The tensor field is purely advected if and only if we have the following **conservation law**:

$$\underbrace{\phi_t^* T_t}_{\langle\text{type}\rangle} \quad \text{is constant over time.}$$

In 3D vector calculus

$$\Big(\frac{\partial}{\partial t} + \underbrace{\mathscr{L}_{\vec{u}}}_{\text{0-form}}\Big) f_{\text{0-form}} = 0$$

material derivative

$$\Big(\frac{\partial}{\partial t} + \nabla_{\mathbf{u}}\Big) f = 0$$

$$\Big(\frac{\partial}{\partial t} + \underbrace{\mathscr{L}_{\vec{u}}}_{\text{1-form}}\Big) \mathbf{w}_{\text{1-form}} = 0$$

Oldroyd's lower-convection

$$\Big(\frac{\partial}{\partial t} + \nabla_{\mathbf{u}}\Big) \mathbf{w} = -(\nabla \mathbf{u})^{\intercal} \mathbf{w}$$

$$\Big(\frac{\partial}{\partial t} + \underbrace{\mathscr{L}_{\vec{u}}}_{\text{2-form}}\Big) \mathbf{w}_{\text{2-form}} = 0$$

stretching & compression

$$\Big(\frac{\partial}{\partial t} + \nabla_{\mathbf{u}}\Big) \mathbf{w} = \nabla_{\mathbf{w}} \mathbf{u} - (\nabla \cdot \mathbf{u}) \mathbf{w}$$

$$\Big(\frac{\partial}{\partial t} + \underbrace{\mathscr{L}_{\vec{u}}}_{\text{3-form}}\Big) q_{\text{3-form}} = 0$$

compression

$$\Big(\frac{\partial}{\partial t} + \nabla_{\mathbf{u}}\Big) q = -(\nabla \cdot \mathbf{u}) q$$

$$\Big(\frac{\partial}{\partial t} + \underbrace{\mathscr{L}_{\vec{u}}}_{\text{vec}}\Big) \mathbf{w}_{\text{vec}} = 0$$

stretching

$$\Big(\frac{\partial}{\partial t} + \nabla_{\mathbf{u}}\Big) \mathbf{w} = \nabla_{\mathbf{w}} \mathbf{u}$$

Continuity equation

- Conservation of mass:

  $$\rho_M = \phi_t^* \rho_t \quad \text{is constant over time.}$$

  n-form

- This translates to $\left(\frac{\partial}{\partial t} + \mathcal{L}_{\vec{u}}\right)\rho = 0$

  n-form

- Using Cartan's magic formula $\frac{\partial}{\partial t}\rho + d i_{\vec{u}}\rho = 0$

- In terms of scalar density $\rho_t = q_t \mu_W \qquad \dot{q} + \mathrm{div}(q\vec{u}) = 0$

- Divergence of vector field $\mathrm{div}(\vec{u}) := \frac{\mathcal{L}_{\vec{u}}\mu_W}{\mu_W}$

- Expand: $\dot{q} + d_{\vec{u}}q = -\mathrm{div}(\vec{u})q$

Divergence free

- Velocity is volume-preserving if

$$\operatorname{div}(\vec{u}) := \frac{\mathcal{L}_{\vec{u}}\, \mu_W}{\mu_W} = 0$$

Vorticity

- Fluid as elastic body
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Vortex

Top Gun: Maverick [Paramount Picture 2022]

Vorticity

- Suppose the velocity field is $\vec{u}$

$$\vec{u} = u^1\vec{e}_1 + u^2\vec{e}_2 + u^3\vec{e}_3$$

- The velocity 1-form is

$$\eta = \quad \flat_W \vec{u} = u_1 dx^1 + u_2 dx^2 + u_3 dx^3$$

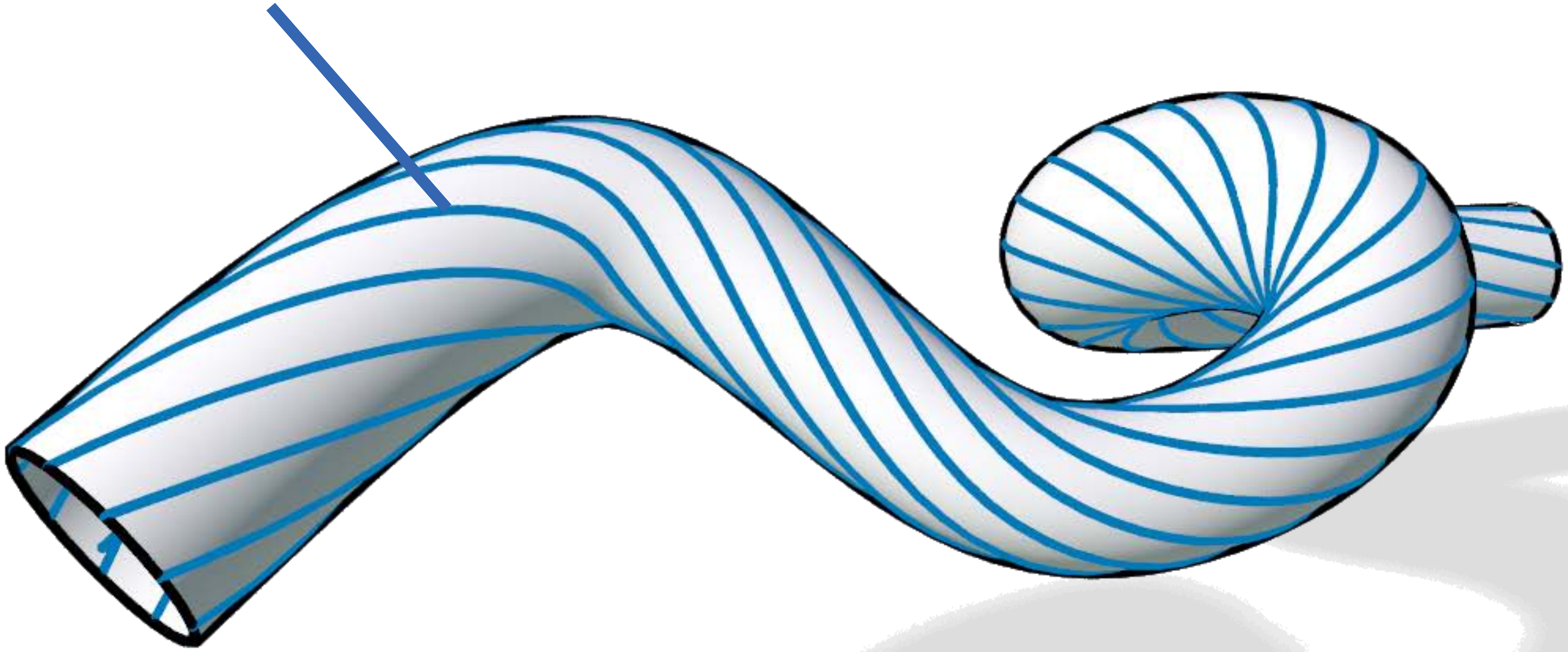
- Take exterior derivative to obtain the **vorticity 2-form**

$$\omega = d\eta$$

- In 2D $\quad w = \frac{\partial u_2}{\partial x^1} - \frac{\partial u_1}{\partial x^2}$
- In 3D $\quad \mathbf{w} = \nabla \times \mathbf{u}$ $\qquad$ trail of **w** is called a **vortex line**.

Vortex tube

Vortex lines are tangent to the tube boundary



- The vorticity fluxes on two cross sections connected by a vortex tube must be the same.
- In fact, the entire vortex geometry (as bundle of vortex lines) is “frozen-in” the fluid as deformable body.

Vorticity

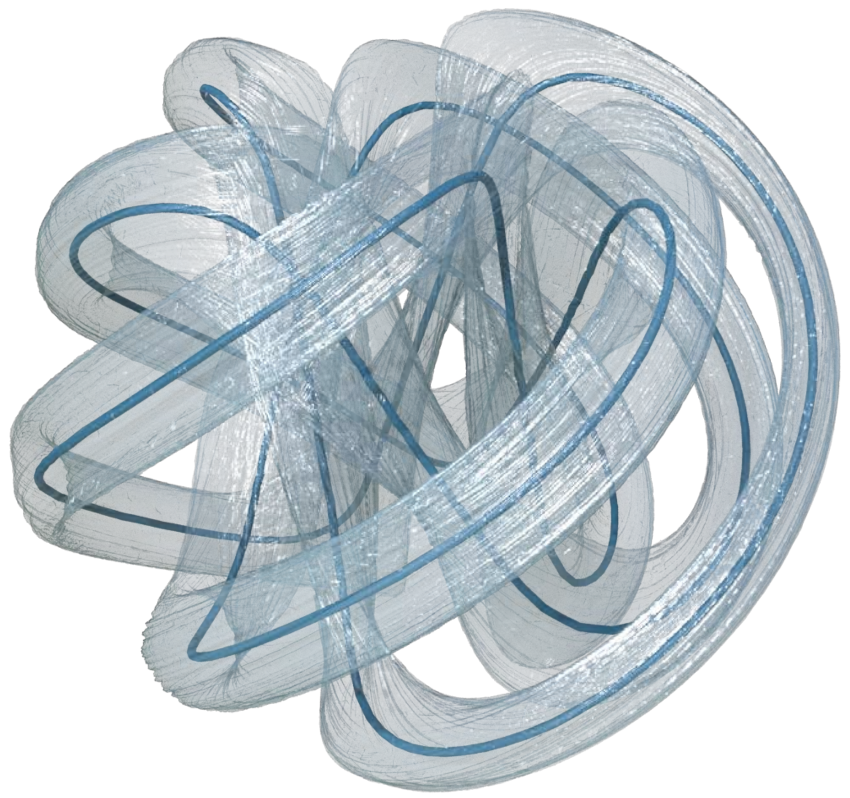
- In barotropic Euler fluid and incompressible fluid:

$$\Big(\frac{\partial}{\partial t} + \underbrace{\mathscr{L}_{\vec{u}}}_{\text{2-form}}\Big)\omega = 0$$

  - Vorticity flux on a flowing surface $\int_{\phi_t(S)} \omega_t = \mathrm{const}$
  - Circulation on a flowing closed loop $\oint_{\phi_t(C)} \eta_t = \mathrm{const}$

More geometric measurements

- For incompressible flows, the following quantities are conserved
- Total self-linking of vortex lines: Helicity



$$\int_W \eta \wedge d\eta$$

$$= \int_W \mathbf{u} \cdot \mathbf{w}$$

- Total area vector (known as impulse; equals to total fluid momentum)

$$\mathbf{A} = q_0 \int_W \tfrac{1}{2}(\mathbf{x} \times \mathbf{w})\, dV$$

Vorticity

- In barotropic Euler fluid and incompressible fluid:

$$\left(\frac{\partial}{\partial t} + \underbrace{\mathscr{L}_{\vec{u}}}_{\text{2-form}}\right)\omega = 0$$

  - First, show $\left(\frac{\partial}{\partial t} + \underbrace{\mathscr{L}_{\vec{u}}}_{\text{1-form}}\right)\eta = -d\left(p - \frac{1}{2}|\vec{u}|^2_{\flat_W}\right)$
  - Take exterior derivative

Vorticity

- In barotropic Euler fluid and incompressible fluid:

$$\left(\frac{\partial}{\partial t} + \underset{\text{2-form}}{\mathscr{L}_{\vec{u}}}\right)\omega = 0$$

- In vector notation,

3D:

$$\tilde{\mathbf{w}} := \mathbf{w}/q$$

$$\dot{\tilde{\mathbf{w}}} + \nabla_{\mathbf{u}}\tilde{\mathbf{w}} = \nabla_{\tilde{\mathbf{w}}}\mathbf{u}$$

2D:

$$\tilde{w} := w/q$$

$$\dot{\tilde{w}} + \nabla_{\mathbf{u}}\tilde{w} = 0$$

- To see it why dividing vorticity by density, note that $\omega = i_{\tilde{\mathbf{w}}}\rho$ and ω, ρ are both Lie advected

History of Euler equation

- Bernoulli principle (1738)
- d'Alembert (1740's) $\operatorname{div} \vec{u} = 0$
- Euler (1750's)

$$\frac{\partial}{\partial t} q + \operatorname{div}(q\vec{u}) = 0$$
$$\frac{\partial}{\partial t} \vec{u} + \nabla_{\vec{u}} \vec{u} = -\frac{1}{q} \operatorname{grad} p$$
$$p = \pi(q)$$

$$\frac{\partial}{\partial t} \vec{u} + \nabla_{\vec{u}} \vec{u} = -\frac{1}{q_0} \operatorname{grad} p$$
$$\operatorname{div} \vec{u} = 0$$

- Lagrange (1788): Least action principle $\rho_M \overset{\nabla}{\ddot{\phi}} = -J(\operatorname{grad} p) \circ \phi$
- Cauchy (1815): $\omega = d\eta$; $\phi^* \omega$ is time-independent
- Helmholtz (1858): $(\frac{\partial}{\partial t} + \mathscr{L}_{\vec{u}})\omega = 0$ in vector form

Helmholtz's vorticity equation is the main point of reference

- Clebsch (1859): Least action principle in Eulerian coordinate, which directly leads to conservation of vorticity.

History of Euler equation

- Kelvin (1868): Circulation theorem $\oint_C \phi^* \eta$ is time-independent
- Lin (1963): More intuitive least action in Eulerian coordinate, can be generalized to many other fluid system with thermal variables.
- Arnold (1966): Euler equation as geodesic on space of flow maps, which is a Lie group. Fluid equation and rigid body rotation are unified as Euler–Arnold equation.
- Moreau (1961) Moffatt (1969): Conservation of helicity $\int_W \eta \wedge d\eta$
- Marsden, Weinstein (1970’s): Conservation laws using group theory
- Arnold–Khesin (1998): Book on topological fluid dynamics