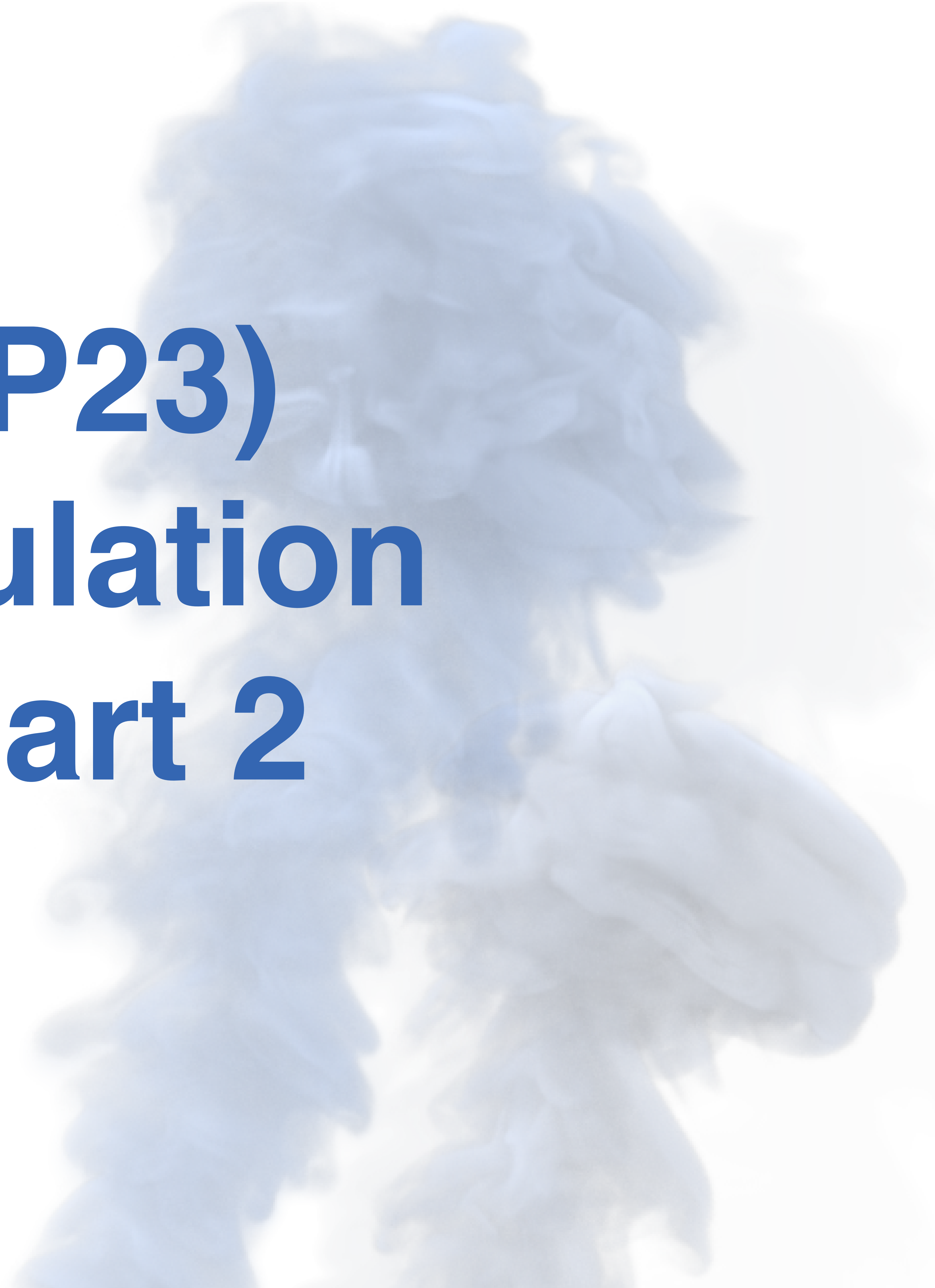
CSE 291 (SP23)
Physical Simulation
Elasticity: Part 2

Albert Chern



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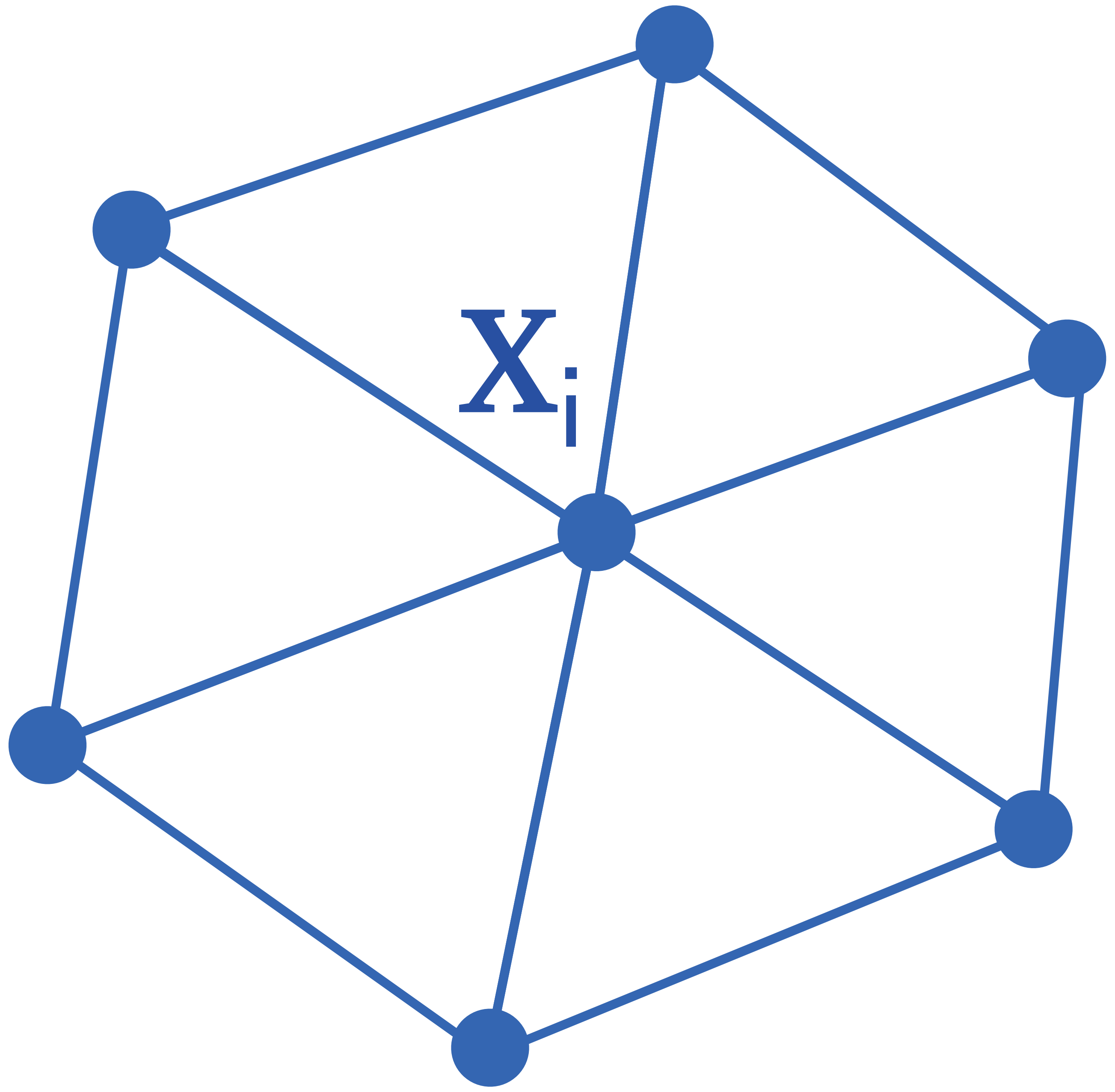
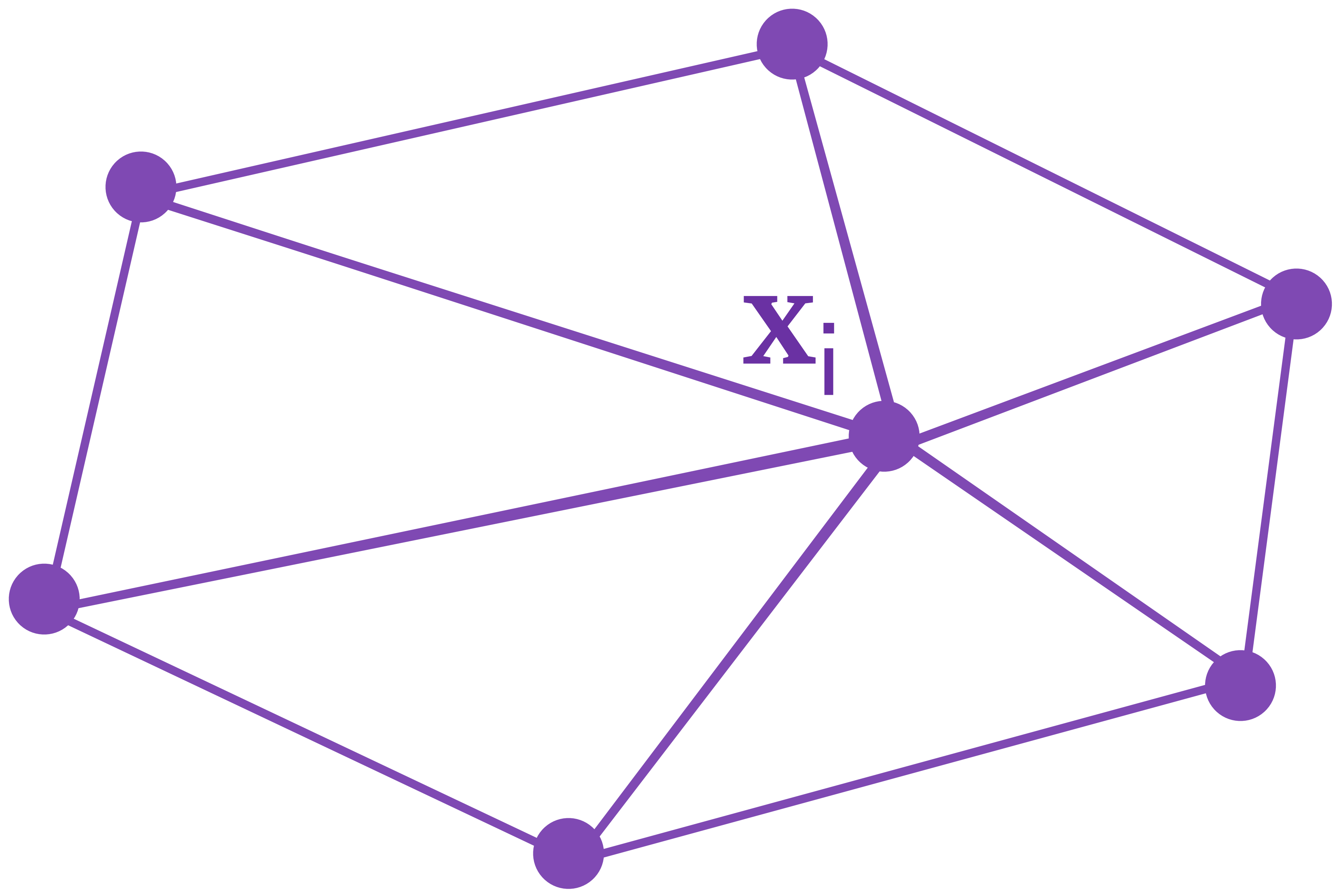
Finite Element Elasticity

- Finite element elasticity
- More on Stress–Strain relation

Finite element simulation

- Discretize the deformable body by triangle mesh (2D) or tetrahedral mesh (3D) ($n = \dim(M)$)
- Each vertex i stores a fixed rest position $\mathbf{X}_i$ (material coordinate) and a variable world position $\mathbf{x}_i$ (representing value of flow map)

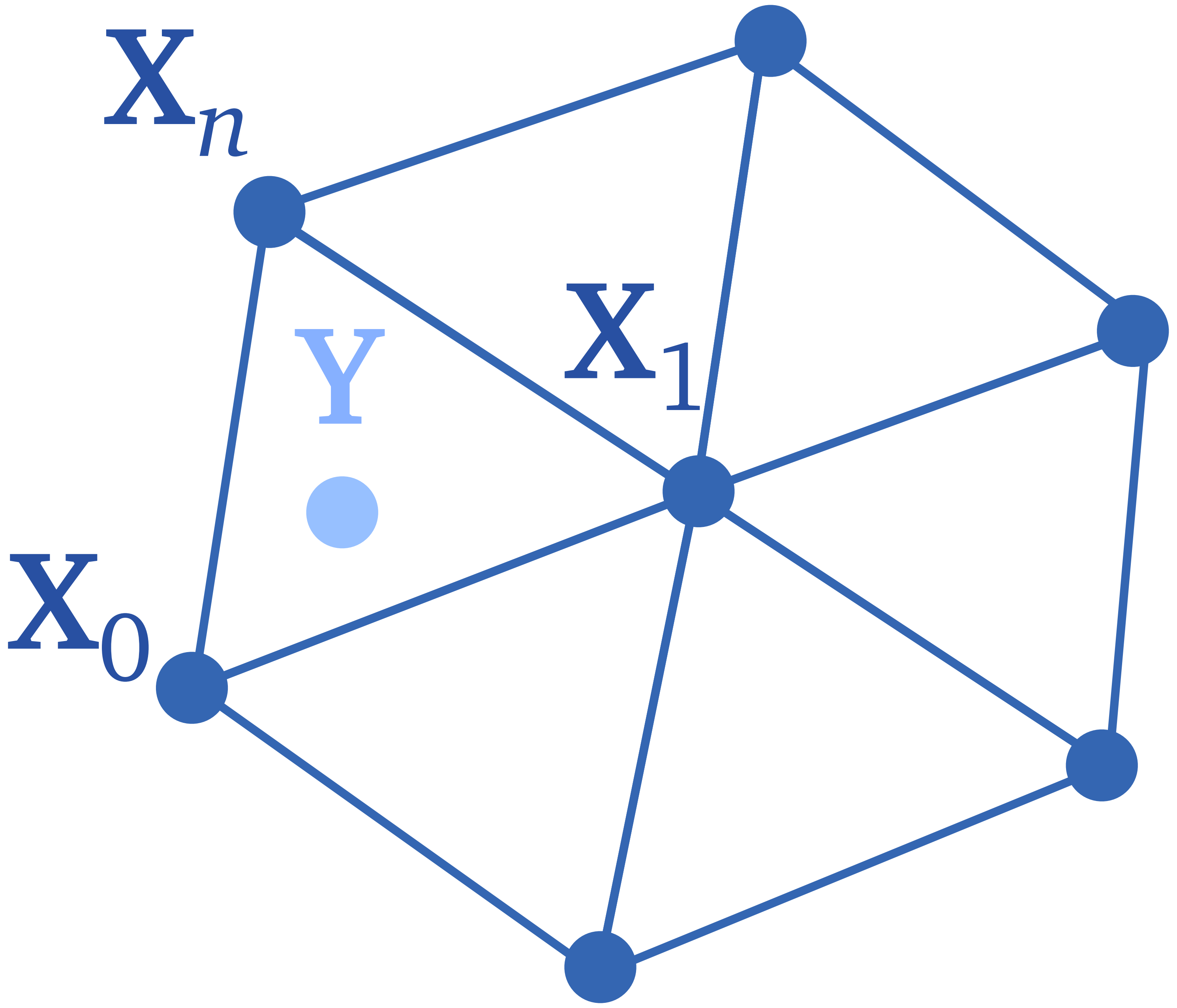


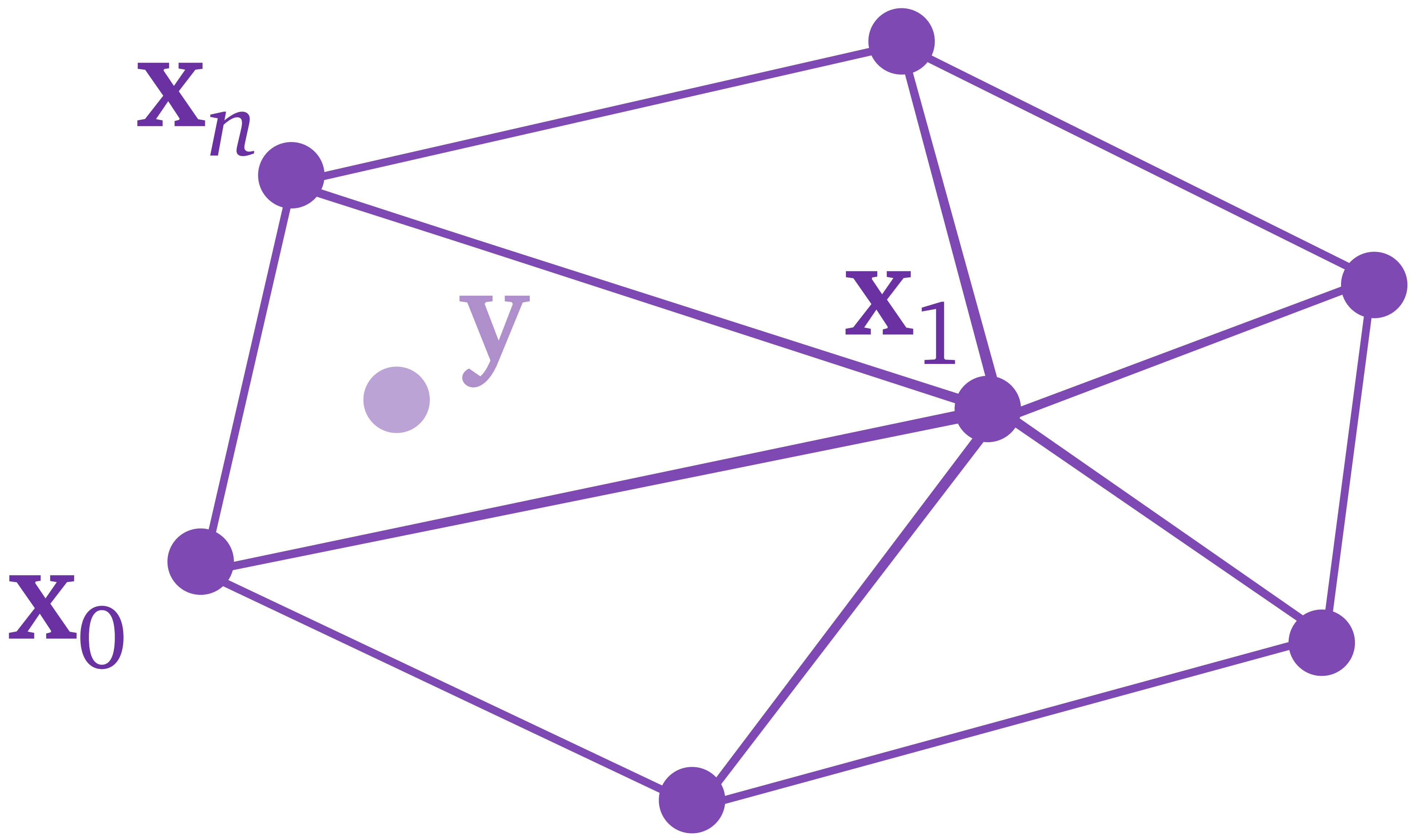


Linear interpolation

- The data on the vertices can be linearly interpolated into a piecewise linear flow map.

$$\begin{bmatrix} | \\ \mathbf{y} \\ | \\ 1 \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \mathbf{x}_0 & \mathbf{x}_1 & \cdots & \mathbf{x}_n \\ | & | & & | \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} | & | & & | \\ \mathbf{X}_0 & \mathbf{X}_1 & \cdots & \mathbf{X}_n \\ | & | & & | \\ 1 & 1 & \cdots & 1 \end{bmatrix}^{-1} \begin{bmatrix} | \\ \mathbf{Y} \\ | \\ 1 \end{bmatrix}$$





Deformation gradient

- The data on the vertices can be linearly interpolated into a piecewise linear flow map.

$$\begin{bmatrix} | \\ \mathbf{y} \\ | \\ 1 \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \mathbf{x}_0 & \mathbf{x}_1 & \cdots & \mathbf{x}_n \\ | & | & & | \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} | & | & & | \\ \mathbf{X}_0 & \mathbf{X}_1 & \cdots & \mathbf{X}_n \\ | & | & & | \\ 1 & 1 & \cdots & 1 \end{bmatrix}^{-1} \begin{bmatrix} | \\ \mathbf{Y} \\ | \\ 1 \end{bmatrix}$$

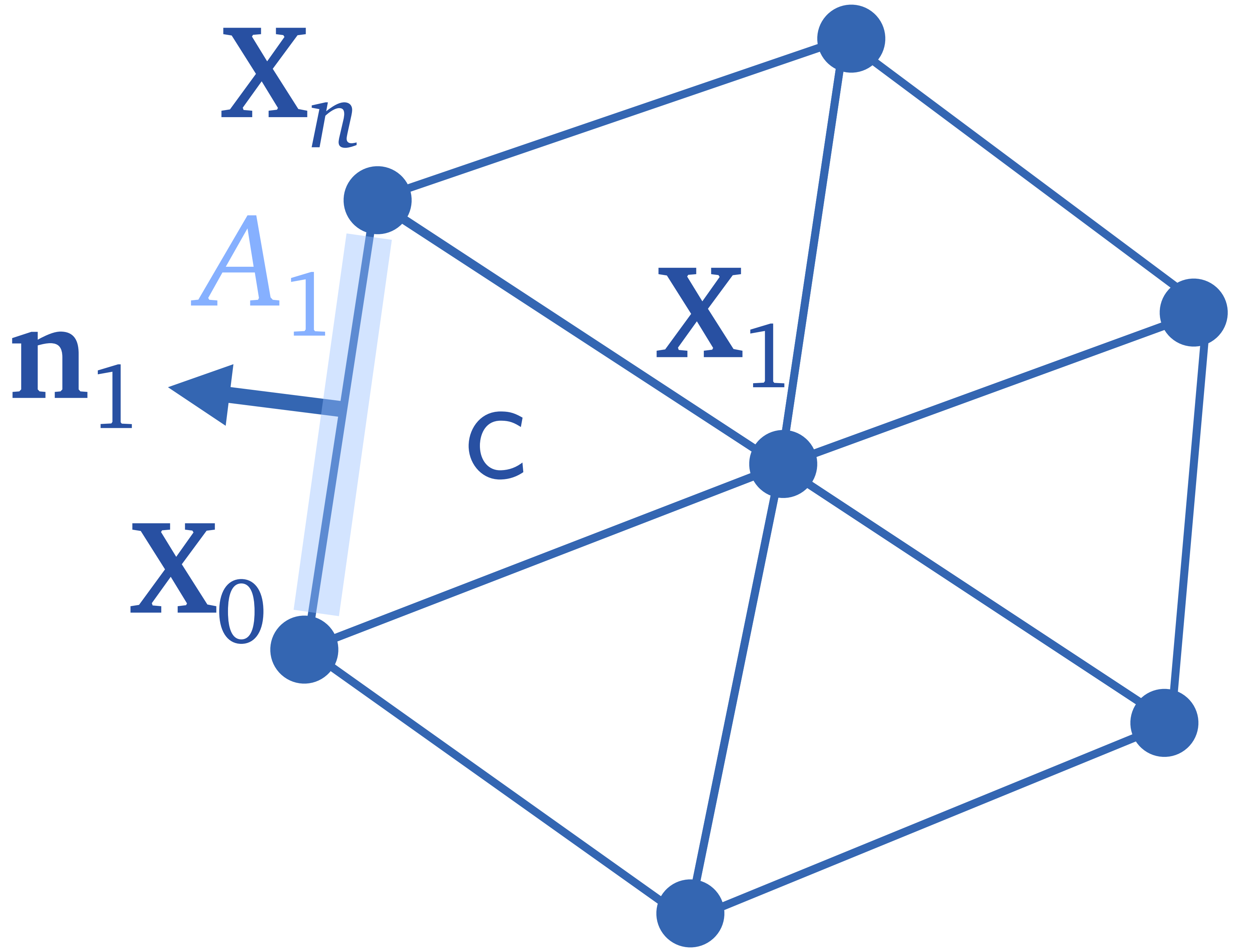
- The deformation gradient is a piecewise constant matrix

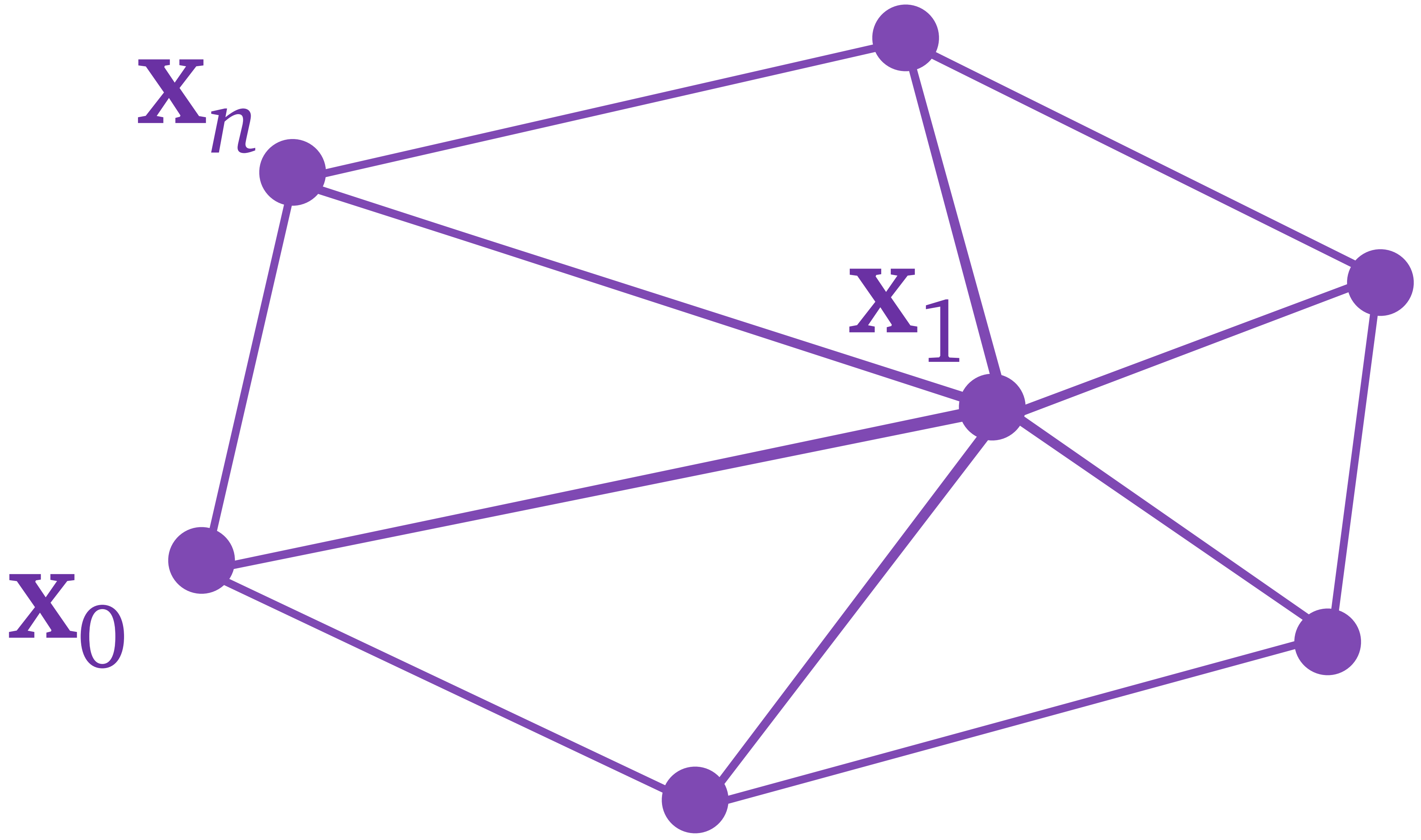
$$\begin{bmatrix} & & & 0 \\ & \mathbf{F} & & | \\ & & & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \mathbf{x}_0 & \mathbf{x}_1 & \cdots & \mathbf{x}_n \\ | & | & & | \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} | & | & & | \\ \mathbf{X}_0 & \mathbf{X}_1 & \cdots & \mathbf{X}_n \\ | & | & & | \\ 1 & 1 & \cdots & 1 \end{bmatrix}^{-1}$$

Deformation gradient

- If $A_j \mathbf{n}_j$ is the area normal of the opposite face of *j*-th vertex and V is the volume of the cell. Then

$$\mathbf{F}_\mathsf{c} = -\frac{1}{nV_\mathsf{c}} \sum_{j=0}^{n} \begin{bmatrix} | \\ \mathbf{x}_j \\ | \end{bmatrix} \begin{bmatrix} - & A_{\mathsf{c},j}\mathbf{n}_{\mathsf{c},j}{}^\intercal & - \end{bmatrix}$$





Strain and stress computation

- Now in each cell we have deformation gradient $\mathbf{F}$
- We can compute the Cauchy–Green tensor per cell $\mathbf{C} = \mathbf{F}^\mathsf{T}\mathbf{F}$
- Like the smooth theory, build $\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I})$
- Look up some stress–strain relation

$$\mathbf{S} = 2\mu\mathbf{E} + \lambda\,\mathrm{tr}(\mathbf{E})\mathbf{I}$$

$$\begin{tikzcd}
\mathbf{x} \text{ per vtx} \arrow[r] & \mathbf{F} \text{ per cell} \arrow[r] & \mathbf{C} \text{ per cell} \arrow[d, dash, dotted] \\
 & & \mathbf{S} \text{ per cell}
\end{tikzcd}$$

Stress computation

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$$\mathbf{S} = 2\mu\mathbf{E} + \lambda\,\mathrm{tr}(\mathbf{E})\mathbf{I}$$

- 1st-Piola stress

$$\mathbf{P} = \mathbf{F}\mathbf{S}$$

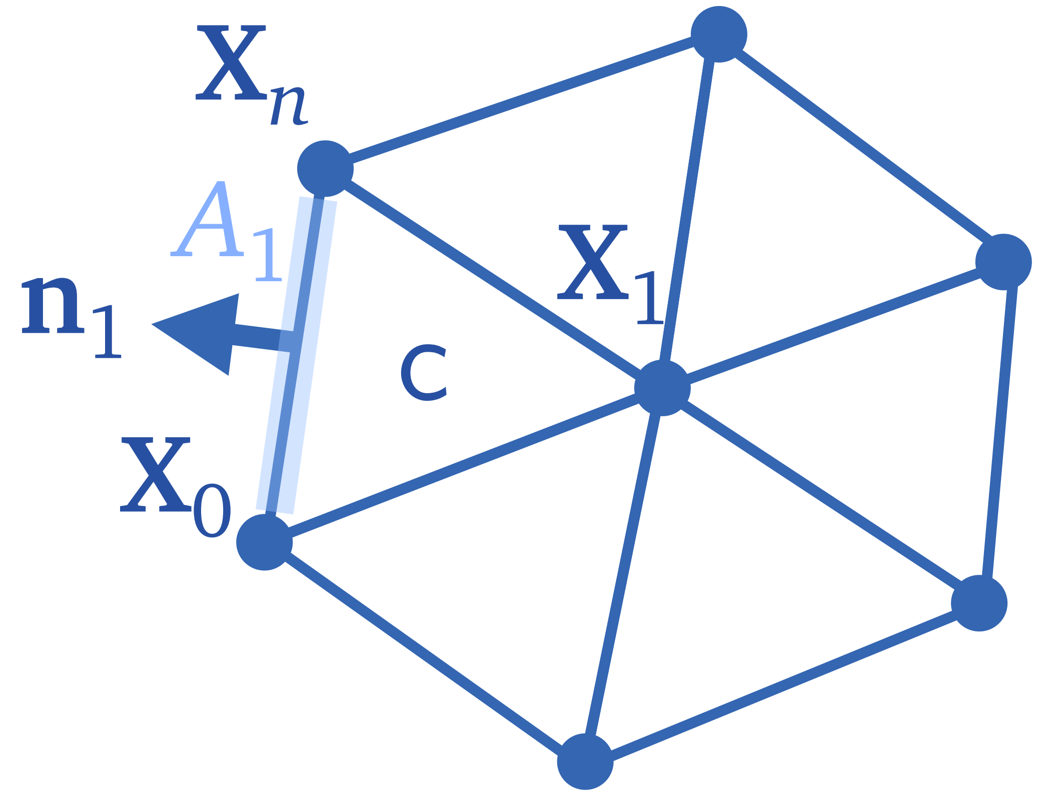
- Compute force by taking adjoint of gradient

$$\begin{tikzcd}
\mathbf{x}\text{ per vtx} \arrow[r] & \mathbf{F}\text{ per cell} \arrow[r] & \mathbf{C}\text{ per cell} \arrow[d, dash, dotted] \\
-\mathbf{f}\text{ per vtx} & \mathbf{P}\text{ per cell} \arrow[l] & \mathbf{S}\text{ per cell} \arrow[l]
\end{tikzcd}$$

Total force computation

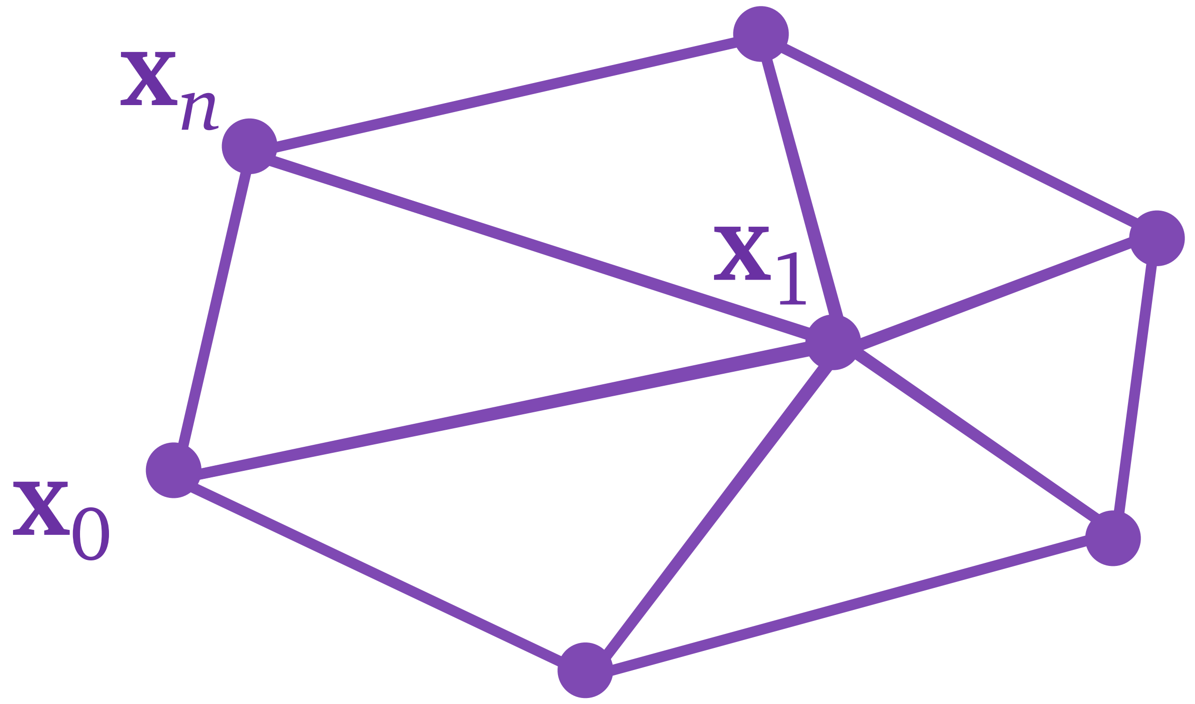
- The differential of $\mathbf{F}$ with respect to $\mathbf{x}$

$$\mathring{\mathbf{F}}_{\mathsf{c}} = -\frac{1}{nV_{\mathsf{c}}}\sum_{\mathsf{v}\prec\mathsf{c}}\begin{bmatrix} | \\ \mathring{\mathbf{x}}_{\mathsf{v}} \\ | \end{bmatrix}\begin{bmatrix} - & A_{\mathsf{c},\mathsf{v}}\mathbf{n}_{\mathsf{c},\mathsf{v}}^{\intercal} & - \end{bmatrix}$$



- Adjoint: accumulate traction force to vertex

$$\sum_{\mathsf{c}}\mathrm{tr}(\mathbf{P}_{\mathsf{c}}\mathring{\mathbf{F}}_{\mathsf{c}}^{\intercal})V_{\mathsf{c}} = \sum_{\mathsf{v}} -\mathbf{f}_{\mathsf{v}}^{\intercal}\mathring{\mathbf{x}}_{\mathsf{v}}$$

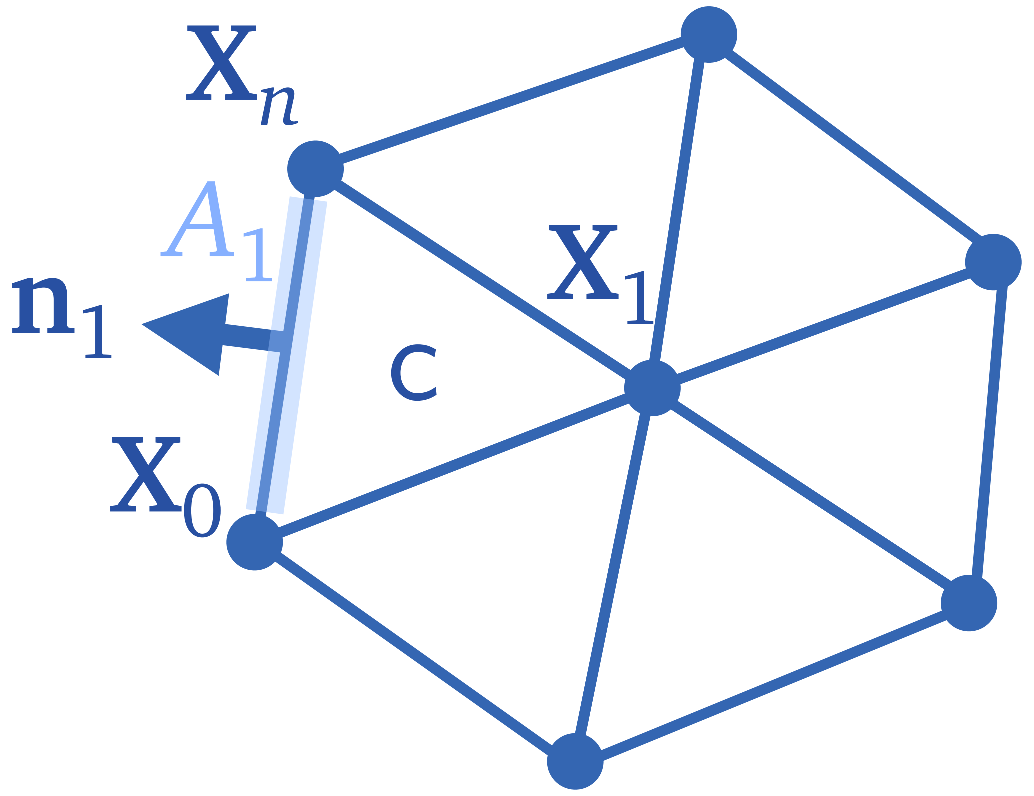


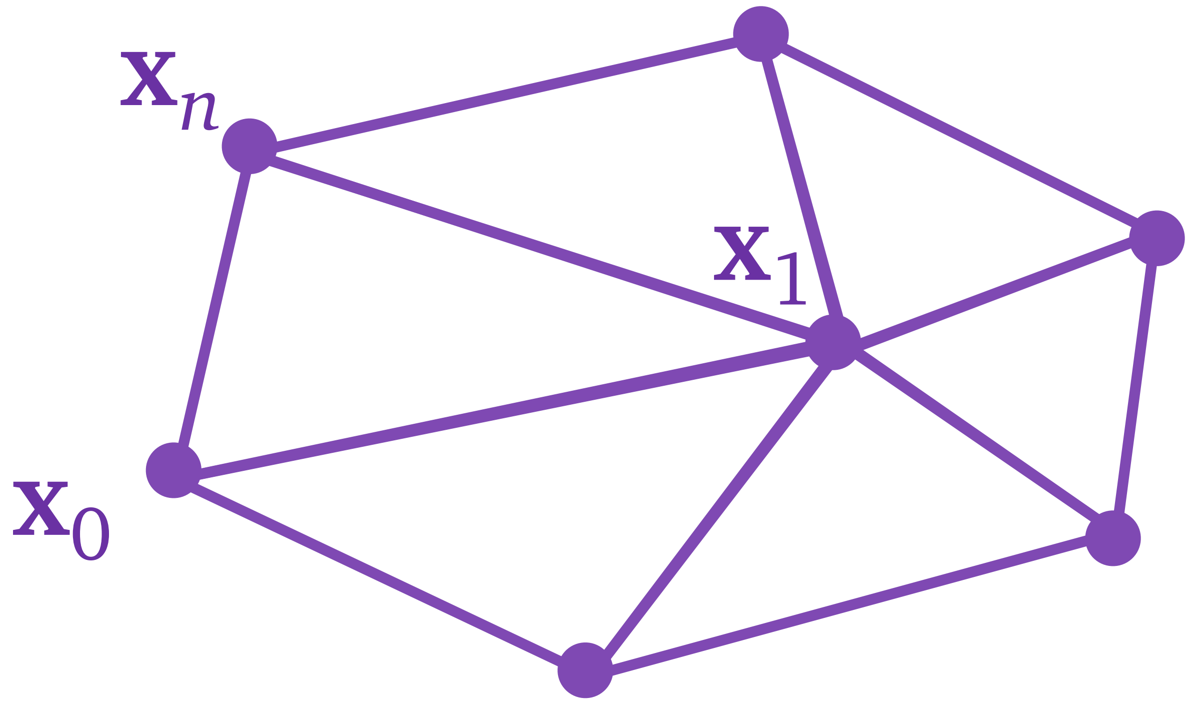
$$\mathbf{f}_{\mathsf{v}} = \frac{1}{n}\sum_{\mathsf{c}\succ\mathsf{v}}\mathbf{P}_{\mathsf{c}}\mathbf{n}_{\mathsf{c},\mathsf{v}}A_{\mathsf{c},\mathsf{v}}$$

Equation of motion

$$\mathbf{f}_{\mathrm{v}} = \frac{1}{n} \sum_{\mathrm{c} \succ \mathrm{v}} \mathbf{P}_{\mathrm{c}} \mathbf{n}_{\mathrm{c,v}} A_{\mathrm{c,v}}$$

$$m_{\mathrm{v}} \ddot{\mathbf{x}}_{\mathrm{v}} = \mathbf{f}_{\mathrm{v}}$$



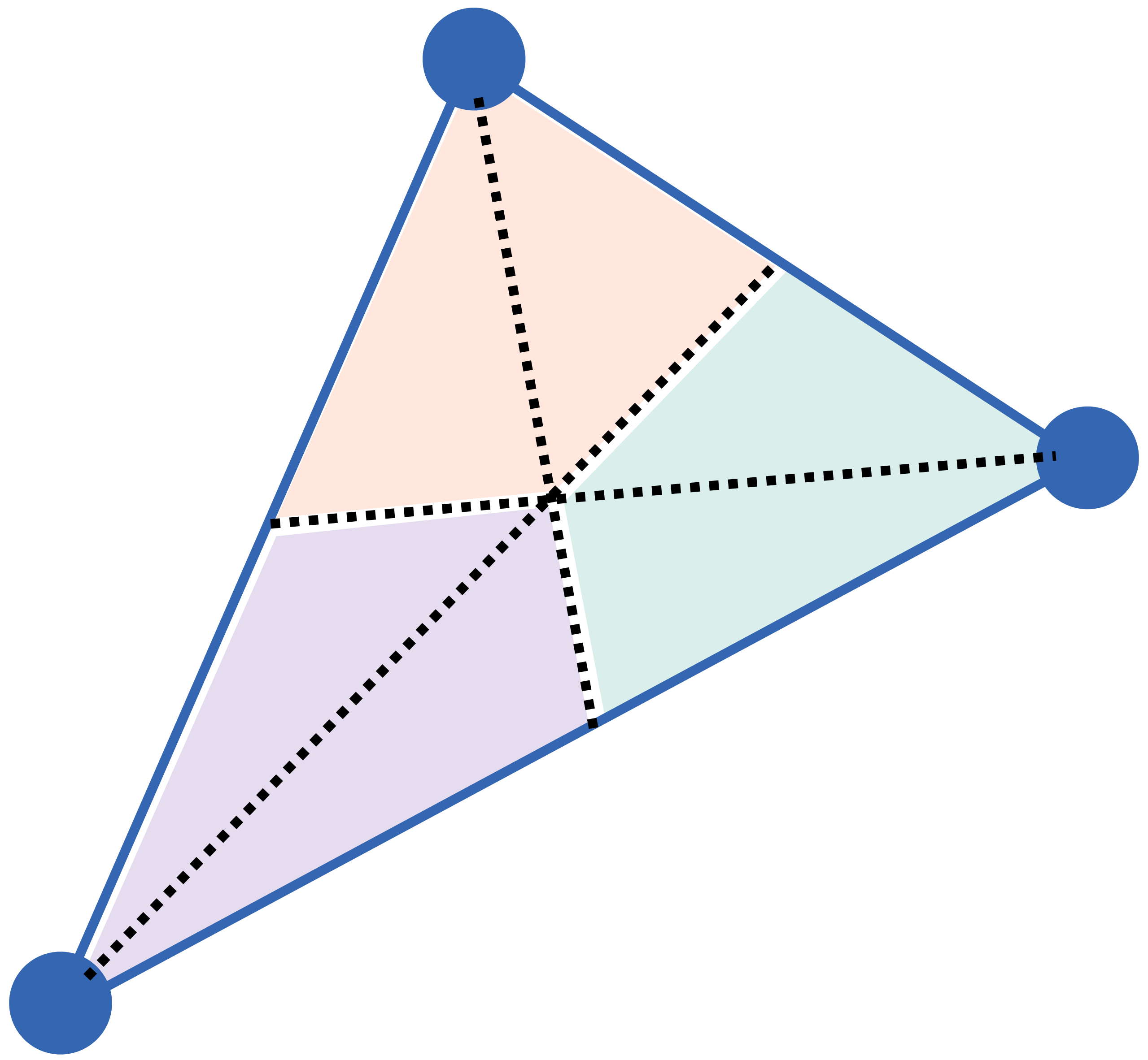


Mass computation

- The total mass of each vertex should be proportional to the vertex volume approximated by

$$m_{\mathrm{v}} = \sum_{\mathrm{c} \succ \mathrm{v}} \frac{1}{n+1} V_{\mathrm{c}}$$

- This is called lumped mass



Time integration

$$m_{\mathsf{v}}\ddot{\mathbf{x}}_{\mathsf{v}} = \mathbf{f}_{\mathsf{v}} + \mathbf{f}_{\mathsf{v}}^{\text{ext}}$$

- RK4 or Symplectic Euler method
  - Just need to evaluate force $(\mathbf{f}_{\mathsf{v}})_{\mathsf{v}}$ given current position $(\mathbf{x}_{\mathsf{v}})_{\mathsf{v}}$
  - Stepsize $\Delta t = O(\text{edge lengths})$

- Implicit Euler (with incremental potential): stable

$$\mathbf{x}^{(n+1)} = \underset{\mathbf{x}\in\mathbb{R}^m}{\operatorname{argmin}} \sum_{\mathsf{v}} \frac{m_{\mathsf{v}}}{2\Delta t^2} |\mathbf{x}_{\mathsf{v}} - \mathbf{x}_{\mathsf{v}}^{\text{pred}}|^2 + \mathcal{U}(\mathbf{x})$$

Time integration

- Implicit Euler (with incremental potential): stable

$$\mathbf{x}^{(n+1)} = \underset{\mathbf{x}\in\mathbb{R}^m}{\operatorname{argmin}} \sum_{\mathsf{v}} \frac{m_{\mathsf{v}}}{2\Delta t^2} |\mathbf{x}_{\mathsf{v}} - \mathbf{x}_{\mathsf{v}}^{\text{pred}}|^2 + \mathcal{U}(\mathbf{x})$$

  - For gradient descent (or Newton) method with line search
  - Need evaluation of $\mathcal{U}(\mathbf{x}) = \sum_{\mathsf{c}} U(\mathbf{F}^{\mathsf{T}}\mathbf{F}) V_{\mathsf{c}}$
  - Need evaluation of differential of potential (same as force evaluation)
  - Need an (approximated) Hessian for the potential

Time integration

- Need an (approximated) Hessian for the potential

$$\begin{tikzcd}
\mathbf{x}\ \text{per vtx} \arrow[r, "{[\mathrm{grad}]}"] & \mathbf{F}\ \text{per cell} \arrow[r, "\mathbf{F}^{\mathsf{T}}(\cdot)+(\cdot)^{\mathsf{T}}\mathbf{F}"] & \mathbf{C}\ \text{per cell} \arrow[d, dotted, dash] \\
-\mathbf{f}\ \text{per vtx} & \mathbf{P}\ \text{per cell} \arrow[l, "{[\mathrm{div}]}"'] & \mathbf{S}\ \text{per cell} \arrow[l, "\mathbf{F}(\cdot)"']
\end{tikzcd}$$

Laplacian

$$\mathbf{L} = \begin{bmatrix} \quad -\mathrm{div} \quad \end{bmatrix} \begin{bmatrix} V_{\mathrm{c}} & \\ & \ddots \end{bmatrix} \begin{bmatrix} \quad \mathrm{grad} \quad \end{bmatrix}$$

can serve as an approximated Hessian

- True Hessian: replace the central term by the cell-wise Hessian of the energy

More on Stress–Strain relation

- Finite element elasticity
- More on Stress–Strain relation

Designing potential energy

- Given the Cauchy–Green $\mathbf{C} = \mathbf{F}^{\intercal}\mathbf{F}$

$$\sharp_M C = \sharp_M F^* \flat_W F \in \Gamma(\mathrm{End}(TM))$$

  - (as endomorphism that measures the deviation between induced metric from the world and the pre-defined material metric)

- Design a potential energy function $U(\mathbf{C})$
  - Note that it's a function on symmetric matrices
  - The energy is said to be *isotropic* if

$$U(\mathbf{C}) = U(\mathbf{R}^{\intercal}\mathbf{C}\mathbf{R})$$

for rotation matrices $\mathbf{R}$ $\qquad R^* \flat_M R = \flat_M$

Designing potential energy

- If the material is isotropic $U(\mathbf{C}) = U(\mathbf{R}^\mathsf{T}\mathbf{C}\mathbf{R})$

  then the energy is only a function of the eigenvalues (modulo permutation)

$$\text{eigenvalues}(\mathbf{C}) = \{\lambda_1, \lambda_2, \lambda_3\}$$

- By the way, these eigenvalues are the square of the eigenvalues of **Y** in polar decomposition $\mathbf{F} = \mathbf{R}\mathbf{Y}$; equivalently, square of singular values of **F**. They are the square of principal stretching.

Designing potential energy

- Can we model *U* like $U(\mathbf{C}) = u(\lambda_1, \lambda_2, \lambda_3)$?
  - Generally this wouldn't respect symmetry under label permutation
  - View the eigenvalues as the roots of a polynomial, and use the coefficient of the polynomial as our parameters

$$\{\lambda_1, \lambda_2, \lambda_3\} = \mathrm{roots}(t^3 - I_1 t^2 + I_2 t - I_3; t)$$

  - These coefficients are called the "invariants":

$$\begin{aligned} I_1 &= \lambda_1 + \lambda_2 + \lambda_3 &&= \mathrm{tr}(\mathbf{C}) \\ I_2 &= \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3 &&= \tfrac{1}{2}(\mathrm{tr}(\mathbf{C})^2 - \mathrm{tr}(\mathbf{C}^2)) \\ I_3 &= \lambda_1\lambda_2\lambda_3 &&= \det(\mathbf{C}) \end{aligned}$$

  - Characteristic polynomial $t^3 - I_1 t^2 + I_2 t - I_3 = \det(t\mathbf{I} - \mathbf{C})$

Designing potential energy

$$\begin{aligned} I_1 &= \lambda_1 + \lambda_2 + \lambda_3 &&= \operatorname{tr}(\mathbf{C}) \\ I_2 &= \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3 &&= \tfrac{1}{2}(\operatorname{tr}(\mathbf{C})^2 - \operatorname{tr}(\mathbf{C}^2)) \\ I_3 &= \lambda_1\lambda_2\lambda_3 &&= \det(\mathbf{C}) \end{aligned}$$

- We model $U(\mathbf{C}) = w(I_1, I_2, I_3)$
  - How do you do chain rule? (Blackboard)
- For example neo-Hookean model

how much material respond to
1D stretch and volume change

$$w(I_1, I_2, I_3) = \tfrac{\mu}{2}(I_1 - 3 - \ln I_3) + \tfrac{\lambda}{2}(\sqrt{I_3} - 1)^2$$

- Approximately

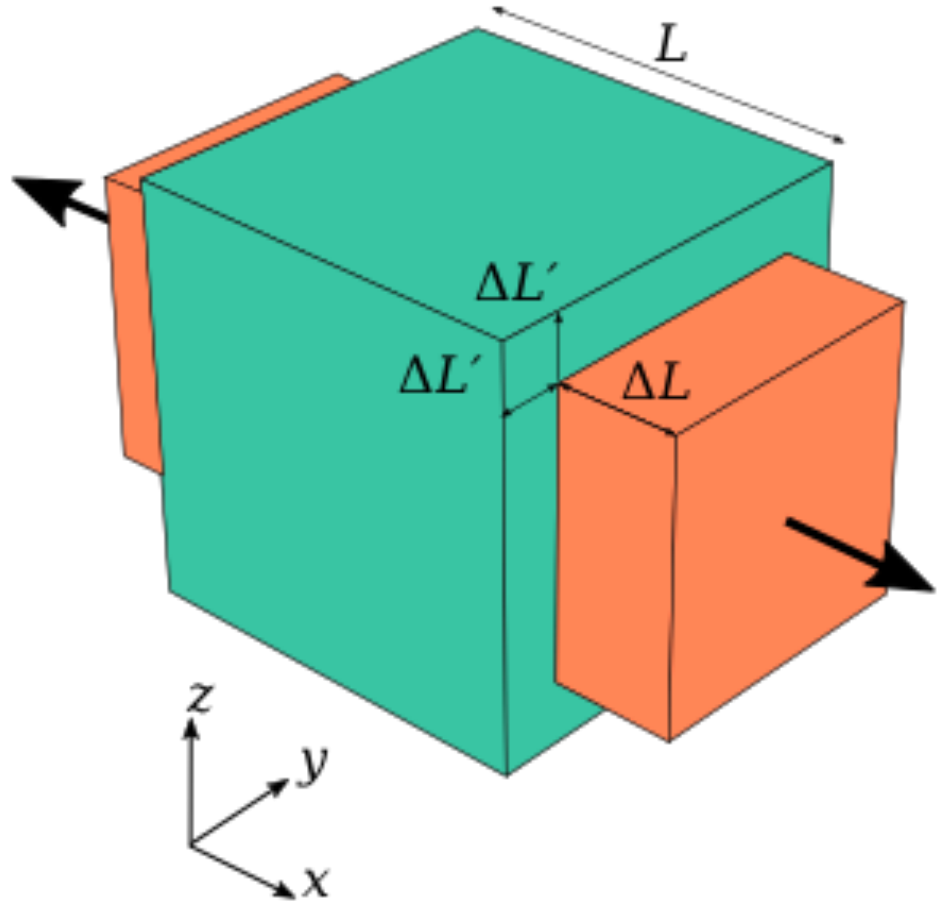
$$U(C) = \left(\tfrac{\lambda}{2}\operatorname{tr}(E)^2 + \mu \operatorname{tr}(E^2)\right) dV_M$$

Designing potential energy

- One can measure Young's modulus *E*, and Poisson ratio ν

$$E = \frac{F/A}{\Delta\ell/\ell}$$

similar to spring constant



$$\nu = -\frac{\Delta L'}{\Delta L}$$

usually between 0 and 0.5; it could be negative.

- Lamé constants

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$$

$$\mu = \frac{E}{2(1+\nu)}$$