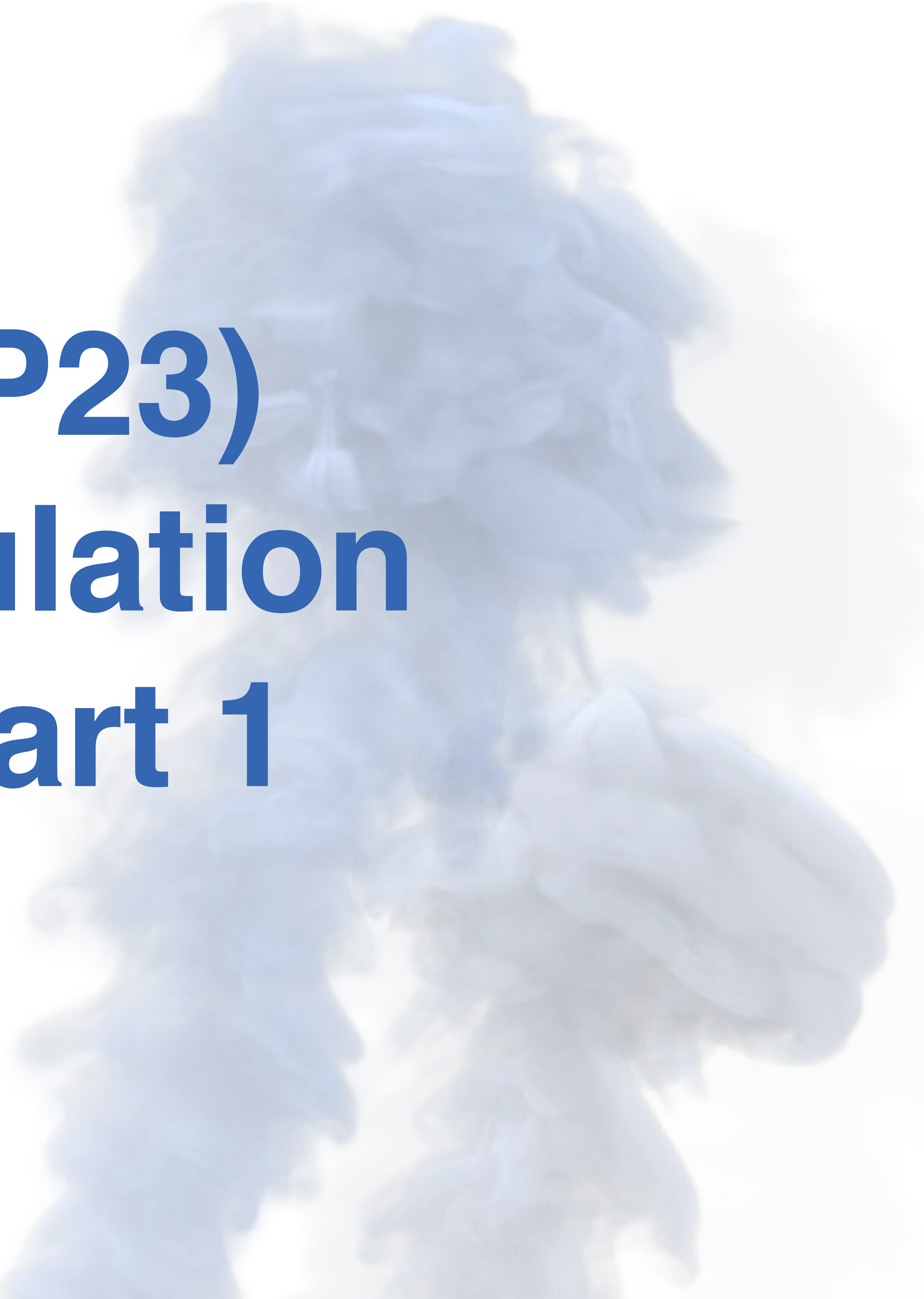
CSE 291 (SP23)
Physical Simulation
Elasticity: Part 1

Albert Chern

UC San Diego
JACOBS SCHOOL OF ENGINEERING

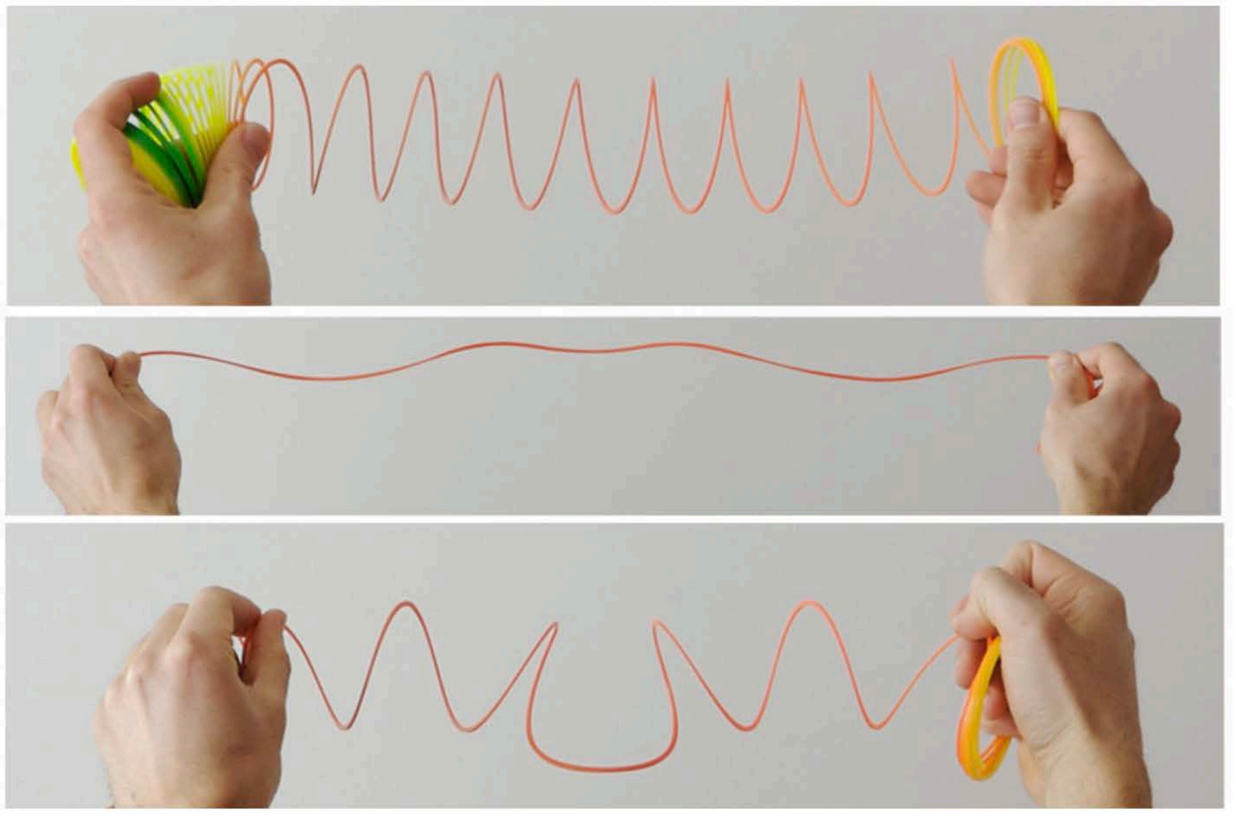
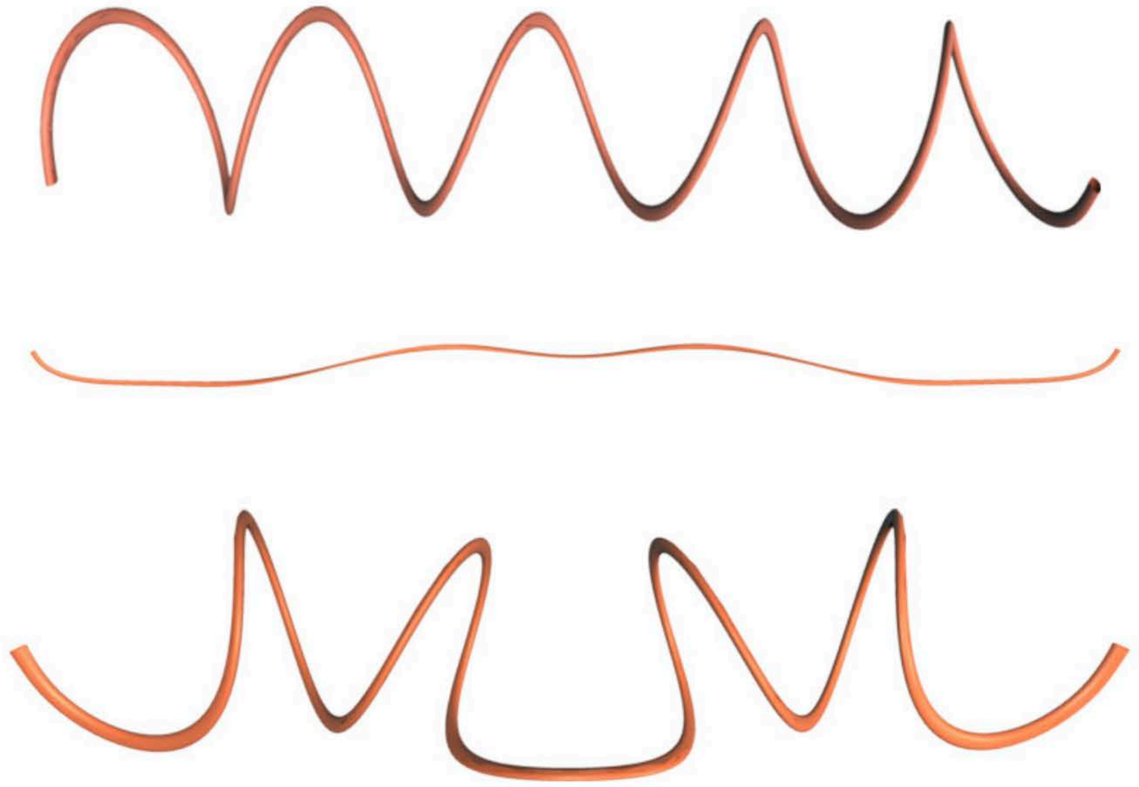
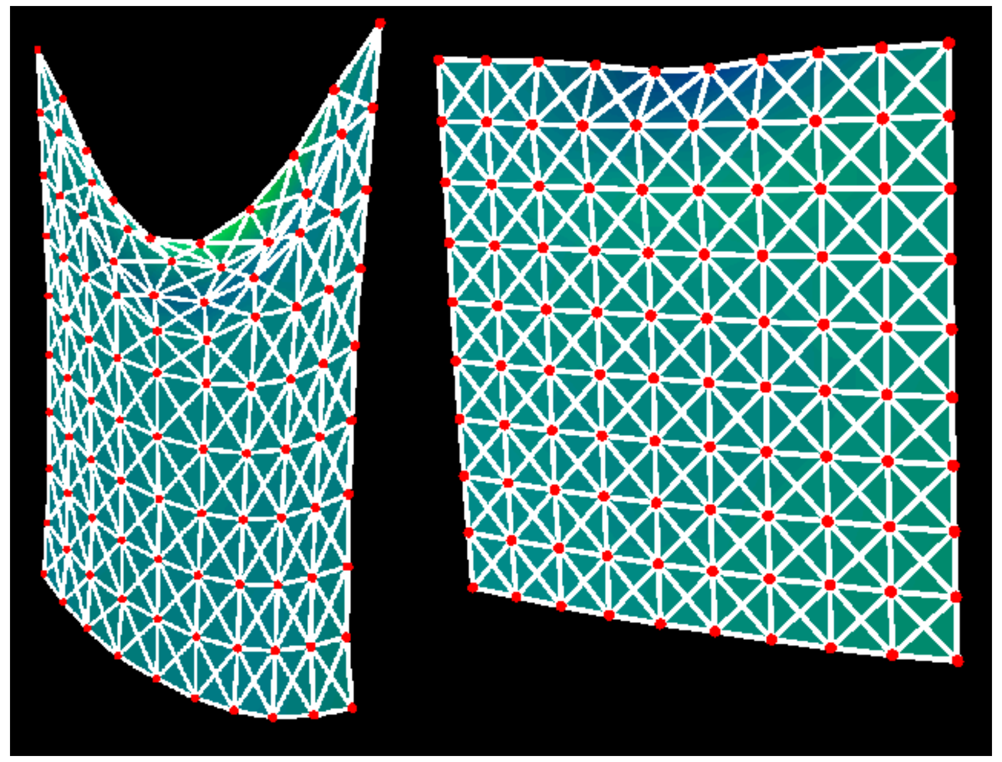


Overview

- Overview
- Postulates and roadmap
  - 2nd Piola tensor
  - 1st Piola tensor
  - Elastic force
  - Summary of derivation
- Cauchy stress and linear elasticity

Elastic solid bodies

- Potential energy is a function of how the body is deformed
- We get restoration force back to undeformed state



Overview

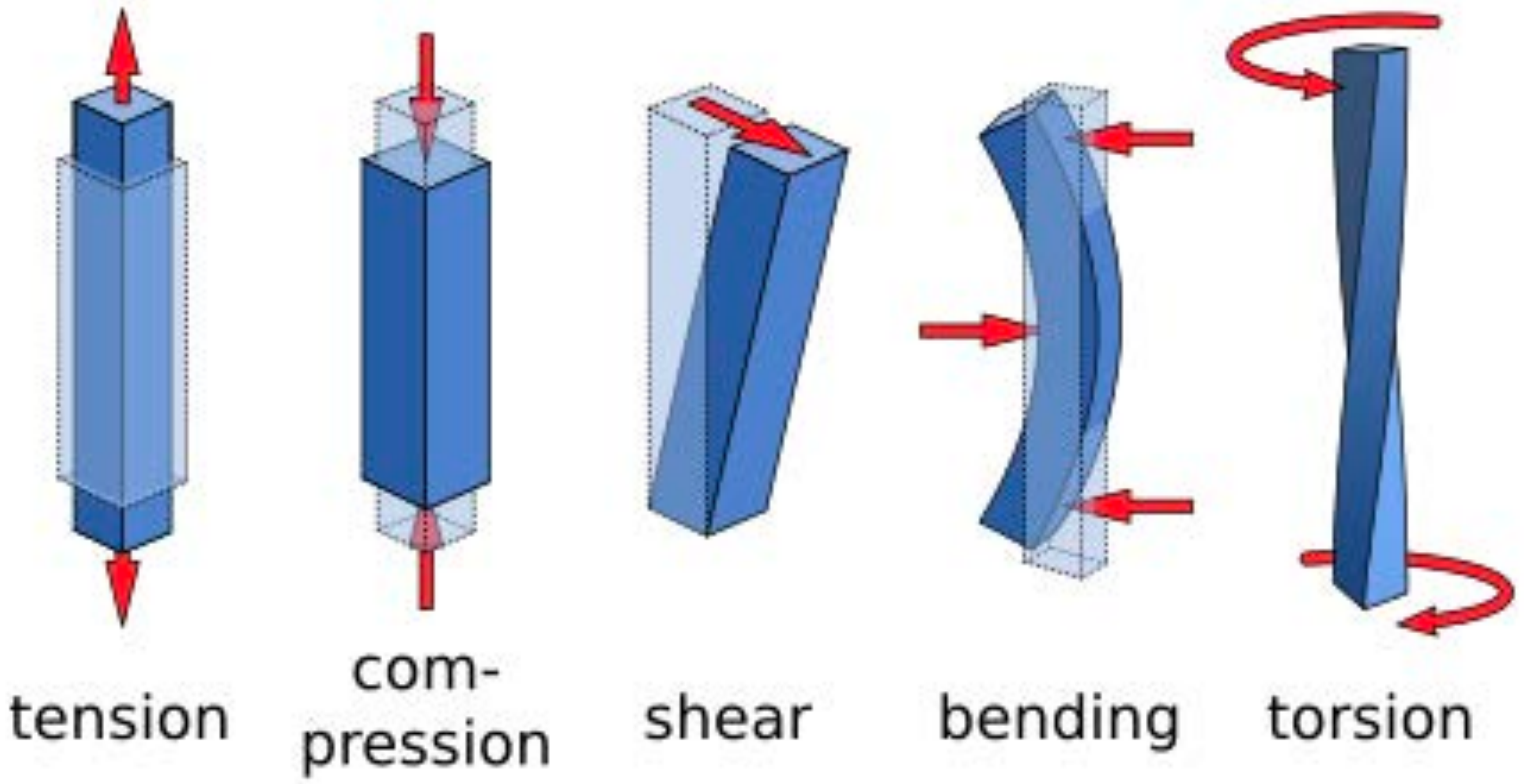
- The setup for deformable body
  - Manifold M : **Material** coordinate, **Lagrangian** coordinate
  - Manifold W : **World** coordinate, **Eulerian** coordinate
  - A state of a deformable body is a map (called **flow map**)

$$\phi : M \rightarrow W$$

- Let (X,Y,Z) denote the Cartesian coordinate for M and (x,y,z) the Cartesian coordinate for W
- Flow map $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \phi^1(X,Y,Z) \\ \phi^2(X,Y,Z) \\ \phi^3(X,Y,Z) \end{bmatrix}$

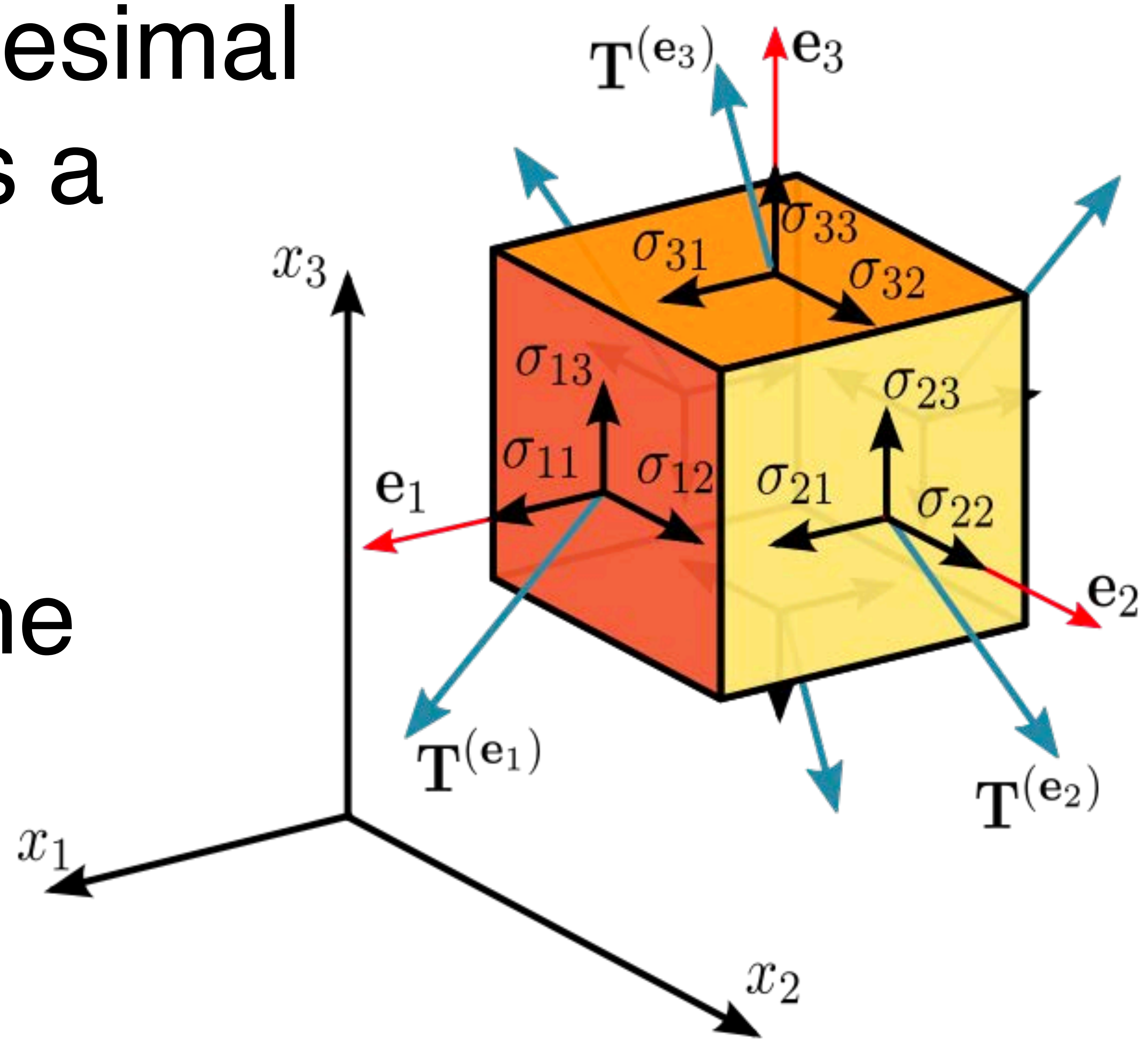
Overview

- What is the force experienced by each point?
- People knew about there can be **stress** inside material



- More rigorously introduced by Cauchy in early 19th century.

Cauchy's stress theory



- There is a traction force $\mathbf{T}^{(\mathbf{n})}$ for infinitesimal plane with normal vector $\mathbf{n}$. It satisfies a reciprocity property

$$\mathbf{T}^{(-\mathbf{n})} = -\mathbf{T}^{(\mathbf{n})}$$

- The total force experienced by a volume is given by the total traction force

$$\iiint_V \mathbf{f}\, dV = \oiint_{\partial V} \mathbf{T}^{(\mathbf{n})}\, dA$$

- **Cauchy's stress theorem:**

  $\mathbf{T}^{(\mathbf{n})}$ is linear in $\mathbf{n}$; i.e., there is a matrix σ called **stress tensor** such that $\mathbf{T}^{(\mathbf{n})} = \sigma\mathbf{n}$.

Cauchy's stress theory

- Net force at each point $\mathbf{f} = \nabla \cdot \sigma$ $f^i = \partial_j \sigma^{ij}$
- Balance equation in equilibrium $\nabla \cdot \sigma + \mathbf{f}_{\text{ext}} = \mathbf{0}$
- At equilibrium, the Cauchy stress tensor σ must be symmetric.
  - This can be shown by requesting zero net torque.
  - When it's not in equilibrium, Cauchy stress tensor can be non-symmetric, such as in viscoelastic materials.
  - For a different reason that we will see next, in pure elastic material, Cauchy stress σ is always symmetric even at non-equilibrium.

Alternative stress tensors

- Deformation gradient $\mathbf{F} = \begin{bmatrix} \frac{\partial \phi^1}{\partial X} & \frac{\partial \phi^1}{\partial Y} & \frac{\partial \phi^1}{\partial Z} \\ \frac{\partial \phi^2}{\partial X} & \frac{\partial \phi^2}{\partial Y} & \frac{\partial \phi^2}{\partial Z} \\ \frac{\partial \phi^3}{\partial X} & \frac{\partial \phi^3}{\partial Y} & \frac{\partial \phi^3}{\partial Z} \end{bmatrix}$ $J = \det(\mathbf{F})$
- 1st Piola–Kirchhoff stress $\mathbf{P} = J \sigma \mathbf{F}^{-\mathsf{T}}$
  - Net world-force in Lagrangian coordinate $\mathbf{f} \circ \phi = \nabla \cdot \mathbf{P}$
- 2nd Piola–Kirchhoff stress $\mathbf{S} = \mathbf{F}^{-1} \sigma$
  - It’s symmetric iff Cauchy is symmetric.

Full equation of motion

- Define a strain $\mathbf{C} = \mathbf{F}^{\intercal}\mathbf{F}$ (right Cauchy–Green)

  $\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I})$ (Green–St Venant)

- Model a stress–strain relation; e.g.

  $\mathbf{S} = \lambda\,\mathrm{tr}(\mathbf{E}) + 2\mu\mathbf{E}$ (2nd Piola) $\lambda, \mu :$ Lamé constants.

- 1st Piola–Kirchhoff tensor $\mathbf{P} = \mathbf{F}\mathbf{S}$
- Pointwise world-force on Lagrangian coord $\mathbf{f} = \nabla \cdot \mathbf{P}$ $f_j = \frac{\partial}{\partial X^i} P_j{}^i$
- Equation of motion $\rho_M \ddot{\phi} = f$

Geometric exposition

- Give various strain and stress matrices **type**.
- Different stresses are related by canonical pullback.
- Equation of motion is derived by least action. All the strain & stress tensors naturally show up as intermediate variables using back-propagation.
- Theorems such as Cauchy stress is symmetric is just the result of type checking.

Postulates & Roadmap

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- Postulates and roadmap
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  - 1st Piola tensor
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State of deformable body is a map

- The setup for deformable body
  - Manifold M : **Material** coordinate, **Lagrangian** coordinate
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  - A state of a deformable body is a map (called **flow map**)

$$\phi : M \to W$$

- Its temporal and spatial derivatives are of the following type

$$\dot{\phi} \in \Gamma(T_\phi W)$$

$$d\phi \in \Gamma(T^*M \otimes T_\phi W)$$

Postulates

- The position of the body is described by a (time-dependent) map

  $$\phi : M \to W$$

  where M has a time-independent mass density $\rho_M \in \Omega^n(M)$

  and W has a time-independent metric $\flat_W \in \Gamma(T^*W \odot T^*W)$

- The elastic potential energy for a map ϕ takes the form

  $$\mathcal{U}(\phi) = \int_M U(\phi^* \flat_W)$$

  for some fiber-wise (nonlinear) mapping (depending on material)

  $$U_p : T_p^*M \odot T_p^*M \xrightarrow{\text{nonlinear}} \wedge^n T_p^*M$$

  i.e. the potential is only a function of the induced metric encoding its notion of distances in the world. (**Frame-indifference**)

Terminology

- We call $F = \phi_* = d\phi \in \Gamma(T^*M \otimes T_\phi W)$ **deformation gradient**.
- The induced metric $\phi^*\flat_W$ can be understood by the diagram

$$\begin{tikzcd}
T_pM \arrow[r, "F"] \arrow[d, dashed, bend left, "F^*\flat_W F = \phi^*\flat_W"'] & T_{\phi(p)}W \arrow[d, "\flat_W"] \\
T_p^*M & T_{\phi(p)}^*W \arrow[l, "F^*"']
\end{tikzcd}$$

- The induced metric $C := F^*\flat_W F \in \Gamma(T^*M \odot T^*M)$ is called the (right)-**Cauchy–Green tensor**.

Notation in 3D Cartesian coordinate

- Let (X, Y, Z) denote the Cartesian coordinate for M and (x, y, z) the Cartesian coordinate for W
- Flow map $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \phi^1(X,Y,Z) \\ \phi^2(X,Y,Z) \\ \phi^3(X,Y,Z) \end{bmatrix}$
- Deformation gradient $\mathbf{F} = \begin{bmatrix} \frac{\partial \phi^1}{\partial X} & \frac{\partial \phi^1}{\partial Y} & \frac{\partial \phi^1}{\partial Z} \\ \frac{\partial \phi^2}{\partial X} & \frac{\partial \phi^2}{\partial Y} & \frac{\partial \phi^2}{\partial Z} \\ \frac{\partial \phi^3}{\partial X} & \frac{\partial \phi^3}{\partial Y} & \frac{\partial \phi^3}{\partial Z} \end{bmatrix}$
- Right Cauchy–Green tensor $\mathbf{C} = \mathbf{F}^\intercal \mathbf{F}$

Deriving elastic force

- Back to our potential energy $\mathcal{U}(\phi) = \int_M U(\phi^*\flat_W) = \int_M U(C)$

  given some model $U : \Gamma(T^*M \odot T^*M) \xrightarrow{\substack{\text{pointwise}\\ \text{nonlinear}}} \Gamma(\wedge^n T^*M)$

- The force is given by $-d\mathcal{U}_\phi$
- Compute it using backpropagation.

Deriving elastic force

$$\mathcal{U}(\phi) = \int_M U(\phi^* \flat_W) = \int_M U(C)$$

$$\begin{tikzcd}
\phi & F & C = F^* \flat_W F & U(C) & \\
C^\infty(M;W) \arrow[r, "d"] & \Gamma(T^*M \otimes T_\phi W) \arrow[r, "\mathcal{G}"] & \Gamma(T^*M \odot T^*M) \arrow[r, "U"] & \Gamma(\wedge^n T^*M) \arrow[r, "\int_M"] & \mathbb{R}
\end{tikzcd}$$

Sequence of linear maps on tangent spaces:

$$\begin{tikzcd}
\Gamma(T_\phi W) \arrow[r, "d^{\nabla^W}"] & \Gamma(T^*M \otimes T_\phi W) \arrow[r, "d\mathcal{G}|_F"] & \Gamma(T^*M \odot T^*M) \arrow[r, "dU|_C"] & \Gamma(\wedge^n T^*M) \arrow[r, "\int_M"] & \mathbb{R}
\end{tikzcd}$$

Pullback:

$$\begin{tikzcd}
d\mathcal{U}_\phi & \frac{\partial U}{\partial F} & \frac{\partial U}{\partial C} & \mathbb{1} & 1 \\
\Gamma(\wedge^n T^*M \otimes T^*_\phi W) & \Gamma(\wedge^{n-1} T^*M \otimes T^*_\phi W) \arrow[l, "(d^\nabla)^*"'] & \Gamma(TM \odot TM \otimes \wedge^n T^*M) \arrow[l, "d\mathcal{G}|_F^*"'] & \Omega^0(M) \arrow[l, "dU|_C^*"'] & \mathbb{R}^* \arrow[l]
\end{tikzcd}$$

Terminology

$$\begin{tikzcd}
 & \frac{\partial U}{\partial F} & \frac{\partial U}{\partial C} & \mathbb{1} & 1 \\
d\mathcal{U}_\phi & \Gamma(\Lambda^{n-1}T^*M \otimes T^*_\phi W) \arrow[dl, "(d^\nabla)^*"'] & \Gamma(TM \odot TM \otimes \Lambda^n T^*M) \arrow[l, "d\mathcal{G}|_F^*"'] & \Omega^0(M) \arrow[l, "dU|_C^*"'] & \mathbb{R}^* \arrow[l] \\
\Gamma(\Lambda^n T^*M \otimes T^*_\phi W) & & & &
\end{tikzcd}$$

2nd Piola–Kirchhoff stress (type is dual to right-Cauchy–Green)

$$S := 2\frac{\partial U}{\partial C} \in \Gamma(TM \odot TM \otimes \Lambda^n T^*M)$$

1st Piola–Kirchhoff stress tensor (type is dual to deformation gradient)

$$P := d\mathcal{G}|_F{}^*S = \frac{\partial U}{\partial F} \in \Gamma(\Lambda^{n-1}T^*M \otimes T^*_\phi W)$$

Terminology

1st Piola tensor

2nd Piola tensor

$\mathbb{1}$

1

–elastic force

$$\begin{tikzcd}
 & \Gamma(\Lambda^{n-1}T^*M \otimes T^*_\phi W) & \Gamma(TM \odot TM \otimes \Lambda^n T^*M) \arrow[l, "d\mathcal{G}|^*_F"'] & \Omega^0(M) \arrow[l, "dU|^*_C"'] & \mathbb{R}^* \arrow[l] \\
\Gamma(\Lambda^n T^*M \otimes T^*_\phi W) & \arrow[l, "(d^\nabla)^*"]
\end{tikzcd}$$

Terminology

$$\begin{tikzcd}
& \text{1st Piola tensor} & \text{2nd Piola tensor} & \mathbb{1} & 1 \\
\text{–elastic force} & \Gamma(\Lambda^{n-1}T^*M \otimes T^*_\phi W) \arrow[dl, "(d^\nabla)^*"'] & \Gamma(TM \odot TM \otimes \Lambda^n T^*M) \arrow[l, "dG|_F^*"'] & \Omega^0(M) \arrow[l, "dU|_C^*"'] & \mathbb{R}^* \arrow[l] \\
\Gamma(\Lambda^n T^*M \otimes T^*_\phi W) & & & &
\end{tikzcd}$$

Just need to derive what these adjoints are

Differential of *U*: the 2nd Piola Tensor

- Overview
- Postulates and roadmap
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  - 1st Piola tensor
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2nd Piola Stress

$$\begin{tikzcd}
 & \text{1st Piola tensor} & \text{2nd Piola tensor} & \mathbb{1} & 1 \\
\text{–elastic force} & \Gamma(\Lambda^{n-1}T^*M \otimes T^*_\phi W) \arrow[dl, "(d^\nabla)^*"] & \Gamma(TM \odot TM \otimes \Lambda^n T^*M) \arrow[l, "d\mathcal{G}|^*_F"'] & \Omega^0(M) \arrow[l, "dU|^*_C"'] & \mathbb{R}^* \arrow[l] \\
\Gamma(\Lambda^n T^*M \otimes T^*_\phi W) & & & &
\end{tikzcd}$$

$$S := 2\frac{\partial U}{\partial C} \in \Gamma(TM \odot TM \otimes \Lambda^n T^*M)$$

- This is just a value look-up depending on the model U
- We call the mapping $S = R(C)$ a stress–strain relation.

An example hyperelastic model

- The energy $U(C) \in \Gamma(\wedge^n T^*M)$ measures how much the induced metric (right-Cauchy–Green)
$$C \in \Gamma(T^*M \odot T^*M)$$
deviates from some rest material metric
$$\flat_M \in \Gamma(T^*M \odot T^*M)$$
- For example, the **Green–Lagrangian** or **Green–St Venant strain** is
$$E := \tfrac{1}{2}(\sharp_M C - I) \in \Gamma(\mathrm{End}(TM))$$

An example hyperelastic model

- For example, the **Green–Lagrangian** or **Green–St Venant strain** is

$$E := \tfrac{1}{2}(\sharp_M C - I) \;\; \in \Gamma(\mathrm{End}(TM))$$

- The **Saint Venant–Kirchhoff** model:

$$U(C) = \Big(\tfrac{\lambda}{2}\,\mathrm{tr}(E)^2 + \mu\,\mathrm{tr}(E^2)\Big)\,dV_M$$

  - where the constants λ, μ are the **Lamé parameters.**
  - 2nd Piola–Kirchhoff tensor

$$S = 2\tfrac{\partial U}{\partial C} \; = (\lambda\,\mathrm{tr}(E)\sharp_M + 2\mu E\sharp_M) \otimes dV_M$$

In 3D (2nd Piola tensor)

- Recall $\mathbf{F} = \begin{bmatrix} \frac{\partial \phi^1}{\partial X} & \frac{\partial \phi^1}{\partial Y} & \frac{\partial \phi^1}{\partial Z} \\ \frac{\partial \phi^2}{\partial X} & \frac{\partial \phi^2}{\partial Y} & \frac{\partial \phi^2}{\partial Z} \\ \frac{\partial \phi^3}{\partial X} & \frac{\partial \phi^3}{\partial Y} & \frac{\partial \phi^3}{\partial Z} \end{bmatrix}$ $\quad \mathbf{C} = \mathbf{F}^\intercal \mathbf{F}$

- The **Green–St Venant strain** is $\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I})$
- In **Saint Venant–Kirchhoff** model: $\mathbf{S} = \lambda \operatorname{tr}(\mathbf{E}) + 2\mu \mathbf{E}$

In 3D (2nd Piola tensor)

- Recall $\mathbf{F} = \begin{bmatrix} \frac{\partial \phi^1}{\partial X} & \frac{\partial \phi^1}{\partial Y} & \frac{\partial \phi^1}{\partial Z} \\ \frac{\partial \phi^2}{\partial X} & \frac{\partial \phi^2}{\partial Y} & \frac{\partial \phi^2}{\partial Z} \\ \frac{\partial \phi^3}{\partial X} & \frac{\partial \phi^3}{\partial Y} & \frac{\partial \phi^3}{\partial Z} \end{bmatrix}$ $\mathbf{C} = \mathbf{F}^\intercal \mathbf{F}$
- The **Green–St Venant strain** is $\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I})$
- In **Saint Venant–Kirchhoff** model: $\mathbf{S} = \lambda \operatorname{tr}(\mathbf{E}) + 2\mu \mathbf{E}$
- Other models use various other strains
  - Biot strain $\mathbf{E} = \mathbf{C}^{1/2} - \mathbf{I}$
  - Hencky strain $\mathbf{E} = \frac{1}{2} \ln \mathbf{C}$

Pullback by the CauchyGreen constructor: the 1st Piola Tensor

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Pullback by $\mathcal{G}$

$$\begin{tikzcd}
& \text{1st Piola tensor} & & \text{2nd Piola tensor} & & \mathbb{1} & 1 \\
\text{–elastic force} & \Gamma(\Lambda^{n-1}T^*M \otimes T_\phi^*W) \arrow[dl, "(d^\nabla)^*"'] & & \Gamma(TM \odot TM \otimes \Lambda^n T^*M) \arrow[ll, "d\mathcal{G}|_F^*"'] & & \Omega^0(M) \arrow[ll, "dU|_C^*"'] & \mathbb{R}^* \arrow[l] \\
\Gamma(\Lambda^n T^*M \otimes T_\phi^*W) & & & & & &
\end{tikzcd}$$

Pullback by $\mathcal{G}$

- Recall

$$\mathcal{G} : \Gamma(T^*M \otimes T_\phi W) \to \Gamma(T^*M \odot T^*M)$$

$$\Gamma(\mathrm{Hom}(TM; T_\phi W)) \qquad \Gamma(\mathrm{Hom}_{\substack{\text{self-}\\ \text{adjoint}}}(TM; T^*M))$$

$$F \longmapsto F^* \flat_W F$$

- Take differential

$$d\mathcal{G}|_F[\![\mathring{F}]\!] = \mathring{F}^* \flat_W F + F^* \flat_W \mathring{F}$$

Pullback by $\mathcal{G}$

- Represent dual pairings in terms of linear maps

$$\begin{tikzcd}
A = A^{ij}\vec{e}_i \otimes \vec{f}_j & U \otimes V \arrow[r, dash, dotted, "\cong"] \arrow[d, dash, dotted, "\text{dual}"] & \mathrm{Hom}(U^*; V) \arrow[d, dash, dotted, "\text{dual}"] \\
B = B_{ij}\alpha^i \otimes \beta^j & U^* \otimes V^* \arrow[r, dash, dotted, "\cong"] & \mathrm{Hom}(U; V^*)
\end{tikzcd}$$

$$\langle B|A\rangle = \sum_{ij} B_{ij}A^{ij}$$

$$\langle B|A\rangle = \mathrm{tr}(B^*A)$$

Pullback by $\mathcal{G}$

- Recall $$\mathcal{G}\colon \underset{\Gamma(\mathrm{Hom}(TM;T_\phi W))}{\Gamma(T^*M\otimes T_\phi W)} \to \underset{\Gamma(\mathrm{Hom}_{\substack{\text{self-}\\ \text{adjoint}}}(TM;T^*M))}{\Gamma(T^*M\odot T^*M)}$$ $$F \mapsto F^*\flat_W F$$
- Take differential $$d\mathcal{G}|_F[\![\mathring{F}]\!] = \mathring{F}^*\flat_W F + F^*\flat_W \mathring{F}$$
- Compute adjoint: for each

  $$S \in \Gamma(TM\odot TM\otimes \wedge^n T^*M) = \Gamma(\mathrm{Hom}_{\substack{\text{self-}\\ \text{adjoint}}}(T^*M;TM)\otimes \wedge^n T^*M)$$

  we have

  $$\left\langle\!\!\left\langle d\mathcal{G}_F{}^*[\![S]\!] \,\middle|\, \mathring{F} \right\rangle\!\!\right\rangle = \left\langle\!\!\left\langle S \,\middle|\, d\mathcal{G}_F[\![\mathring{F}]\!] \right\rangle\!\!\right\rangle$$

  i.e. $$\int_M \mathrm{tr}\Big((d\mathcal{G}_F{}^*[\![S]\!])^*\,\mathring{F}\Big) = \int_M \mathrm{tr}\Big(S^*\,d\mathcal{G}_F[\![\mathring{F}]\!]\Big)$$

Pullback by $\mathcal{G}$

$$d\mathcal{G}|_F[\![\mathring{F}]\!] = \mathring{F}^*\flat_W F + F^*\flat_W \mathring{F}$$

$$\int_M \mathrm{tr}\left((d\mathcal{G}_F{}^*[\![S]\!])^* \mathring{F}\right) = \int_M \mathrm{tr}\left(S^* d\mathcal{G}_F[\![\mathring{F}]\!]\right)$$

$$\begin{aligned}\mathrm{tr}\left(S^* d\mathcal{G}_F[\![\mathring{F}]\!]\right) &= \mathrm{tr}\left(S^*(\mathring{F}^*\flat_W F + F^*\flat_W \mathring{F})\right)\\ &= \mathrm{tr}\left((2\flat_W F S)^* \mathring{F}\right)\end{aligned}$$

- Therefore

$$\boxed{d\mathcal{G}_F{}^*[\![S]\!] = 2\flat_W F S}$$

Pullback by $\mathcal{G}$

$$\begin{tikzcd}
 & \overset{\frac{\partial U}{\partial F}}{\Gamma(\Lambda^{n-1}T^*M \otimes T^*_\phi W)} \arrow[dl, "(d^\nabla)^*"'] & \overset{\frac{\partial U}{\partial C}}{\Gamma(TM \odot TM \otimes \Lambda^n T^*M)} \arrow[l, "d\mathcal{G}|_F^*"'] & \overset{\mathbb{1}}{\Omega^0(M)} \arrow[l, "dU|_C^*"'] & \overset{1}{\mathbb{R}^*} \arrow[l] \\
\overset{d\mathcal{U}_\phi}{\Gamma(\Lambda^n T^*M \otimes T^*_\phi W)} & & & &
\end{tikzcd}$$

$$d\mathcal{G}_F{}^*\llbracket S \rrbracket = 2\flat_W F S$$

2nd Piola–Kirchhoff stress $S := 2\frac{\partial U}{\partial C} \in \Gamma(TM \odot TM \otimes \Lambda^n T^*M)$

1st Piola–Kirchhoff stress $P := d\mathcal{G}|_F{}^* S = \frac{\partial U}{\partial F} \in \Gamma(\Lambda^{n-1}T^*M \otimes T^*_\phi W)$

$$\boxed{P = \flat_W F S}$$

2nd piola tensor is defined with an additional factor of 2 to absorb the 2 in the pullback.

In 3D (1st Piola Tensor)

- Recall $\mathbf{F} = \begin{bmatrix} \frac{\partial \phi^1}{\partial X} & \frac{\partial \phi^1}{\partial Y} & \frac{\partial \phi^1}{\partial Z} \\ \frac{\partial \phi^2}{\partial X} & \frac{\partial \phi^2}{\partial Y} & \frac{\partial \phi^2}{\partial Z} \\ \frac{\partial \phi^3}{\partial X} & \frac{\partial \phi^3}{\partial Y} & \frac{\partial \phi^3}{\partial Z} \end{bmatrix}$ (deformation gradient)

$\mathbf{C} = \mathbf{F}^{\mathsf{T}}\mathbf{F}$ (right Cauchy–Green)

- In **Saint Venant–Kirchhoff** model: $\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I})$ (Green–St Venant)

  $\mathbf{S} = \lambda\,\mathrm{tr}(\mathbf{E}) + 2\mu\mathbf{E}$ (2nd Piola)

- 1st Piola–Kirchhoff tensor $\mathbf{P} = \mathbf{F}\mathbf{S}$

Pullback by *d*: the final elastic force

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Pullback by d

- Recall $$d : C^\infty(M;W) \to \Gamma(T^*M \otimes T_\phi W)$$ $$\phi \mapsto d\phi$$
- Its differential

$$\begin{tikzcd}
d^\nabla : \Gamma(T_\phi W) \ni \vec{v} \arrow[r] \arrow[d, dotted, no head] & \Gamma(T^*M \otimes T_\phi W) \ni F \arrow[d, dotted, no head] \\
\Gamma(\wedge^n T^*M \otimes T^*_\phi W) \ni f & \Gamma(TM \otimes \wedge^n T^*M \otimes T^*_\phi W)
\end{tikzcd}$$

$$\mathring{\phi} \mapsto d^\nabla \mathring{\phi}$$

- Dual pairing

$$\cong \Gamma(\wedge^{n-1} T^*M \otimes T^*_\phi W) \ni P$$

$$\langle\!\langle f \,|\, \vec{v} \rangle\!\rangle = \int_M \langle f \,|\, \vec{v} \rangle \qquad \langle\!\langle P \,|\, F \rangle\!\rangle = \int_M \langle F \wedge P \rangle$$

Pullback by d

$$d^{\nabla} : \Gamma(T_\phi W) \to \Gamma(T^*M \otimes T_\phi W)$$

$$\mathring{\phi} \mapsto d^{\nabla} \mathring{\phi}$$

- Compute its adjoint: For each $P \in \Gamma(\wedge^{n-1} T^*M \otimes T_\phi^* W)$ we have

$$\langle\!\langle (d^{\nabla})^* P | \mathring{\phi} \rangle\!\rangle = \langle\!\langle P | d^{\nabla} \mathring{\phi} \rangle\!\rangle$$

i.e.

$$\int_M \langle \mathring{\phi} | (d^{\nabla})^* P \rangle = \int_M \langle d^{\nabla} \mathring{\phi} \wedge P \rangle$$

Pullback by d

$$\int_M \langle \mathring{\phi} | (d^\nabla)^* P \rangle = \int_M \langle d^\nabla \mathring{\phi} \wedge P \rangle$$

$$= \int_M d \langle \mathring{\phi} | P \rangle - \int_M \langle \mathring{\phi} | d^\nabla P \rangle$$

$$= \oint_{\partial M} \langle \mathring{\phi} | P \rangle - \int_M \langle \mathring{\phi} | d^\nabla P \rangle$$

- Therefore

$$(d^\nabla_{\text{1-form}})^* = -d^\nabla_{\text{(n-1)-form}}$$

and boundary term

Pullback by d

$$\begin{tikzcd}
 & \text{1st Piola tensor} & \text{2nd Piola tensor} & \mathbb{1} & 1 \\
\text{–elastic force} & \Gamma(\Lambda^{n-1}T^*M \otimes T^*_\phi W) \arrow[dl, "(d^\nabla)^*"] & \Gamma(TM \odot TM \otimes \Lambda^n T^*M) \arrow[l, "d\mathcal{G}|^*_F"'] & \Omega^0(M) \arrow[l, "dU|^*_C"'] & \mathbb{R}^* \arrow[l] \\
\Gamma(\Lambda^n T^*M \otimes T^*_\phi W) & & & &
\end{tikzcd}$$

- Variation of potential energy

$$\boxed{\frac{\partial \mathcal{U}}{\partial \phi} = -d^\nabla P}$$

Summary of the derivation

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  - Summary of derivation
- Cauchy stress and linear elasticity

Summary and Eq of motion

flow map

deformation gradient

right Cauchy–Green

energy density

total energy

$$\begin{tikzcd}
\overset{\phi}{C^\infty(M;W)} \arrow[r, "d"] & \overset{F}{\Gamma(T^*M \otimes T_\phi W)} \arrow[r, "\mathcal{G}"] & \overset{C = F^*\flat_W F}{\Gamma(T^*M \odot T^*M)} \arrow[r, "U"] & \overset{U(C)}{\Gamma(\wedge^n T^*M)} \arrow[r, "\int_M"] & \mathbb{R} \\
 & & & \overset{\mathbb{1}}{\Omega^0(M)} \arrow[dl, "dU|_C^*"'] & \overset{1}{\mathbb{R}^*} \arrow[l] \\
\Gamma(\wedge^n T^*M \otimes T^*_\phi W) & \Gamma(\wedge^{n-1} T^*M \otimes T^*_\phi W) \arrow[l, "(d^\nabla)^*"'] & \Gamma(TM \odot TM \otimes \wedge^n T^*M) \arrow[l, "d\mathcal{G}|_F^*"'] & &
\end{tikzcd}$$

1st Piola–Kirchhoff

$$P = \flat_W F S$$

$$d\mathcal{U}_\phi = -d^\nabla P$$

–force

$$S = 2\frac{\partial U}{\partial C}$$

2nd Piola–Kirchhoff

- Equation of motion: $\rho_M \overset{\nabla}{\ddot{\phi}} = d^\nabla P$

In 3D (Equation of motion)

- Recall $\mathbf{F} = \begin{bmatrix} \frac{\partial \phi^1}{\partial X} & \frac{\partial \phi^1}{\partial Y} & \frac{\partial \phi^1}{\partial Z} \\ \frac{\partial \phi^2}{\partial X} & \frac{\partial \phi^2}{\partial Y} & \frac{\partial \phi^2}{\partial Z} \\ \frac{\partial \phi^3}{\partial X} & \frac{\partial \phi^3}{\partial Y} & \frac{\partial \phi^3}{\partial Z} \end{bmatrix}$ (deformation gradient)

  $\mathbf{C} = \mathbf{F}^\intercal \mathbf{F}$ (right Cauchy–Green)

- In **Saint Venant–Kirchhoff** model: $\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I})$ (Green–St Venant)

  $\mathbf{S} = \lambda \operatorname{tr}(\mathbf{E}) + 2\mu \mathbf{E}$ (2nd Piola)

- 1st Piola–Kirchhoff tensor $\mathbf{P} = \mathbf{F}\mathbf{S}$ $\quad P_j{}^i = \delta_{jk} F^k{}_\ell S^{\ell i}$
- Pointwise elastic force $\mathbf{f} = \nabla \cdot \mathbf{P}$ $\quad f_j = \frac{\partial}{\partial X^i} P_j{}^i$
- Equation of motion $\rho_M \ddot{\phi} = f$

Cauchy stress tensor

- Overview
- Postulates and roadmap
    - 2nd Piola tensor
    - 1st Piola tensor
    - Elastic force
    - Summary of derivation
- Cauchy stress and linear elasticity

Change coordinate to world

- The 1st Piola tensor is a world-covector-valued (n-1)-form on M

$$P \in \Gamma(\wedge^{n-1} T^* M \otimes T^*_{\phi} W)$$

- Define a world-covector-valued (n-1)-form on world, relating to P by pullback on the (n-1)-form part.

$$\sigma \in \Gamma(\wedge^{n-1} T^* W \otimes T^* W)$$

$$P = \phi^* \sigma \qquad P = (\phi^*)_{\wedge^{n-1} T^* W} \sigma$$

- This is Cauchy stress: force assigned on infinitesimal planes.
- In 3D: $\mathbf{P} = J \sigma \mathbf{F}^{-\mathsf{T}}$

Type conversion

$$\sigma \in \Gamma(\wedge^{n-1} T^*W \otimes T^*W)$$

- Consider $\tilde{\sigma} := \sharp_W \sigma \in \Gamma(\wedge^{n-1} T^*W \otimes TW)$

$$= \Gamma(TW \otimes TW \otimes \wedge^n T^*W)$$

- This doesn't change the matrix entries. But now it is a (vector ⊗ vector)-valued measure, which we ask about symmetry.

$$\tilde{\sigma} = \sharp_W \sigma \;=\; \frac{1}{\det(F)} \sharp_W P F^* \;=\; \frac{1}{\det(F)} \sharp_W (\flat_W F S) F^*$$

$$= \frac{1}{\det(F)} F S F^*$$

- Since $S \in \Gamma(TM \odot TM \otimes \wedge^n T^*M)$
we must have $\tilde{\sigma} \in \Gamma(TW \odot TW \otimes \wedge^n T^*W)$

Symmetry of Cauchy stress

- **Theorem** The Cauchy stress tensor is symmetric either when the stress is in static equilibrium, or that the stress is induced from a pure elastic (hyperelastic) model

Linear elasticity

- Overview
- Postulates and roadmap
    - 2nd Piola tensor
    - 1st Piola tensor
    - Elastic force
    - Summary of derivation
- Cauchy stress and linear elasticity

Linear elasticity

- Assume small deformation: $M = W \qquad \phi = \mathrm{id} + \mathbf{u}$
- Deformation gradient $\mathbf{F} = \mathbf{I} + \nabla\mathbf{u}$
- Right Cauchy–Green $\mathbf{C} = \mathbf{F}^{\mathsf{T}}\mathbf{F} = (\mathbf{I} + \nabla\mathbf{u})^{\mathsf{T}}(\mathbf{I} + \nabla\mathbf{u})$
  $\approx \mathbf{I} + \nabla\mathbf{u}^{\mathsf{T}} + \nabla\mathbf{u}$
- Linearized Green–StVenant $\boldsymbol{\varepsilon} = \frac{1}{2}(\mathbf{C} - \mathbf{I}) \approx \frac{1}{2}(\nabla\mathbf{u}^{\mathsf{T}} + \nabla\mathbf{u})$
- 2nd Piola: some linear mapping $\mathbf{S} = \mathbf{c} : \boldsymbol{\varepsilon} \qquad S_{ij} = c_{ijk\ell}\,\varepsilon_{k\ell}$
  $\mathbf{S} = 2\mu\boldsymbol{\varepsilon} + \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon})\mathbf{I}$

- 1st Piola, Cauchy stress $\sigma \approx \mathbf{P} \approx \mathbf{S}$
- Force $\mathbf{f} = \nabla \cdot \sigma$, motion: $\rho\ddot{\mathbf{u}} = \nabla \cdot \sigma + \mathbf{f}_{\mathrm{ext}}$