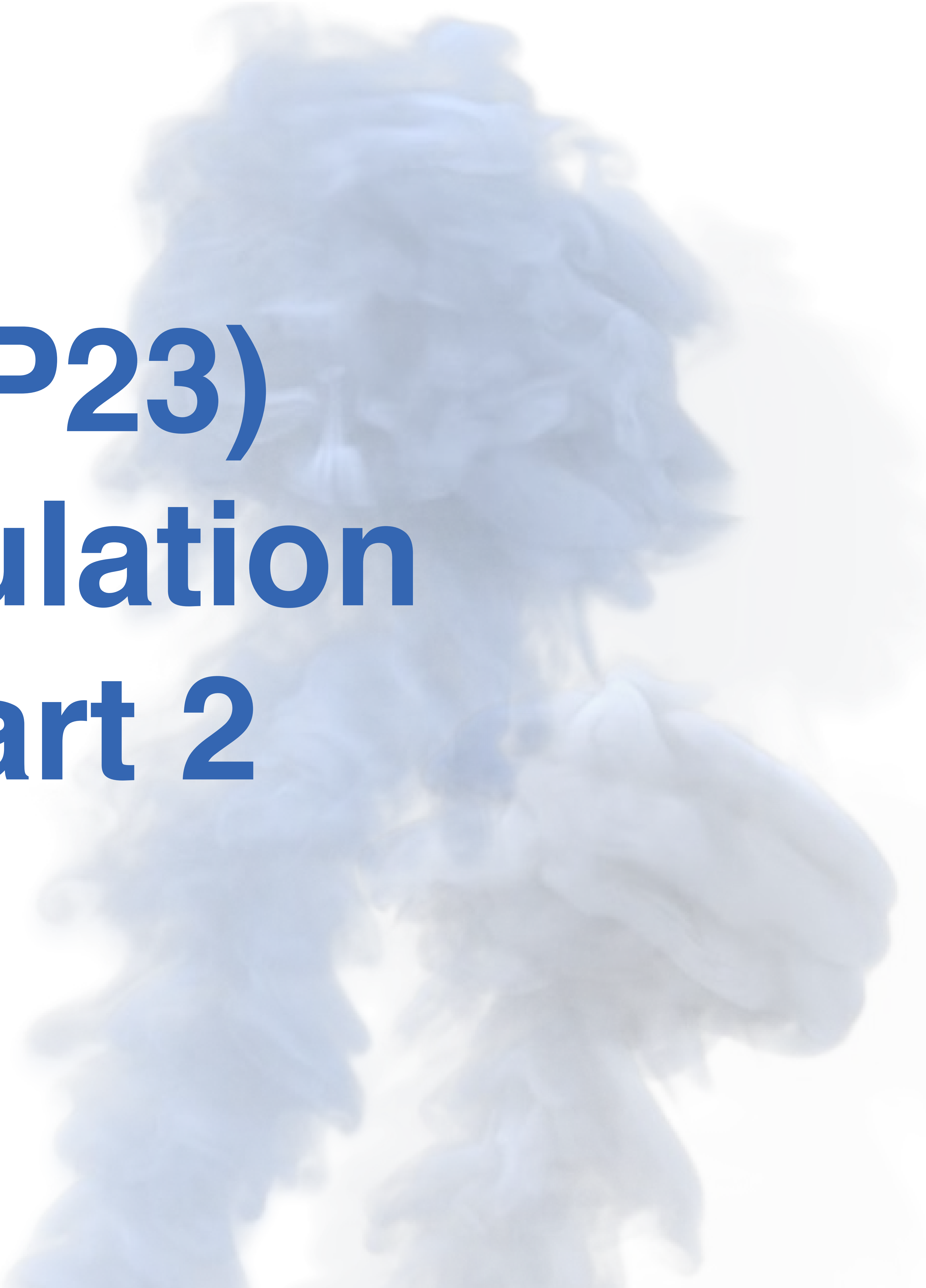


CSE 291 (SP23)
Physical Simulation
Tensors: Part 2

Albert Chern

UC San Diego
JACOBS SCHOOL OF ENGINEERING

Tensors for continuum mechanics

- Tensors for continuum mechanics
- Tensor duality
- Elasticity

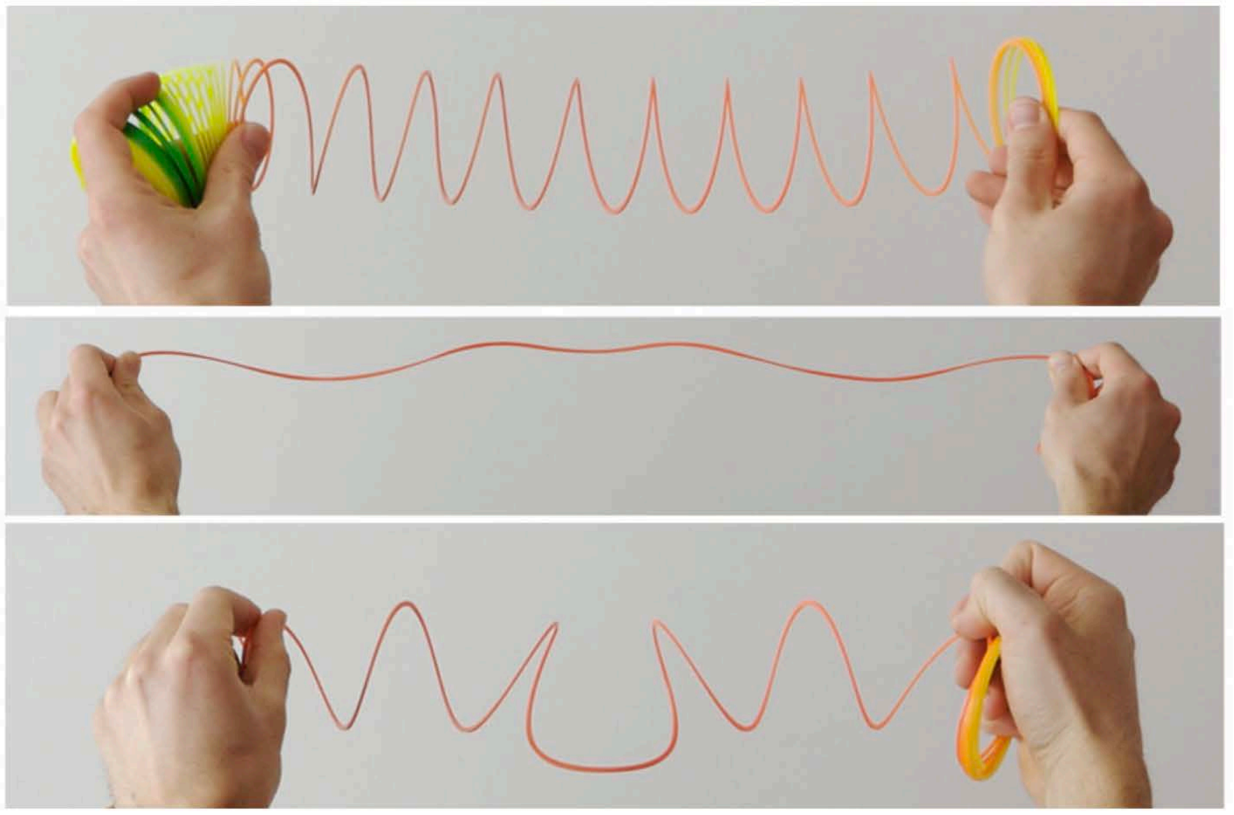
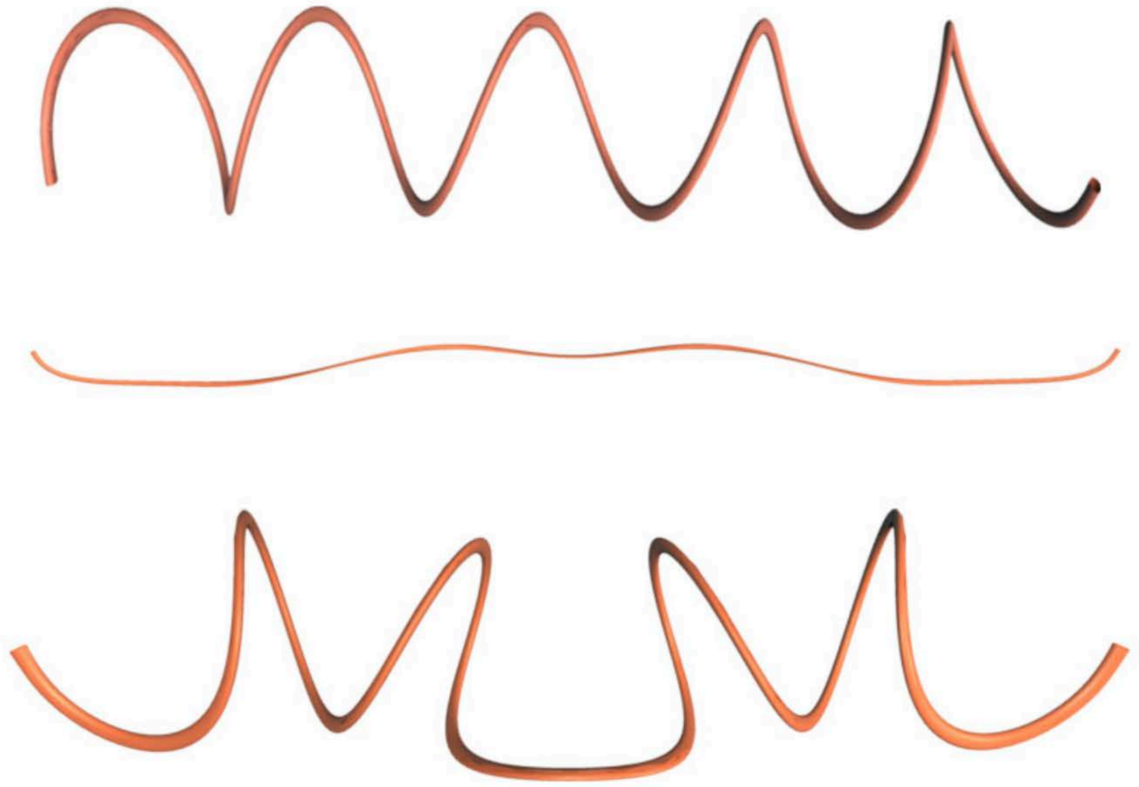
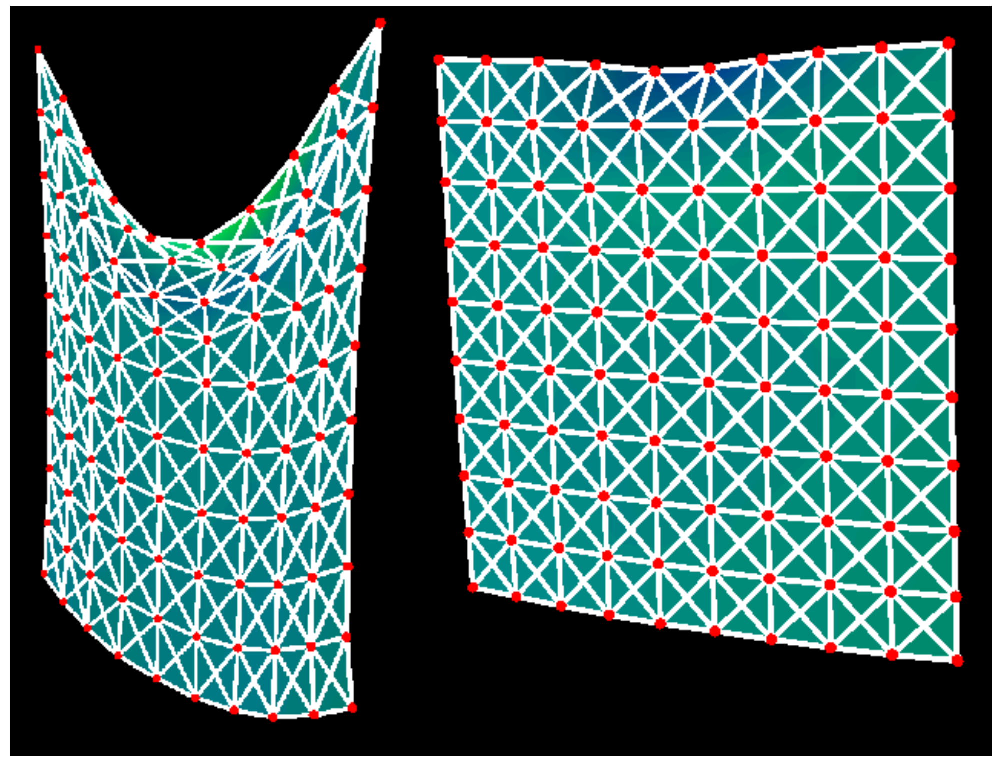
Continuum mechanics

- In continuum mechanics, we study the statics or dynamics for deformable body.

- Examples include elastic solid bodies, fluids, elastoplastic bodies, viscoelastic fluids, ferromagnetic fluids, plasmas, etc

Elastic solid bodies

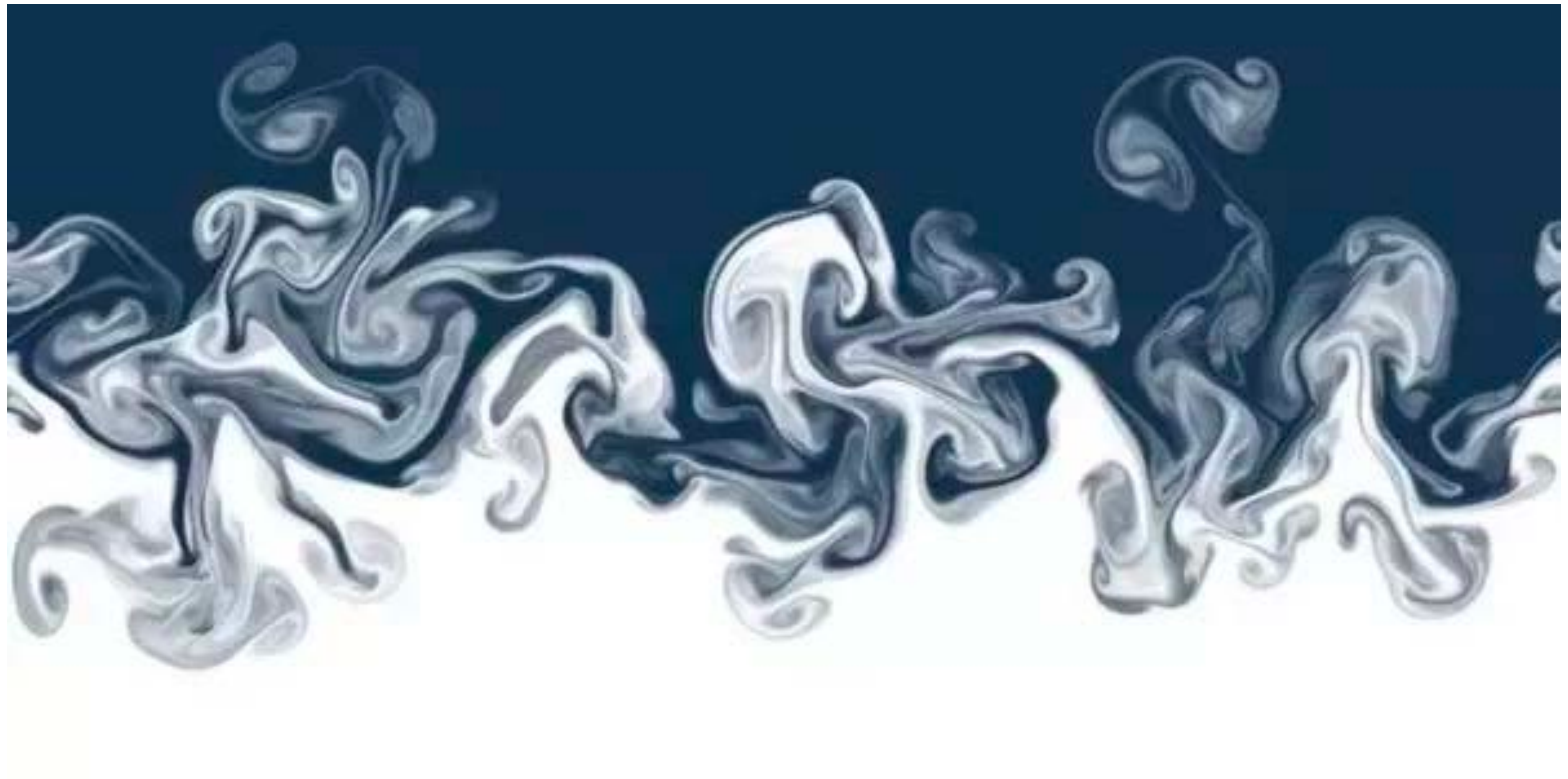
- Potential energy is a function of how the body is deformed
- We get restoration force back to undeformed state



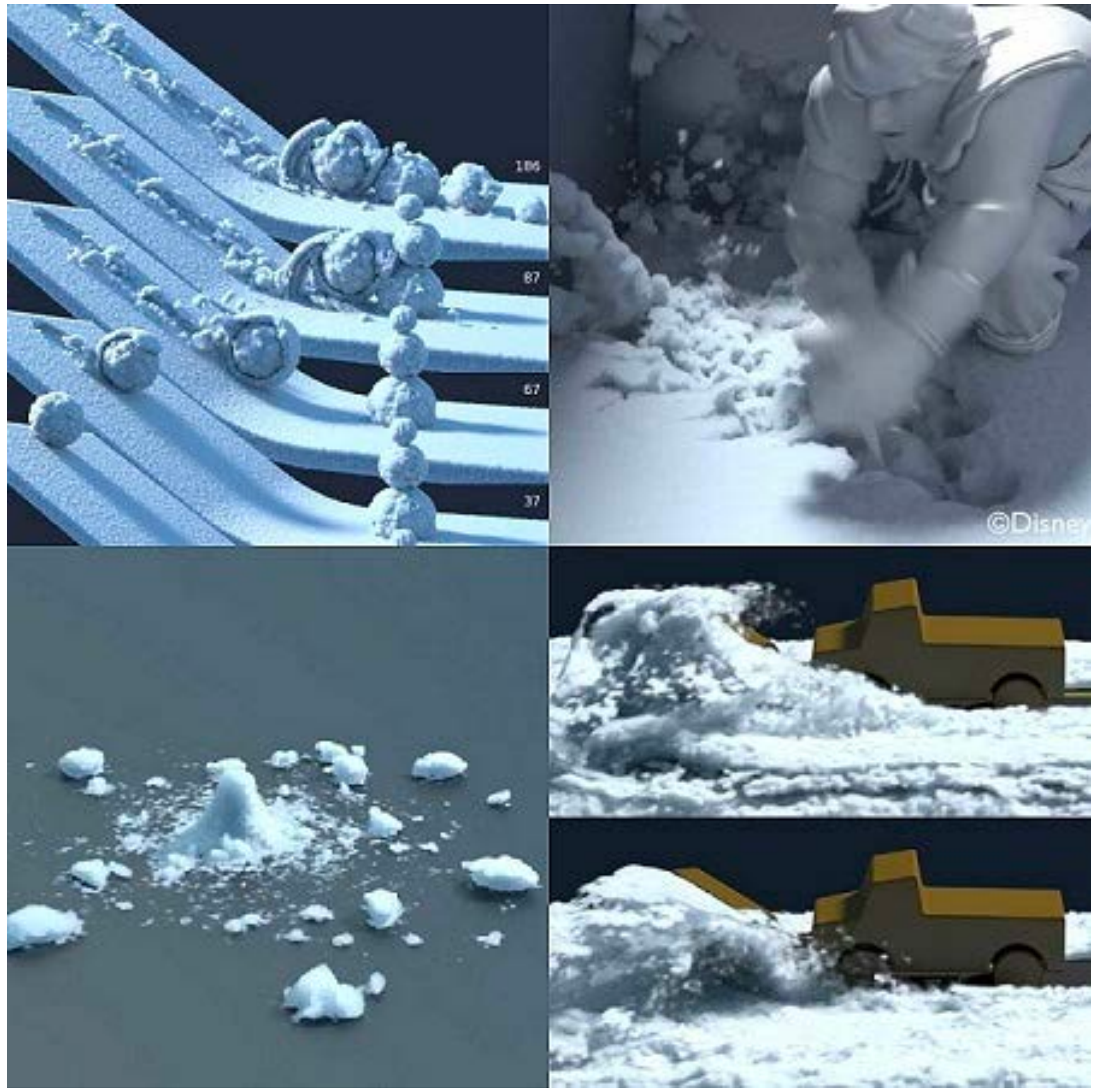
Fluids

- Potential energy is only a function of how the volume is changed
- No restoration force for volume-preserving deformation such as shearing



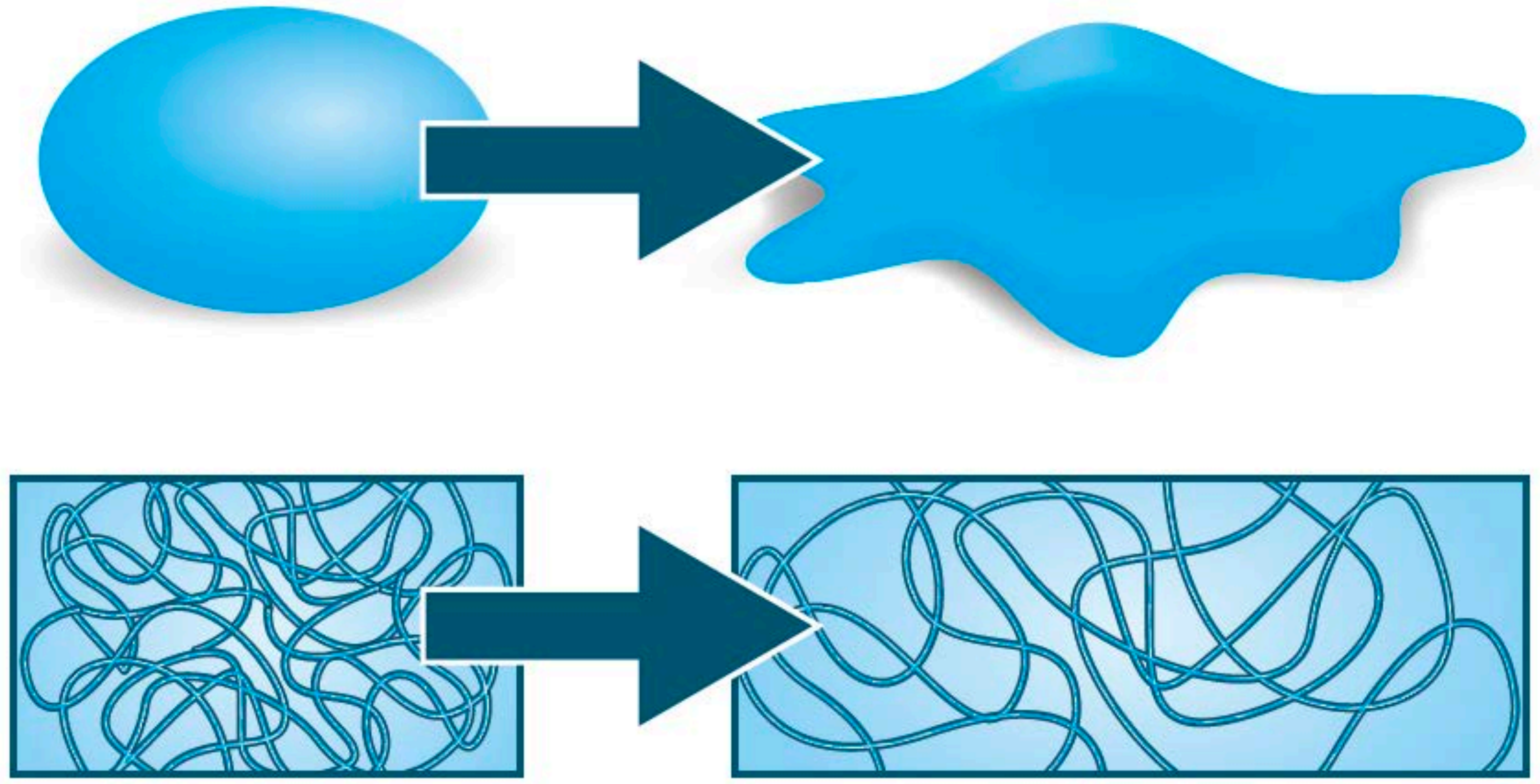
Elastoplastic materials

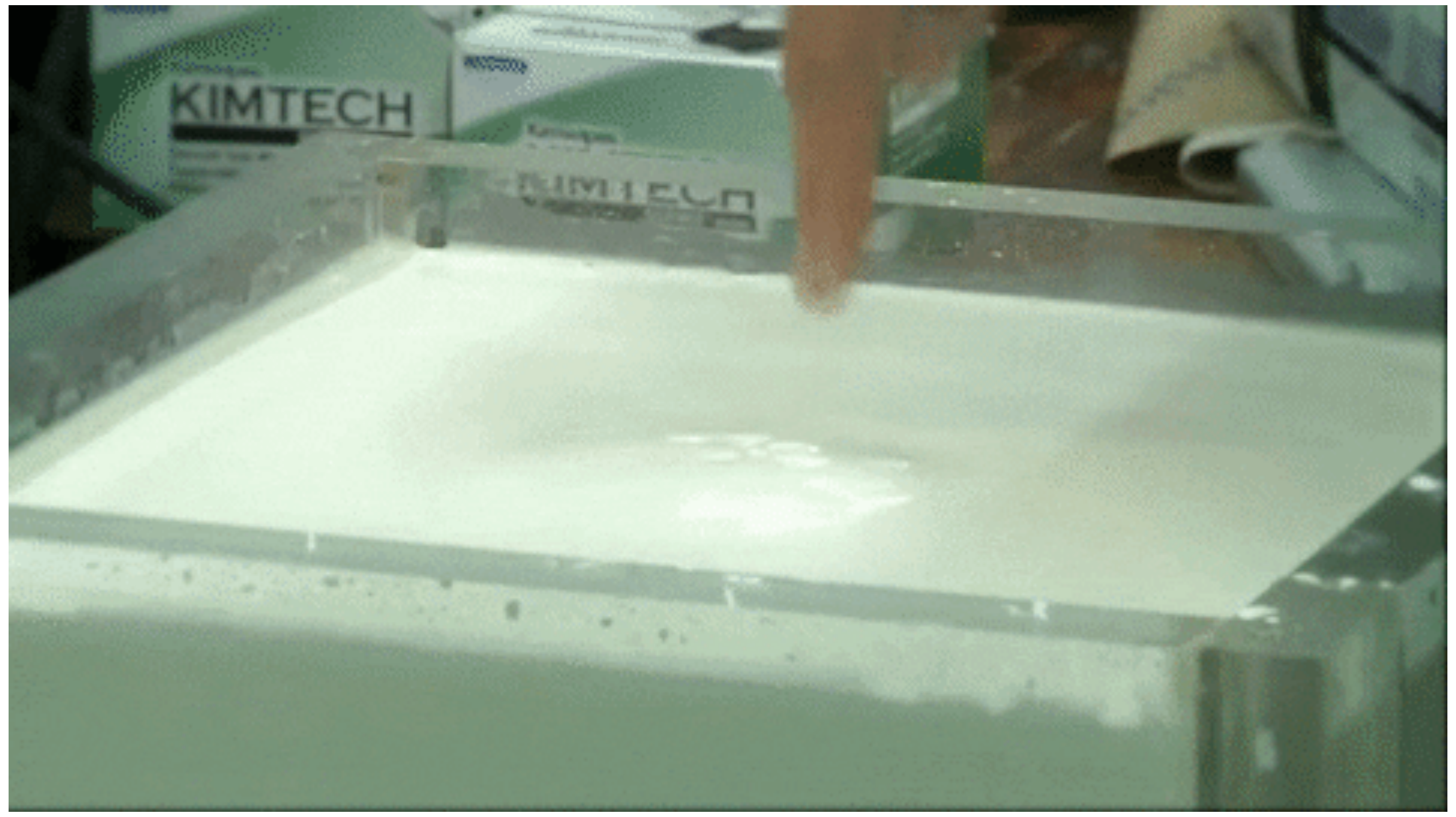
- Potential energy is a function of deformation, but the undeformed reference can change over time.



Viscoelastic materials

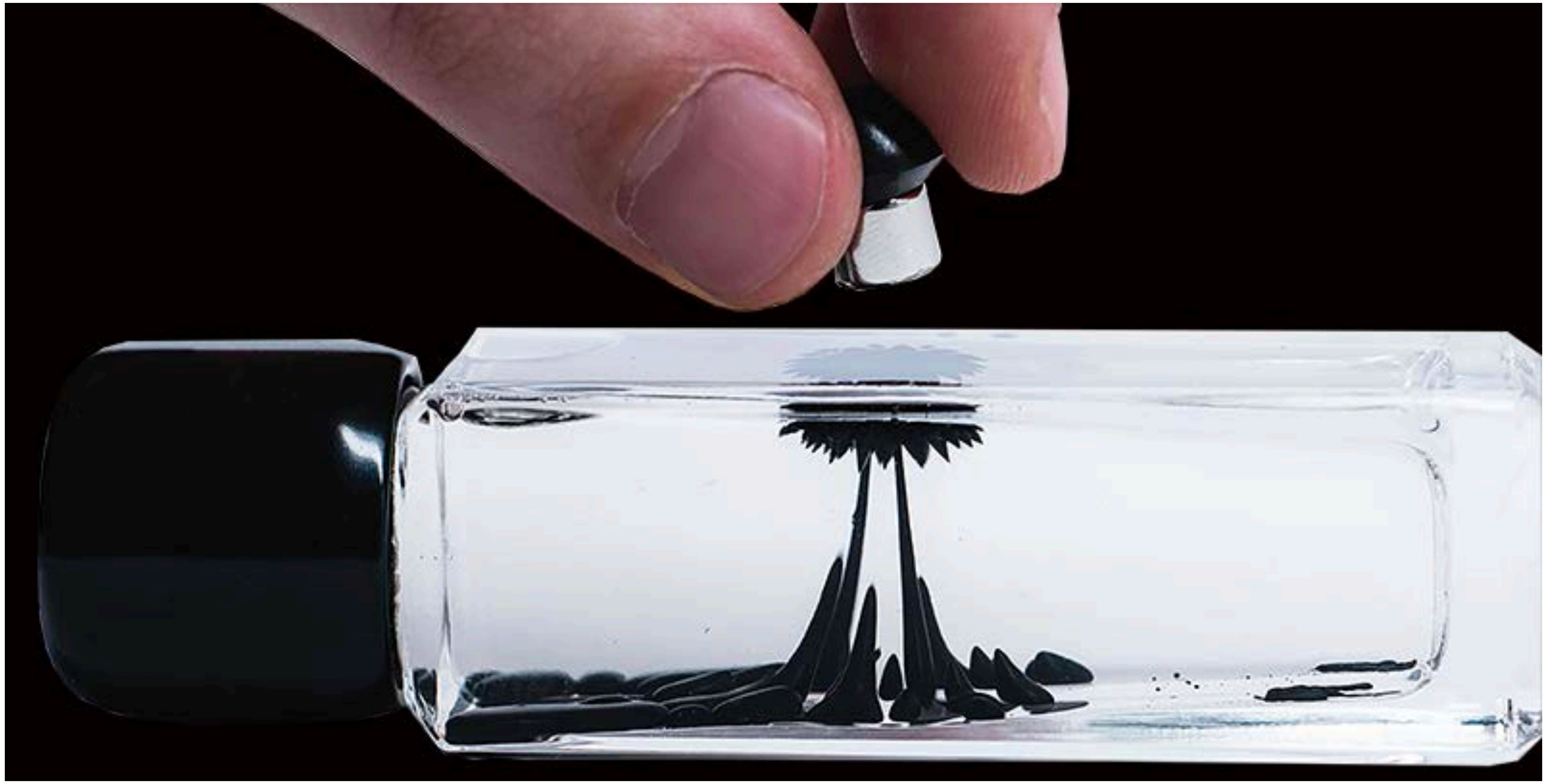
- Potential energy is a function of deformation and the rate of change of deformation





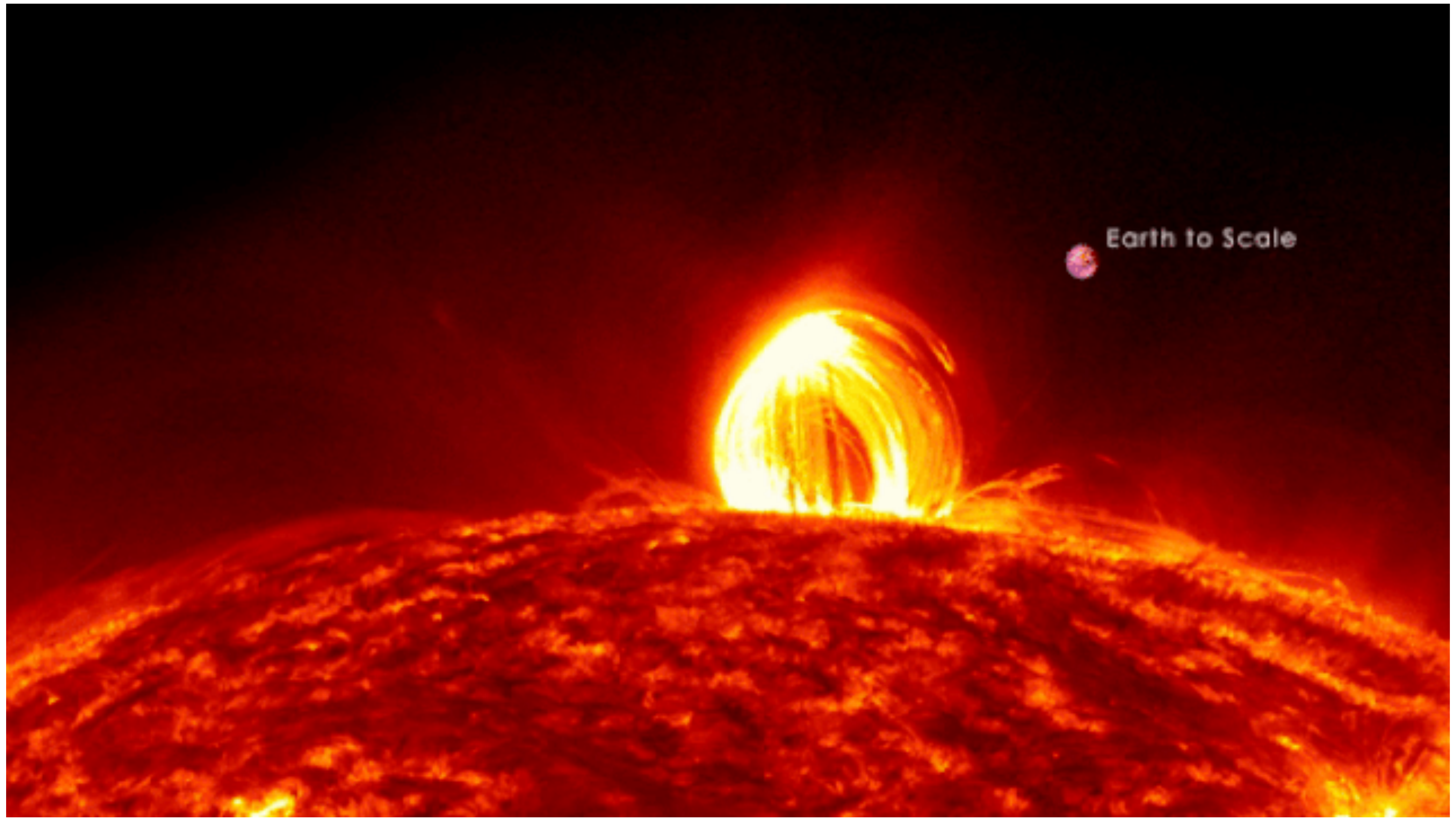
Ferrofluids

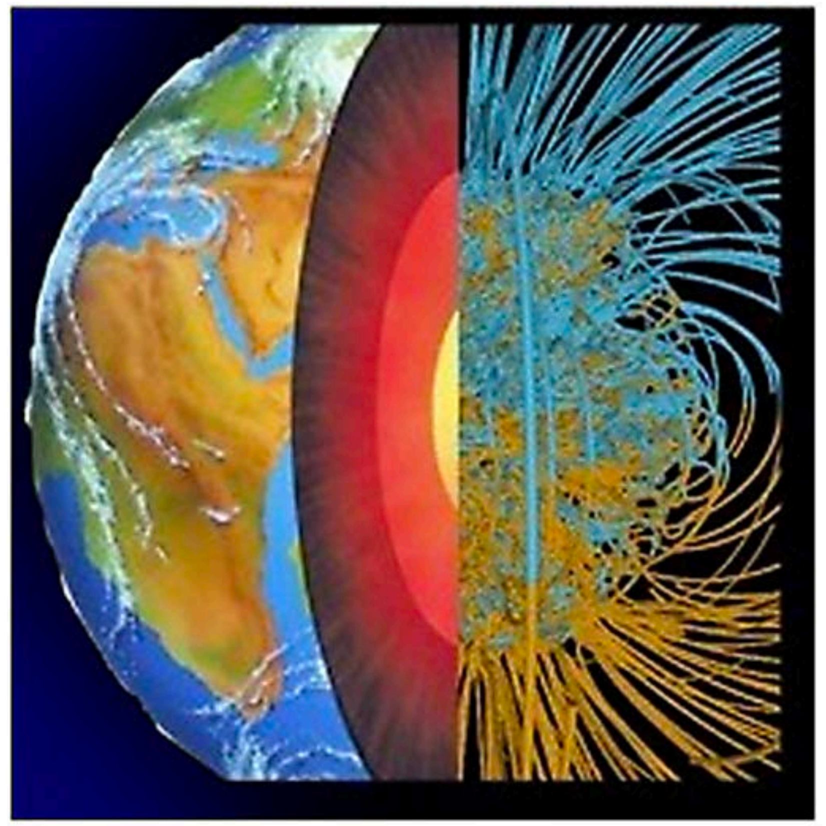
- The fluids made of little magnets

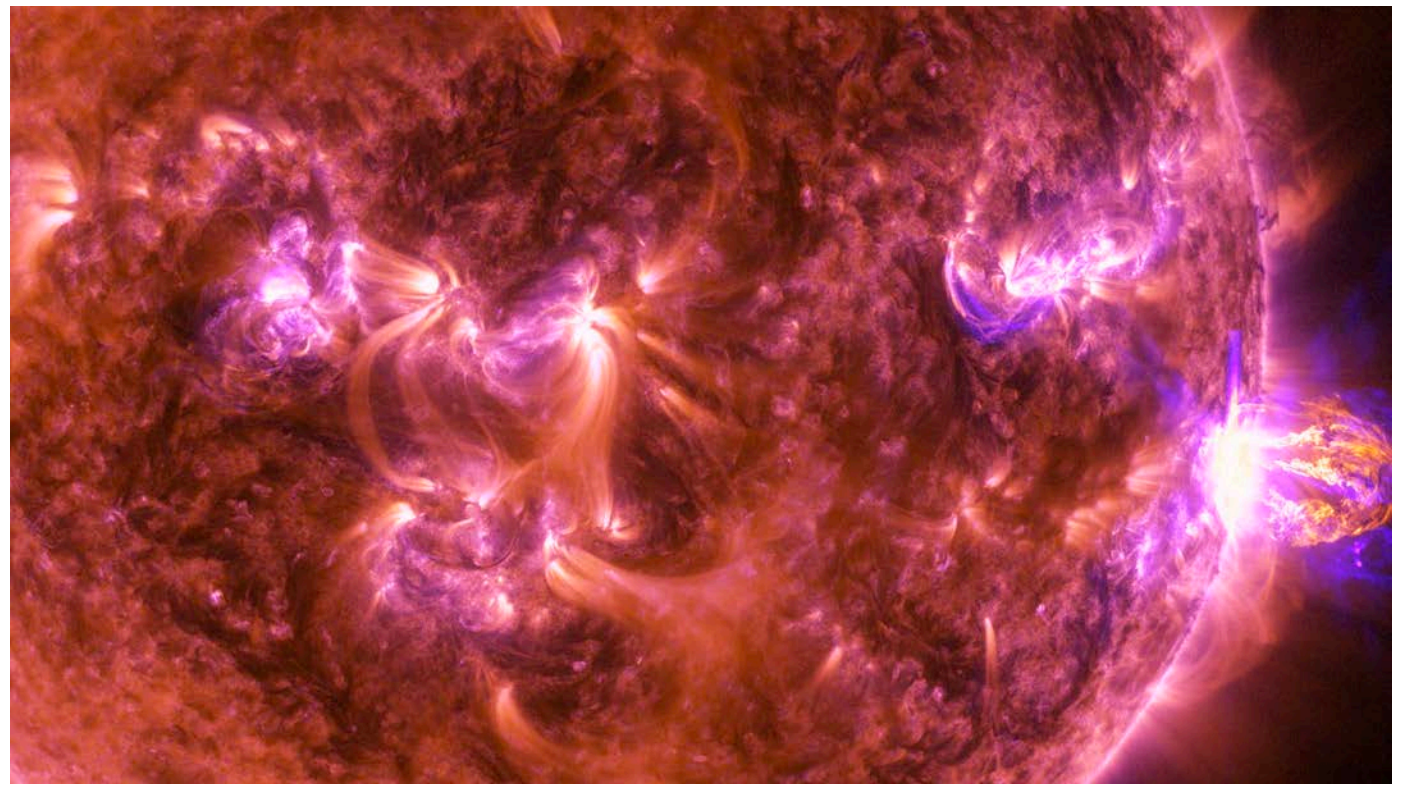


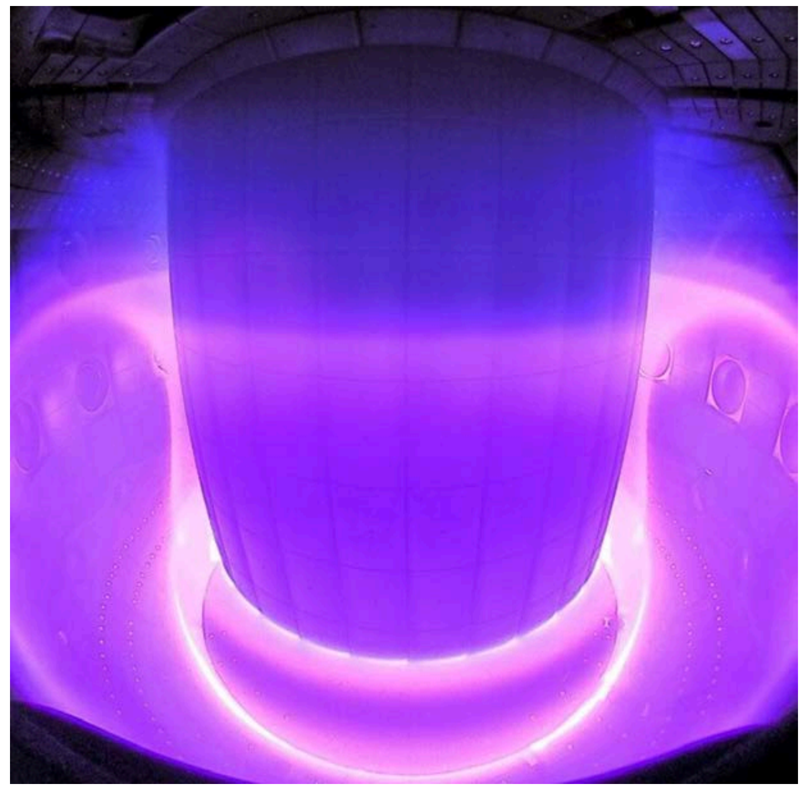
Plasma / magnetohydrodynamics

- The fluids made of electrically conducting material









State of deformable body is a map

- The setup for deformable body
  - Manifold M : **Material** coordinate, **Lagrangian** coordinate
  - Manifold W : **World** coordinate, **Eulerian** coordinate
  - A state of a deformable body is a map (called **flow map**)

$$\phi : M \to W$$

- The flow map assigns each material point its world position
- For a dynamical system, the flow map is time-dependent

$$\phi : \mathbb{R} \times M \to W \qquad \phi(t) : M \to W$$

Some assumptions on M, W

$$\phi(t)\colon M \to W$$

- To define inertia, we need a few structures on Material and World:
  - A time-independent n-form on material describing mass density

$$\rho_M \in \Omega^n(M)$$

$$\mathrm{TotalMass}(\mathrm{SomeRegion} \subset M) := \int_{\mathrm{Region}} \rho_M$$

  - A time-independent metric on the world

$$\flat_W \in \Gamma(T^*W \odot T^*W)$$

Kinetic energy and potential energy

let's understand the tensor types of derivatives of flow maps

- Kinetic energy

$$\mathcal{K}(\phi,\dot{\phi}) := \tfrac{1}{2}\int_M |\boxed{\dot{\phi}}|^2_{\flat_W}\,\rho_M$$

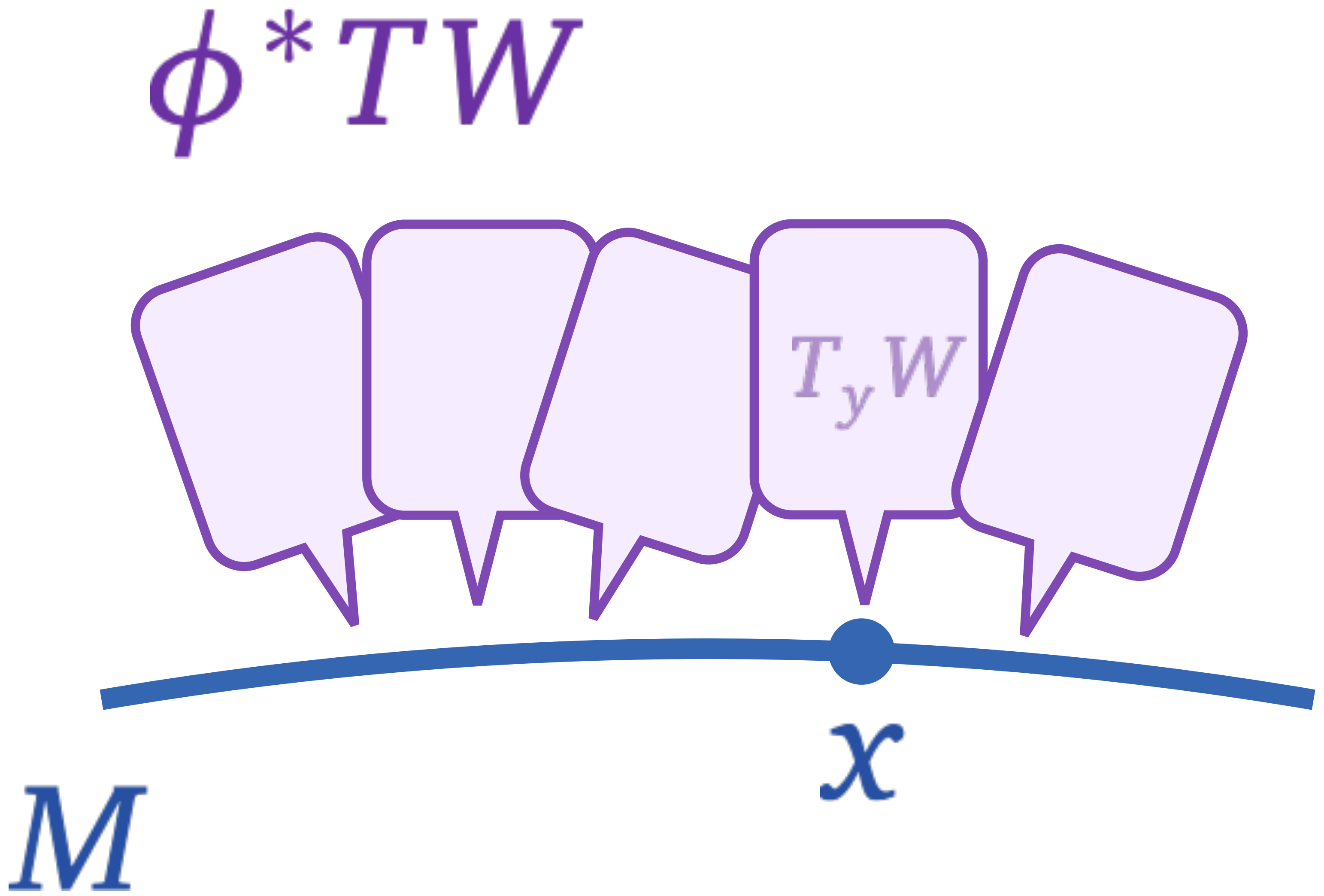
- Potential energy

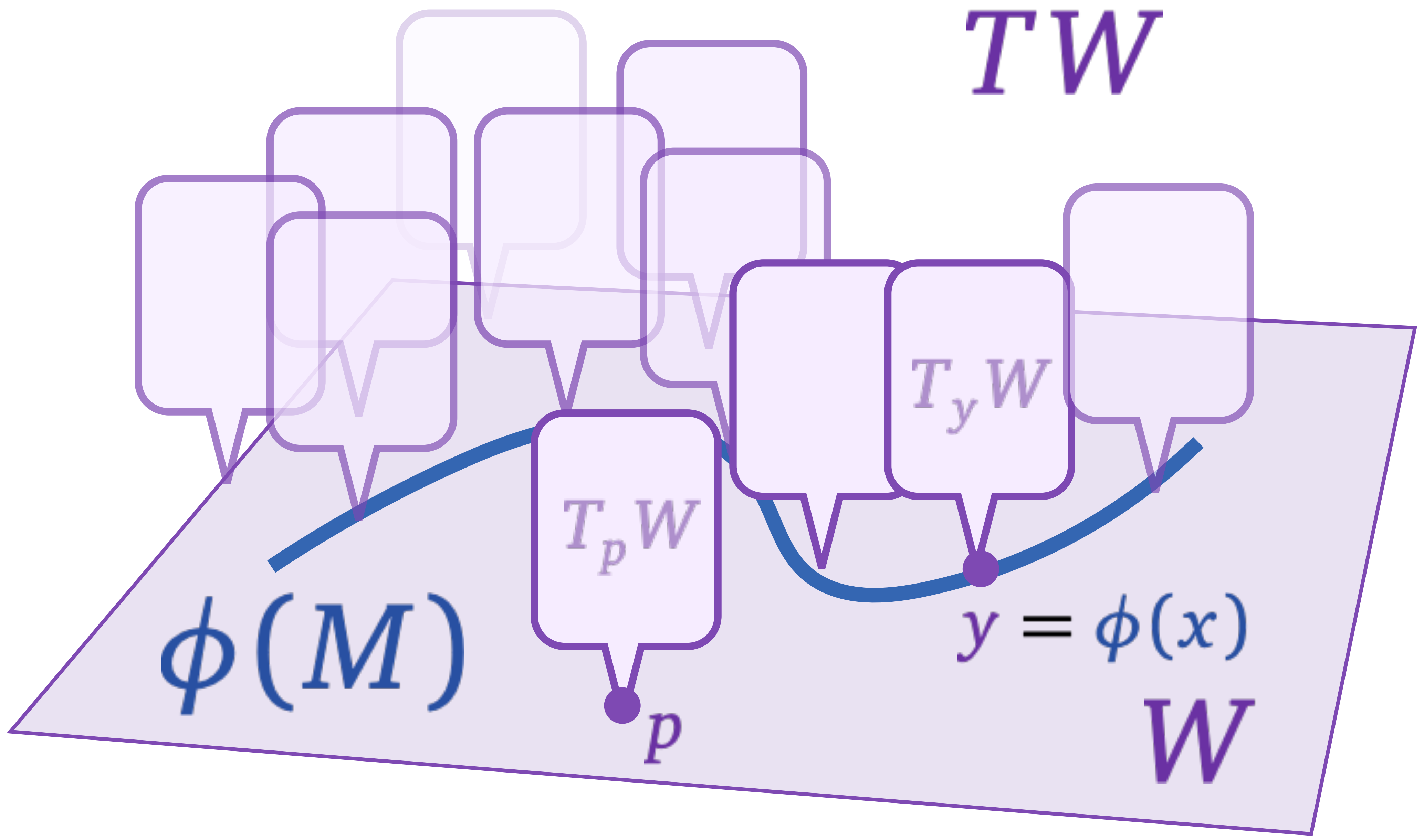
$$\mathcal{U}(\phi) := \int_M U(\phi,\boxed{d\phi})$$

Pullback bundle

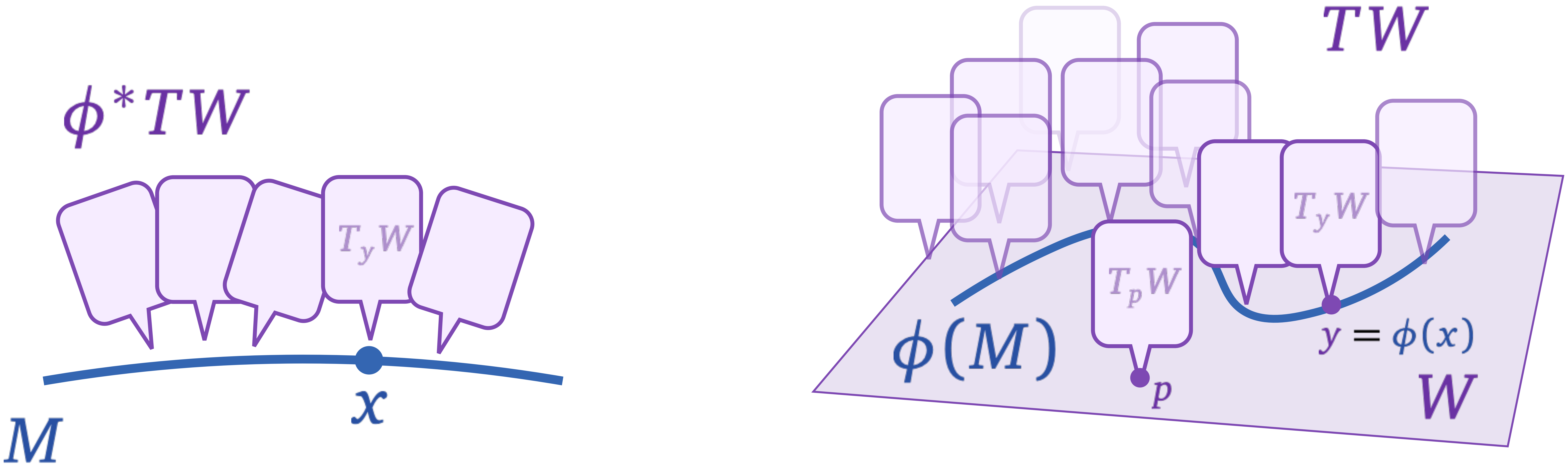
- Consider a bundle over W, say the tangent bundle TW
- Suppose there is a map into W, say a flow map $\phi : M \to W$
- Then we can construct a pullback bundle over M called ϕ^*TW so that the fibers are given by

$$(\phi^*TW)_x = T_{\phi(x)}W$$





Pullback bundle



- We will also use the following notation $T_\phi W = \phi^* TW$

$$(T_\phi W)_x = T_{\phi(x)} W$$

Pullback bundle

- With $\phi : M \longrightarrow W$
  - TM is a bundle over M
  - TW is a bundle over W
  - $T_\phi W$ is a bundle over M

Time derivative of flow map

- For a time dependent flow map $\phi(t)\colon M \to W$

$$\dot{\phi} = \frac{\partial \phi}{\partial t} \in \ ?$$

Time derivative of flow map

- For a time dependent flow map $\phi(t): M \longrightarrow W$

$$\dot{\phi} = \frac{\partial \phi}{\partial t} \in \Gamma(T_{\phi} W)$$

Differential (Jacobian) of flow map

- Given a flow map

$$\phi : M \to W$$

- Its differential, at each point, is

$$d\phi|_p : T_pM \xrightarrow{\text{linear}} T_{\phi(p)}W$$

- The differential is of type “(world)vector-valued 1-form over M”

$$\phi_* = d\phi \in ?$$

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$$\phi_* = d\phi \in \Gamma(T^*M \otimes T_\phi W)$$

$$= \Omega^1(M; T_\phi W)$$ for short

Two-point tensor

- In general, in continuum mechanics, **two-point tensors** are of type

$$\Gamma\Big(\underbrace{T^*M\otimes\cdots\otimes T^*M}_{p\text{ copies}}\otimes\underbrace{TM\otimes\cdots\otimes TM}_{q\text{ copies}}$$

$$\otimes\underbrace{T_\phi^*W\otimes\cdots\otimes T_\phi^*W}_{\ell\text{ copies}}\otimes\underbrace{T_\phi W\otimes\cdots\otimes T_\phi W}_{m\text{ copies}}\Big)$$

- Term coined by Erickson 1960. See "*Mathematical Foundation of Elasticity*" by J. Marsden and T. Hughes 1983.
- Special shorthand: $\Gamma\Big(\underbrace{T^*M\wedge\cdots\wedge T^*M}_{k\text{ copies}}\otimes E\Big)=\Omega^k(M;E)$

E-valued k-form

Two-point tensor

- Special shorthand: $\Gamma\Big(\underbrace{T^*M \wedge \cdots \wedge T^*M}_{k \text{ copies}} \otimes E\Big) = \Omega^k(M;E)$

  E-valued k-form

- For a tensor $\tau \in \Omega^k(M;E)$
  - As a k-form, it's to be measured/evaluated over infinitesimal k-dim geometries
  - The measurement is a vector/tensor of type *E*.

Tensor Duality

- Tensors for continuum mechanics
- Tensor duality
- Elasticity

Dual space of tensor product

- Dual of tensor:

$$(U \otimes V)^* = U^* \otimes V^*$$

$$\langle \alpha \otimes \beta | u \otimes v \rangle = \langle \alpha | u \rangle \langle \beta | v \rangle$$

- In basis, this looks like Frobenius inner product for matrices

$$\mathbf{A} : \mathbf{B} = \mathrm{tr}(\mathbf{A}^\intercal \mathbf{B}) = \sum_{ij} A_{ij} B_{ij}$$

- For tensor fields, the dual requires additionally tensoring an n-form

$$\Gamma(E)^* = \Gamma(E^* \otimes \wedge^n T^* M)$$

$$\langle\!\langle \tau | \sigma \rangle\!\rangle = \int_M \underbrace{\langle \tau | \sigma \rangle}_{\text{n-form}}$$

Dual space of tensor product

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  - Example

$$\Omega^0(M)^* = \Omega^n(M)$$

  - Example

$$\Gamma(T^*M \otimes T_\phi W)^* = \Gamma(TM \otimes T_\phi^* W \otimes \wedge^n T^*M)$$

Vector ⊗ n-form = (n-1)-form

- Let V be any vector space
- We have a canonical isomorphism

$$V \otimes (\wedge^n V^*) \cong \wedge^{n-1} V^*$$

$$\vec{v} \otimes \mu \mapsto i_{\vec{v}} \mu$$

  - This mapping is linearly bijective.

Vector ⊗ n-form = (n-1)-form

- More generally

$$(\wedge^k V) \otimes (\wedge^n V^*) \cong \wedge^{n-k} V^*$$

  - Example:

$$\begin{aligned}\Omega^k(M)^* &= \Gamma(\wedge^k T^*M)^* \\ &= \Gamma((\wedge^k TM) \otimes (\wedge^n T^*M)) \\ &= \Gamma(\wedge^{n-k} T^*M) \\ &= \Omega^{n-k}(M)\end{aligned}$$

$$\langle\!\langle \underbrace{\alpha}_{k\text{-form}} \mid \underbrace{\beta}_{(n-k)\text{-form}} \rangle\!\rangle = \int_M \underbrace{\alpha \wedge \beta}_{n\text{-form}}$$

More examples

- What is type of $d\phi$ and what is the dual object?

$$d\phi \in \Gamma(T^*M \otimes T_\phi W)$$

$$\Gamma(T^*M \otimes T_\phi W)^* = \Gamma(TM \otimes T^*_\phi W \otimes \wedge^n T^*M)$$

$$= \Gamma(\wedge^{n-1} T^*M \otimes T^*_\phi W)$$

momentum flux

which is stress

Elasticity

- Tensors for continuum mechanics
- Tensor duality
- Elasticity

Postulates

- The position of the body is described by a (time-dependent) map

  $$\phi : M \to W$$

  where M has a time-independent mass density $\rho_M \in \Omega^n(M)$

  and W has a time-independent metric $\flat_W \in \Gamma(T^*W \odot T^*W)$

- The elastic potential energy for a map ϕ takes the form

  $$\mathcal{U}(\phi) = \int_M U(\phi^*\flat_W)$$

  for some fiber-wise (nonlinear) mapping (depending on material)

  $$U_p : T_p^*M \odot T_p^*M \xrightarrow{\text{nonlinear}} \wedge^n T_p^*M$$

  i.e. the potential is only a function of the induced metric encoding its notion of distances in the world. (**Frame-indifference**)

Terminology

- We call $F = \phi_* = d\phi \in \Gamma(T^*M \otimes T_\phi W)$ **deformation gradient**.
- The induced metric $\phi^*\flat_W$ can be understood by the diagram

$$\begin{tikzcd}
T_pM \arrow[r, "F"] \arrow[d, dashed, bend left, "F^*\flat_W F = \phi^*\flat_W"'] & T_{\phi(p)}W \arrow[d, "\flat_W"] \\
T_p^*M & T^*_{\phi(p)}W \arrow[l, "F^*"']
\end{tikzcd}$$

- The induced metric $C := F^*\flat_W F \in \Gamma(T^*M \odot T^*M)$ is called the (right)-**Cauchy–Green tensor**.

Notation in 3D Cartesian coordinate

- Let (X, Y, Z) denote the Cartesian coordinate for M and (x, y, z) the Cartesian coordinate for W
- Flow map $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \phi^1(X,Y,Z) \\ \phi^2(X,Y,Z) \\ \phi^3(X,Y,Z) \end{bmatrix}$
- Deformation gradient $\mathbf{F} = \begin{bmatrix} \frac{\partial \phi^1}{\partial X} & \frac{\partial \phi^1}{\partial Y} & \frac{\partial \phi^1}{\partial Z} \\ \frac{\partial \phi^2}{\partial X} & \frac{\partial \phi^2}{\partial Y} & \frac{\partial \phi^2}{\partial Z} \\ \frac{\partial \phi^3}{\partial X} & \frac{\partial \phi^3}{\partial Y} & \frac{\partial \phi^3}{\partial Z} \end{bmatrix}$
- Right Cauchy–Green tensor $\mathbf{C} = \mathbf{F}^{\mathsf{T}}\mathbf{F}$

Deriving elastic force

- Back to our potential energy $\mathcal{U}(\phi) = \int_M U(\phi^* \flat_W) = \int_M U(C)$

  given some model $U : \Gamma(T^*M \odot T^*M) \xrightarrow[\text{nonlinear}]{\text{pointwise}} \Gamma(\wedge^n T^*M)$

- The force is given by $-d\mathcal{U}_\phi$
- Compute it using backpropagation.

Deriving elastic force

$$\mathcal{U}(\phi) = \int_M U(\phi^* \flat_W) = \int_M U(C)$$

$$\begin{tikzcd}
\phi & F & C = F^* \flat_W F & U(C) & \\
C^\infty(M; W) \arrow[r, "d"] & \Gamma(T^*M \otimes T_\phi W) \arrow[r, "\mathcal{G}"] & \Gamma(T^*M \odot T^*M) \arrow[r, "U"] & \Gamma(\Lambda^n T^*M) \arrow[r, "\int_M"] & \mathbb{R}
\end{tikzcd}$$

Sequence of linear maps on tangent spaces:

$$\begin{tikzcd}
\Gamma(T_\phi W) \arrow[r, "d^{\nabla^W}"] & \Gamma(T^*M \otimes T_\phi W) \arrow[r, "d\mathcal{G}|_F"] & \Gamma(T^*M \odot T^*M) \arrow[r, "dU|_C"] & \Gamma(\Lambda^n T^*M) \arrow[r, "\int_M"] & \mathbb{R}
\end{tikzcd}$$

Pullback:

$$\begin{tikzcd}
 & \frac{\partial U}{\partial F} & \frac{\partial U}{\partial C} & \mathbb{1} & 1 \\
 & \Gamma(\Lambda^{n-1} T^*M \otimes T^*_\phi W) \arrow[dl, "(d^\nabla)^*"'] & \Gamma(TM \odot TM \otimes \Lambda^n T^*M) \arrow[l, "d\mathcal{G}|^*_F"'] & \Omega^0(M) \arrow[l, "dU|^*_C"'] & \mathbb{R}^* \arrow[l] \\
d\mathcal{U}_\phi & & & & \\
\Gamma(\Lambda^n T^*M \otimes T^*_\phi W) & & & &
\end{tikzcd}$$

Terminology

$$\begin{tikzcd}
 & \overset{\frac{\partial U}{\partial F}}{\Gamma(\Lambda^{n-1}T^*M \otimes T_\phi^* W)} \arrow[dl, "(d^\nabla)^*"'] & \overset{\frac{\partial U}{\partial C}}{\Gamma(TM \odot TM \otimes \Lambda^n T^*M)} \arrow[l, "d\mathcal{G}|_F^*"'] & \overset{\mathbb{1}}{\Omega^0(M)} \arrow[l, "dU|_C^*"'] & \overset{1}{\mathbb{R}^*} \arrow[l] \\
\overset{d\mathcal{U}_\phi}{\Gamma(\Lambda^n T^*M \otimes T_\phi^* W)} & & & &
\end{tikzcd}$$

2nd Piola–Kirchoff stress (type is dual to right-Cauchy–Green)

$$S := 2\frac{\partial U}{\partial C} \in \Gamma(TM \odot TM \otimes \Lambda^n T^*M)$$

1st Piola–Kirchoff stress tensor (type is dual to deformation gradient)

$$P := d\mathcal{G}|_F{}^* S = \frac{\partial U}{\partial F} \in \Gamma(\Lambda^{n-1}T^*M \otimes T_\phi^* W)$$