

# **CSE 291 (SP23)**

# **Physical Simulation**

# **Tensors: Part 1**

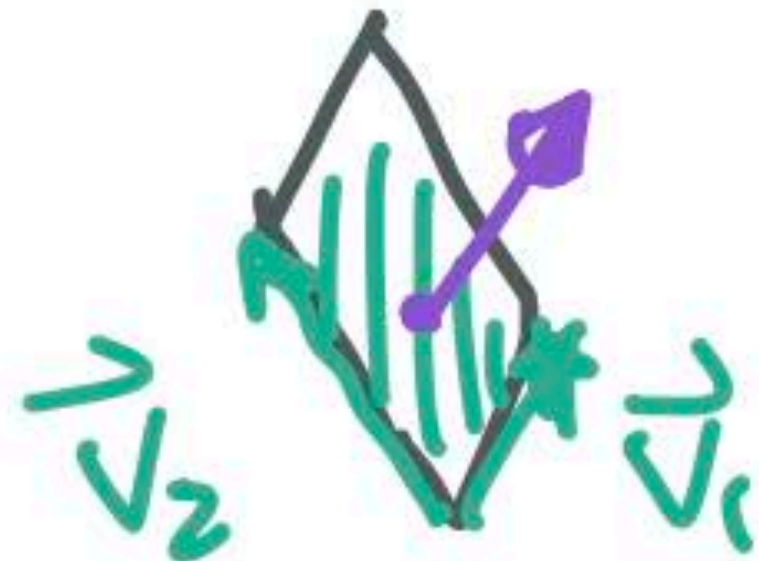
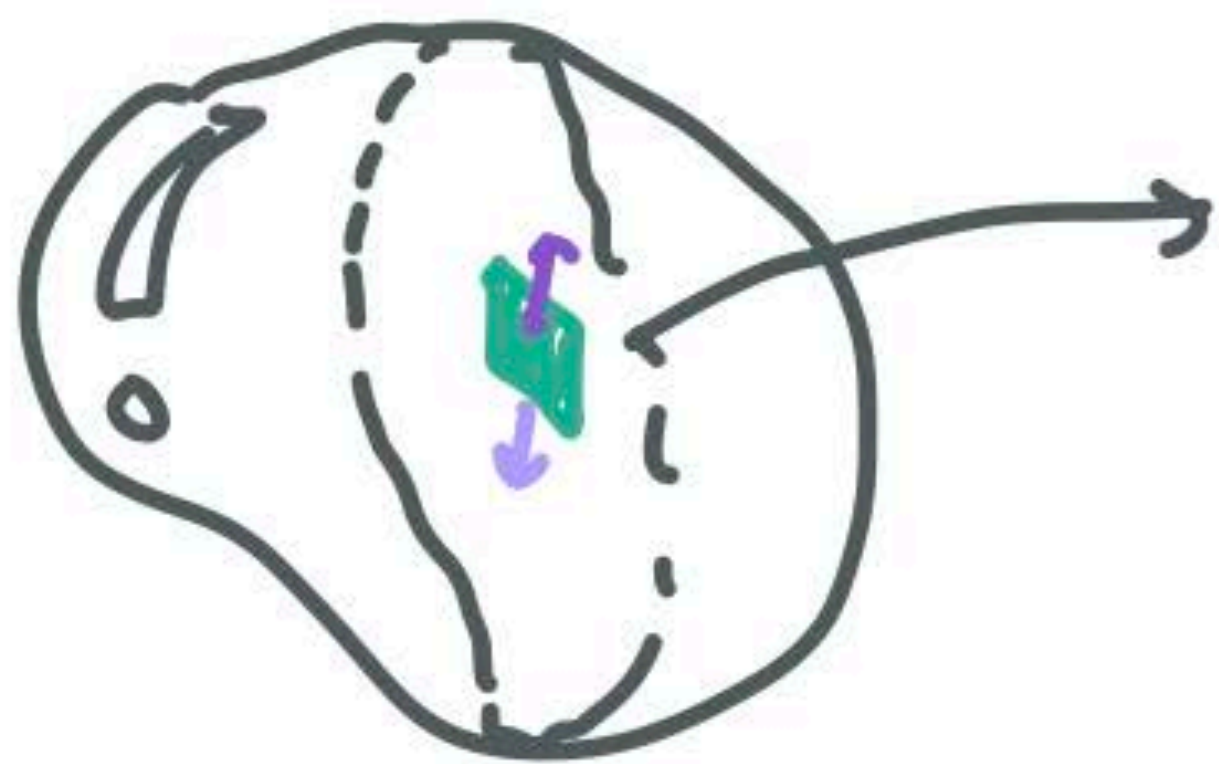
**Albert Chern**

# Tensor product spaces

- Tensor product spaces
- Differential forms
- Exterior algebra
- Exterior derivative

# Tensor: introduction

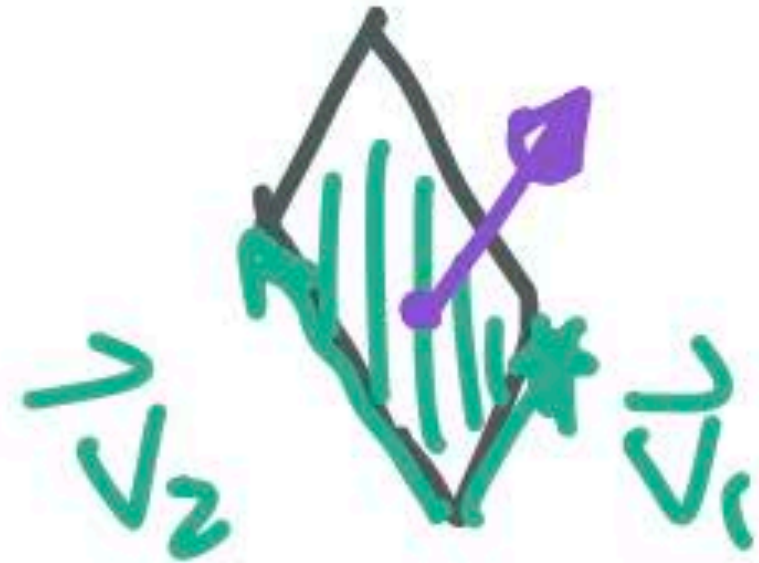
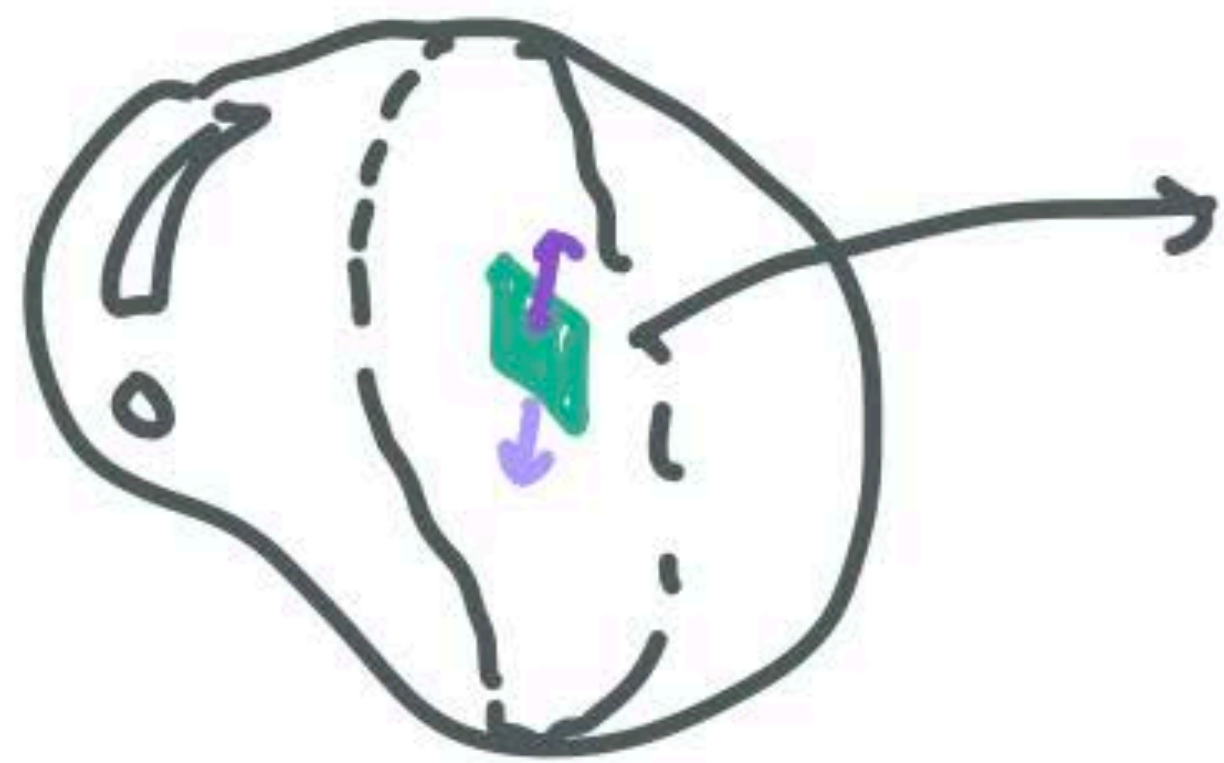
- Vectors, covectors, scalars, bilinear forms, endomorphisms, etc are special examples of **tensors**.
- Tensors are produced by combining vector spaces and covector spaces through tensor product  $\otimes$ .
- Example: stress tensor



$$\sigma : \underbrace{\mathbb{R}^3 \times \mathbb{R}^3}_{\text{area vector}} \xrightarrow{\text{bilinear}} \underbrace{\mathbb{R}^3}_{\text{force (co)vector}}$$

# Tensor: introduction

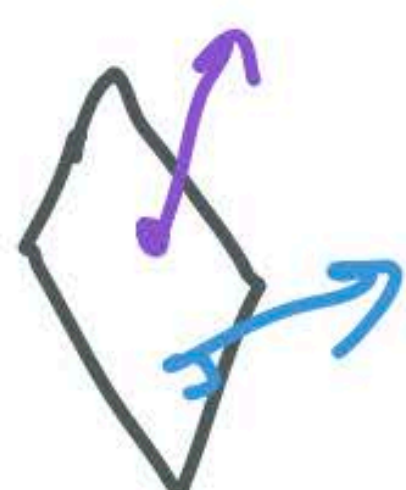
- Example: stress tensor



$$\sigma : \underbrace{\mathbb{R}^3 \times \mathbb{R}^3}_{\text{area vector}} \xrightarrow{\text{bilinear}} \underbrace{\mathbb{R}^3}_{\text{force (co)vector}}$$

$$\begin{aligned} \vec{\sigma}(\vec{v}_1, \vec{v}_2) \\ = -\vec{\sigma}(\vec{v}_2, \vec{v}_1). \end{aligned}$$

Typical exposition



A hand-drawn diagram of a parallelogram representing a surface element. A blue arrow points outwards from the surface, representing the normal vector. A purple arrow points along the surface, representing the force vector.

$$\sigma : \underbrace{\mathbb{R}^3}_{\text{normal vector}} \rightarrow \underbrace{\mathbb{R}^3}_{\text{force vector}}$$

# Tensor product space

- Given two vector spaces  $U, V$
- We can construct a new vector space  $W = U \otimes V$  called their **tensor product space**.

- Suppose we have basis for each of  $U, V$

$$\vec{e}_1, \dots, \vec{e}_m \in U \quad \vec{f}_1, \dots, \vec{f}_n \in V$$

we construct  $W$  as the span of  $\vec{e}_i \otimes \vec{f}_j$   $\begin{matrix} i=1, \dots, m \\ j=1, \dots, n \end{matrix}$

# Tensor product space

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- What's the dimension of  $W = U \otimes V$ ?

$$\dim(U \otimes V) = \dim(U) \dim(V)$$

- The symbol  $\otimes$  between the basis vectors extends into a bilinear map

$$\otimes : U \times V \xrightarrow{\text{bilinear}} W$$

# Tensor product space

- The symbol  $\otimes$  between the basis vectors extends into a bilinear map

$$\otimes : U \times V \xrightarrow{\text{bilinear}} W$$

$$\vec{u} \otimes \vec{v} = (u^i \vec{e}_i) \otimes (v^j \vec{f}_j) = u^i v^j (\vec{e}_i \otimes \vec{f}_j) = u^i v^j (\vec{e}_i \otimes \vec{f}_j)$$

- In general every element  $\tau \in W$  takes the form

$$\tau = \sum_{i=1}^m \sum_{j=1}^n \tau^{ij} (\vec{e}_i \otimes \vec{f}_j)$$

- Most of  $\tau$  cannot be written as  $\tau^{ij} = u^i v^j$
- If  $\tau^{ij} = u^i v^j$  then we say  $\tau$  is a decomposable tensor.

# Tensor product space

- Examples (matrices)

$$\text{End}(U) = \{A: U \xrightarrow{\text{linear}} U\} = U^* \otimes U$$

$$\{B: U \xrightarrow{\text{linear}} U^*\} = \{B: U \times U \xrightarrow{\text{bilinear}} \mathbb{R}\} = U^* \otimes U^*$$

- Vector-valued k-linear forms

$$\underbrace{U^* \otimes \cdots \otimes U^*}_k \otimes U$$

# Symmetric and skewsymmetric product

- We often consider tensors with additional symmetry or skew-symmetry assumption; e.g. symmetric bilinear form:

$$\{B : U \xrightarrow{\text{linear}} U^* \mid B^* = B\} \subset U^* \otimes U^*$$

- We could have built the tensor product with additional symmetry rule
  - ▶ Let  $\vec{e}_1, \dots, \vec{e}_m \in U$  be a basis.
  - ▶ Consider the subspace  $U \odot U \subset U \otimes U$  of only symmetric tensors:  
spanned by  $\vec{e}_i \odot \vec{e}_j$ ,  $1 \leq i \leq j \leq m$ , with  $\vec{e}_i \odot \vec{e}_j = \vec{e}_j \odot \vec{e}_i$
  - ▶ Consider the subspace  $U \wedge U \subset U \otimes U$  of skewsymmetric ones:  
spanned by  $\vec{e}_i \wedge \vec{e}_j$ ,  $1 \leq i < j \leq m$ , with  $\vec{e}_i \wedge \vec{e}_j = -\vec{e}_j \wedge \vec{e}_i$

# Differential Forms

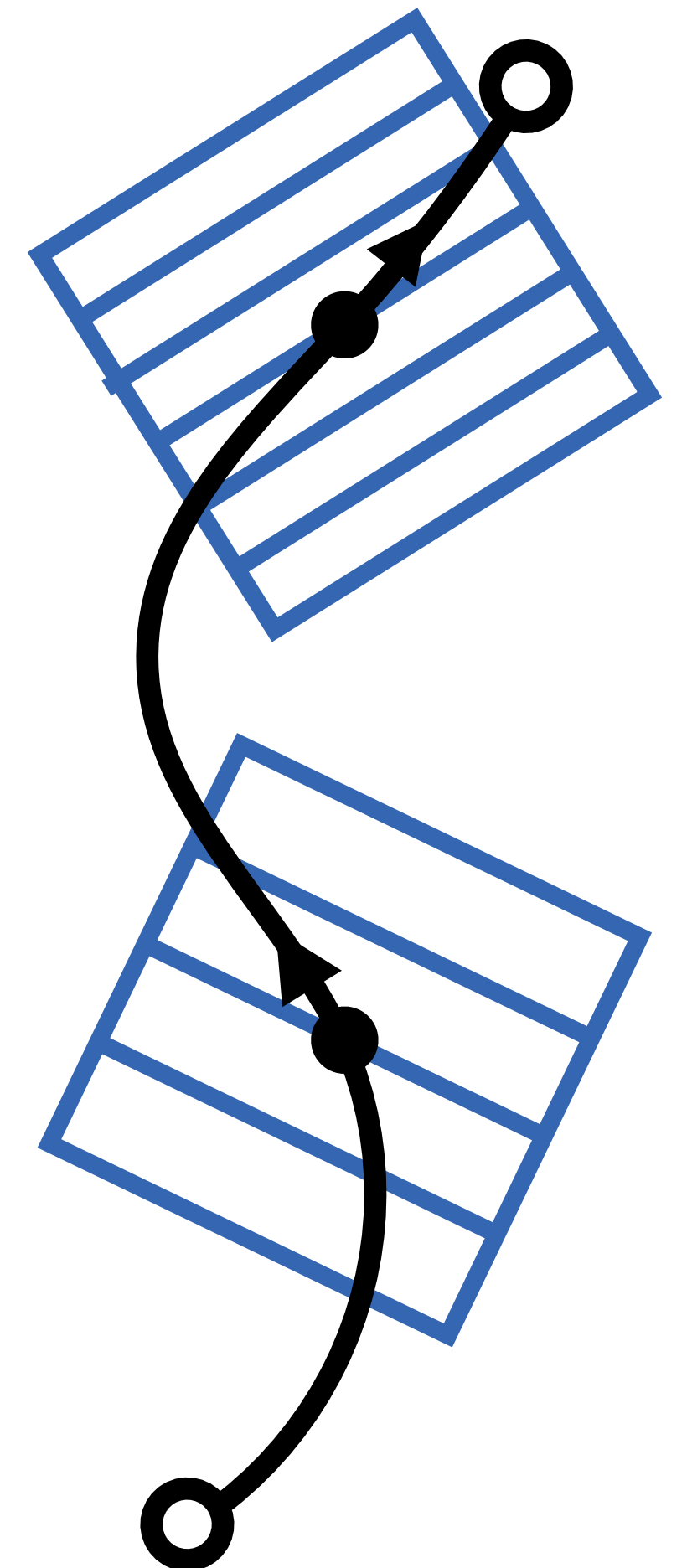
- Tensor product spaces
- Differential forms
- Exterior algebra
- Exterior derivative

# Differential forms

- Differential k-forms are objects to-be-integrated over k-dimensional oriented surfaces
- Line integral

$$\int_{\Gamma} \mathbf{v} \cdot d\mathbf{l} = \int_{\Gamma} \underbrace{v_1 dx + v_2 dy + v_3 dz}_{\text{differential 1-form = covector field}}$$

differential 1-form = covector field



# Differential forms

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$$\int_{\Gamma} \mathbf{v} \cdot d\mathbf{l} = \int_{\Gamma} \underbrace{v_1 dx + v_2 dy + v_3 dz}$$

differential 1-form = covector field

- Flux integral

$$\int_S \mathbf{w} \cdot \mathbf{n} dA = \int_S \underbrace{w_1 dy dz + w_2 dz dx + w_3 dx dy}$$

$dx dy = -dy dx$  Skew-symmetric tensor  $T^*M \wedge T^*M$

# Differential forms

- Line integral

$$\int_{\Gamma} \mathbf{v} \cdot d\mathbf{l} = \int_{\Gamma} \underbrace{v_1 dx + v_2 dy + v_3 dz}_{\text{differential 1-form = covector field}}$$

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$$\int_S \mathbf{w} \cdot \mathbf{n} dA = \int_S \underbrace{w_1 dy dz + w_2 dz dx + w_3 dx dy}_{\text{Skew-symmetric tensor } T^*M \wedge T^*M}$$

$$dx dy = -dy dx \quad \text{Skew-symmetric tensor } T^*M \wedge T^*M$$

- Volume integral  $\int_V f dx dy dz \quad dx dy dz = -dy dx dz$

# Differential forms

- Let  $V = T_p M$
- A differential k-form is a skew-symmetric tensor product of covectors

$$\omega \in \underbrace{V^* \wedge \cdots \wedge V^*}_k = \wedge^k V^* \subset \otimes^k V^*$$

- If the basis for  $V^*$  is the differentials  $dx^1, \dots, dx^n$  of coordinate then the basis for  $\wedge^k V^*$  are

$$dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_k}, \quad i_1 < \cdots < i_k \quad I = (i_1, \dots, i_k)$$

- $\dim(\wedge^k V^*) = \binom{n}{k} = \frac{n!}{(n-k)!k!}$

# Pairing with vectors

- Just like a covector can pair with a vector, a differential k-form can be paired with k vectors

$$\underbrace{\alpha}_{1\text{-form}} \llbracket \underbrace{X}_{\text{vec}} \rrbracket$$

$$\underbrace{\omega}_{k\text{-form}} \llbracket X_1, \dots, X_k \rrbracket$$

- And it will satisfy skew symmetry

$$\omega \llbracket X_1, \dots, X_i, \dots, X_j, \dots, X_k \rrbracket = -\omega \llbracket X_1, \dots, X_j, \dots, X_i, \dots, X_k \rrbracket$$

# Pairing with vectors

- Let  $\alpha^1, \dots, \alpha^k \in V^*$  be covectors
- Let  $X_1, \dots, X_k \in V$  be vectors
- Then the (decomposable) k-form  $\alpha^1 \wedge \dots \wedge \alpha^k$  can be paired with k vectors using the following formula

$$(\alpha^1 \wedge \dots \wedge \alpha^k)[X_1, \dots, X_k] = \det \begin{bmatrix} \langle \alpha^1 | X_1 \rangle & \cdots & \langle \alpha^1 | X_k \rangle \\ \vdots & \ddots & \vdots \\ \langle \alpha^k | X_1 \rangle & \cdots & \langle \alpha^k | X_k \rangle \end{bmatrix}$$

- This is skew symmetric both in the vector part and the covector part.
- Later, we will derive this formula from first-principles.

# In 3D

- In 3D,  $V = \mathbb{R}^3$ ,  $V^* = \mathbb{R}^{3*} = \text{span}\{dx, dy, dz\}$
- $\bigwedge^0 V^* = \mathbb{R}$
- $\bigwedge^1 V^* = \underbrace{\{u_1 dx + u_2 dy + u_3 dz\}}_{\mathbf{u}_{1\text{-form}}} \mid \mathbf{u} \in \mathbb{R}^3 \}$
- $\bigwedge^2 V^* = \underbrace{\{w_1 dy \wedge dz + w_2 dz \wedge dx + w_3 dx \wedge dy\}}_{\mathbf{w}_{2\text{-form}}} \mid \mathbf{w} \in \mathbb{R}^3 \}$
- $\bigwedge^3 V^* = \underbrace{\{f dx \wedge dy \wedge dz\}}_{f_{3\text{-form}}} \mid f \in \mathbb{R} \}$

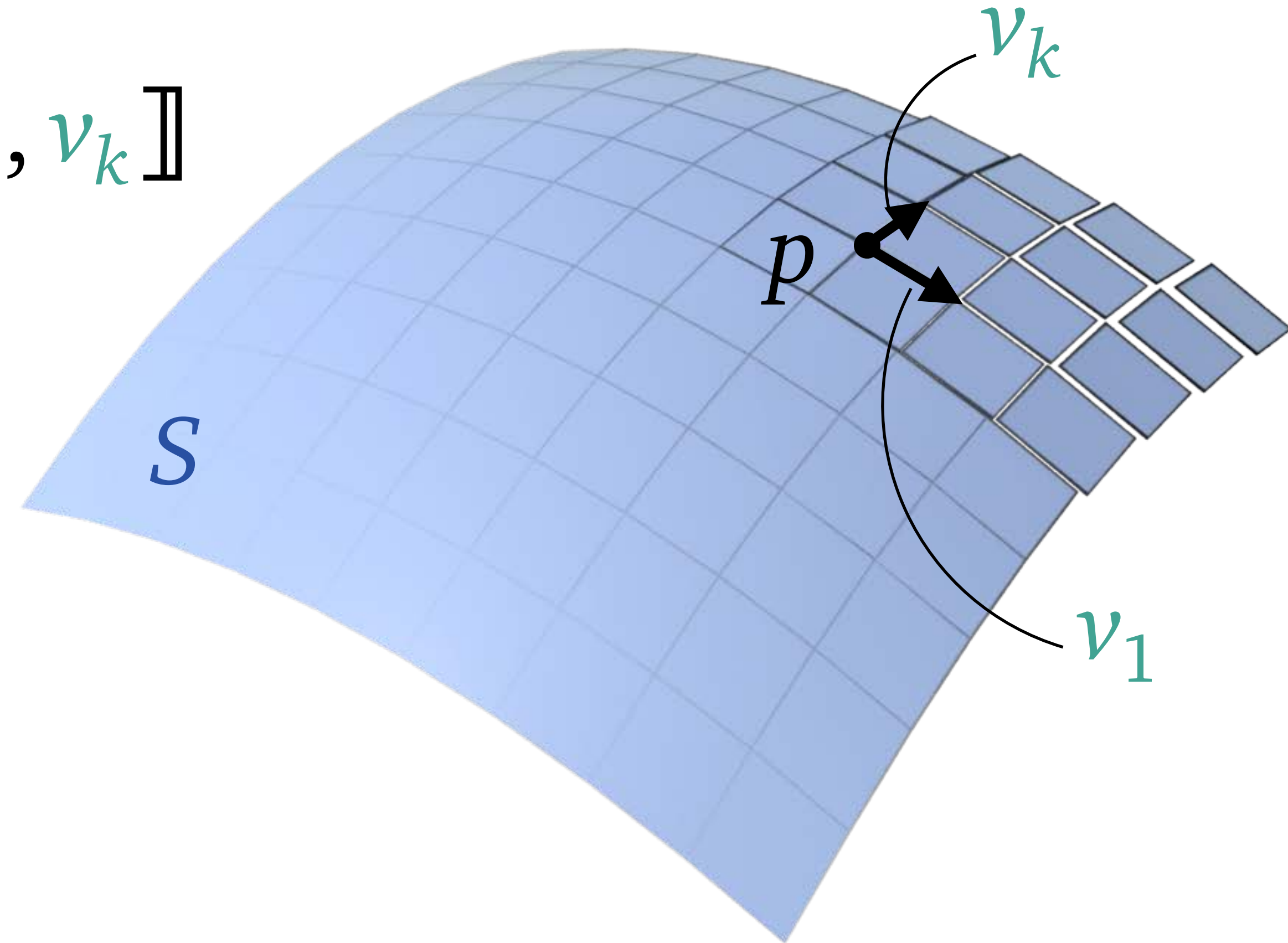
# In 3D

- $\mathbf{u}_{1\text{-form}}[\mathbf{v}_{\text{vec}}] = \mathbf{u} \cdot \mathbf{v} = [u_1 \ u_2 \ u_3] \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$
- $\mathbf{w}_{2\text{-form}}[\mathbf{u}_{\text{vec}}, \mathbf{v}_{\text{vec}}] = \det \begin{bmatrix} | & | & | \\ \mathbf{w} & \mathbf{u} & \mathbf{v} \\ | & | & | \end{bmatrix}$
- $f_{3\text{-form}}[\mathbf{u}_{\text{vec}}, \mathbf{v}_{\text{vec}}, \mathbf{w}_{\text{vec}}] = f \det \begin{bmatrix} | & | & | \\ \mathbf{u} & \mathbf{v} & \mathbf{w} \\ | & | & | \end{bmatrix}$

# Integration of k-form field

- Let  $\omega \in \Gamma(\wedge^k T^*M) = \Omega^k(M)$
- Let  $S$  be a k-dim surface.

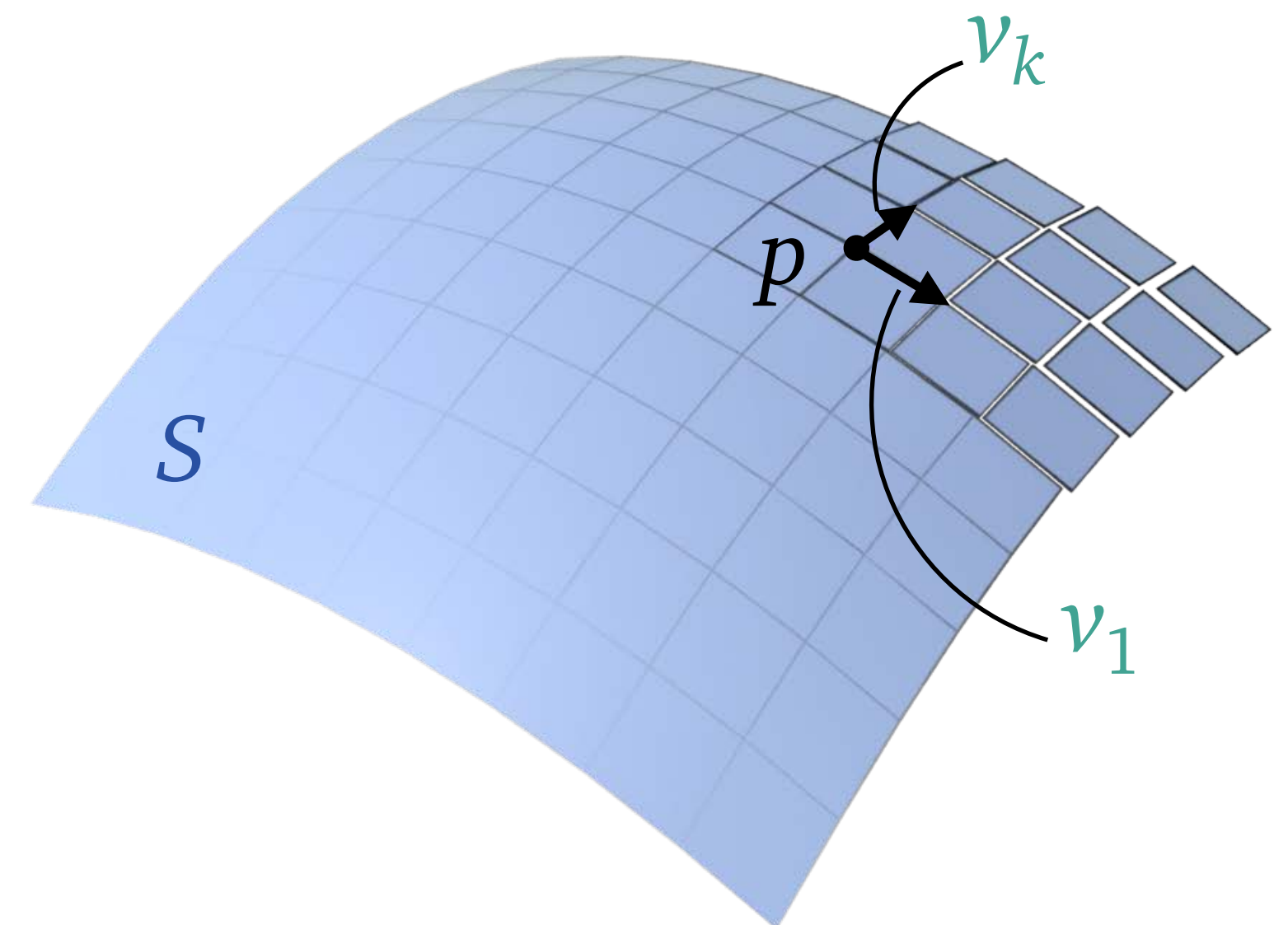
$$\int_S \omega = \lim_{\text{partition refines}} \sum_{\text{all elements}} \omega|_p \llbracket v_1, \dots, v_k \rrbracket$$



# In 3D

- $\mathbf{u}_{1\text{-form}}[\mathbf{v}_{\text{vec}}] = \mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$
- $\mathbf{w}_{2\text{-form}}[\mathbf{u}_{\text{vec}}, \mathbf{v}_{\text{vec}}] = \det \begin{bmatrix} | & | & | \\ \mathbf{w} & \mathbf{u} & \mathbf{v} \\ | & | & | \end{bmatrix}$
- $f_{3\text{-form}}[\mathbf{u}_{\text{vec}}, \mathbf{v}_{\text{vec}}, \mathbf{w}_{\text{vec}}] = f \det \begin{bmatrix} | & | & | \\ \mathbf{u} & \mathbf{v} & \mathbf{w} \\ | & | & | \end{bmatrix}$

$$f(p) = \int_p f_{0\text{-form}}$$
$$\int_{\Gamma} \mathbf{u} \cdot d\mathbf{l} = \int_{\Gamma} \mathbf{u}_{1\text{-form}}$$
$$\int_S \mathbf{w} \cdot \mathbf{n} dA = \int_S \mathbf{w}_{2\text{-form}}$$
$$\int_{\Omega} f dV = \int_{\Omega} f_{3\text{-form}}$$

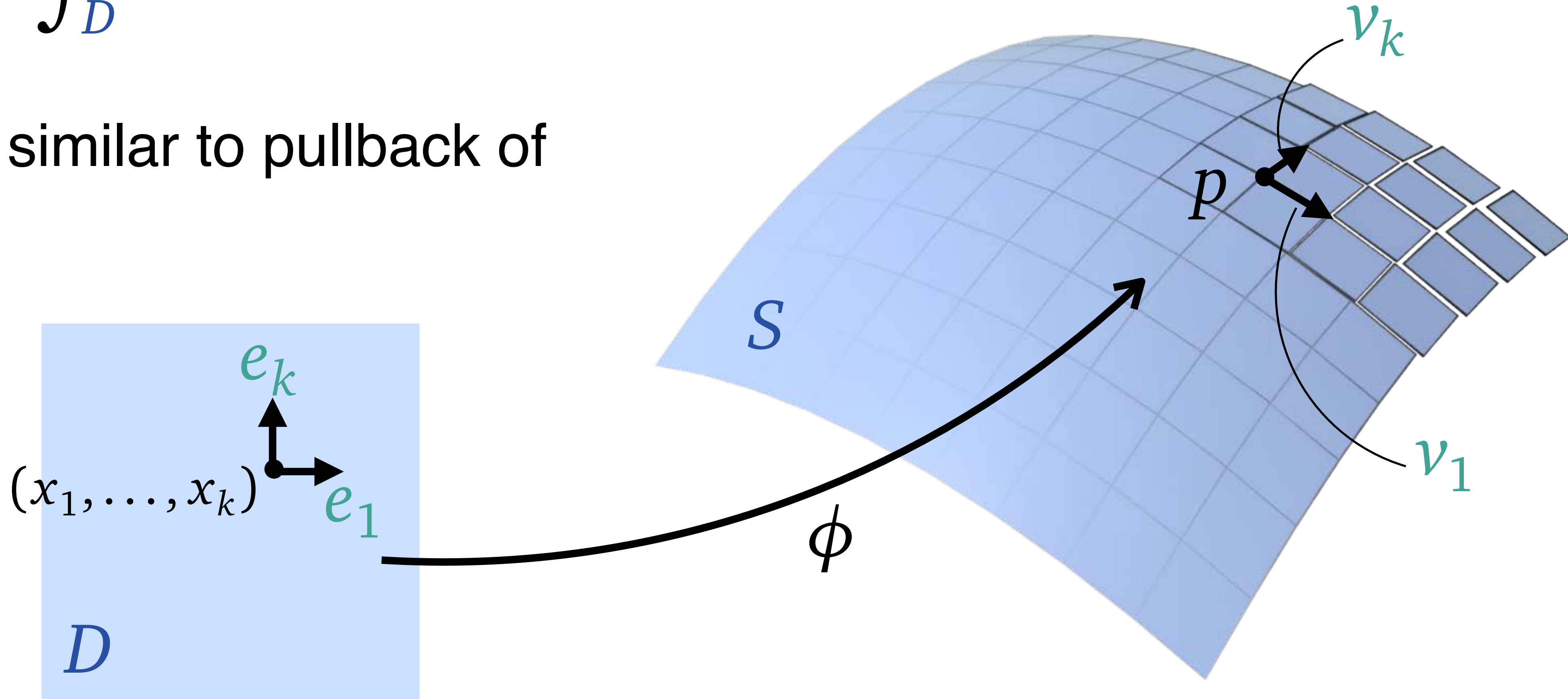


# Change of coordinates

- Let  $S$  be a  $k$ -dim surface parameterized as  $S = \phi(D)$

$$\int_S \omega = \int \cdots \int_D \omega_{\phi(\mathbf{x})} [\phi_* e_1, \dots, \phi_* e_k] dx_1 \cdots dx_k$$

- This looks similar to pullback of covectors!



# Pullback

- Given any linear map  $A: U \xrightarrow{\text{linear}} V$
- we have a canonical pullback for k-forms (like adjoint)

$$A^*: \wedge^k V^* \xrightarrow{\text{linear}} \wedge^k U^*$$

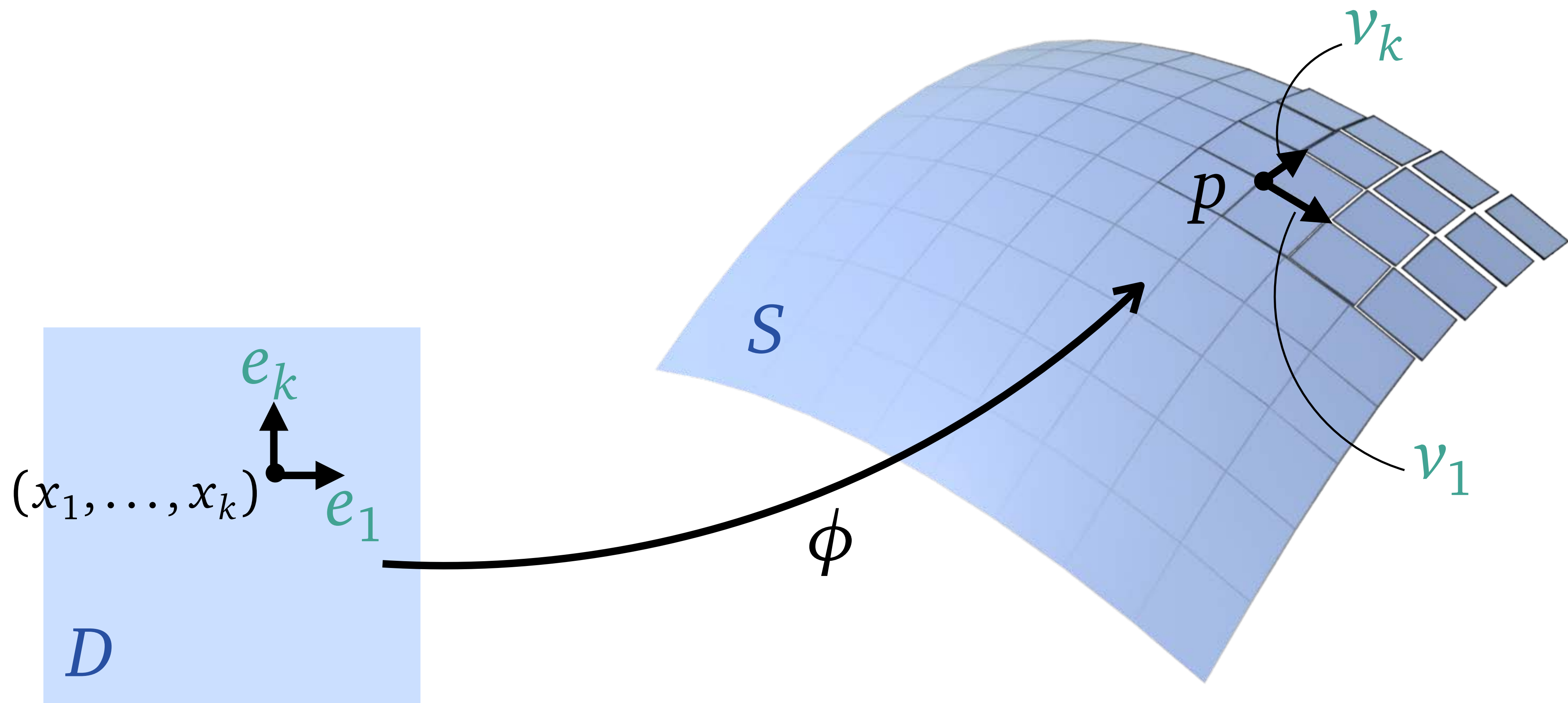
$$(A^* \omega)[X_1, \dots, X_k] := \omega[AX_1, \dots, AX_k]$$

- Given any nonlinear map between domains  $\phi: M \rightarrow N$
- we have a canonical pullback for k-form fields (like back-propagation)

$$\phi^*: \Omega^k(N) \xrightarrow{\text{linear}} \Omega^k(M)$$

$$(\phi^* \omega)_p[X_1, \dots, X_k] := \omega_{\phi(p)}[\phi_* X_1, \dots, \phi_* X_k]$$

# Change of coordinates using pullback



# Change of coordinates using pullback

$$\int_{\phi(D)} \omega = \int_D \phi^* \omega$$

# Exterior Algebra

- Tensor product spaces
- Differential forms
- Exterior algebra
- Exterior derivative

# Exterior algebra

- In addition to the linear combination operator on differential forms as tensors, there is a natural multiplication we can operate on differential forms.
- This is the **wedge product**, also known as **exterior product**.
- It makes the space of all differential forms into an exterior algebra.
- Another operation that conveniently comes with exterior algebra on differential forms is the **interior product**, also known as **contraction**. It is the extension of covector-vector dual pairing.

# Wedge product

- The wedge product

$$\wedge : \bigwedge^k V^* \times \bigwedge^l V^* \xrightarrow{\text{bilinear}} \bigwedge^{k+l} V^*$$

- It is defined axiomatically by the following two rules:

- ▶ Associativity  $(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$
- ▶ Skew on 1-forms: If  $\alpha \in V^* = \bigwedge^1 V^*$  then  $\alpha \wedge \alpha = 0$

- Consequently:

- ▶ If  $\alpha \in \bigwedge^k V^*$ ,  $\beta \in \bigwedge^l V^*$  then  $\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$

# Interior product

- Interior product is the extension of covector-vector pairing

$$i_{(\#1)}(\#2): V \times \bigwedge^k V^* \xrightarrow{\text{bilinear}} \bigwedge^{k-1} V^*$$

- It is defined axiomatically by the following three rules:

▶ Dual pairing: For  $\alpha \in V^* = \bigwedge^1 V^*$ ,  $i_X \alpha = \langle \alpha | X \rangle$

▶ Nilpotent:  $i_X i_X \omega = 0$

▶ Exterior Leibniz rule:

$$i_X (\alpha \wedge \beta) = (i_X \alpha) \wedge \beta + (-1)^{\deg(\alpha)} \alpha \wedge (i_X \beta)$$

- The pairing between k-form and k vectors is defined via interior prod.

$$\omega [X_1, \dots, X_k] := i_{X_k} \cdots i_{X_1} \omega$$

# Interior product

▶ Dual pairing: For  $\alpha \in V^* = \wedge^1 V^*$ ,  $i_X \alpha = \langle \alpha | X \rangle$

▶ Nilpotent:  $i_X i_X \omega = 0$

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$$i_X (\alpha \wedge \beta) = (i_X \alpha) \wedge \beta + (-1)^{\deg(\alpha)} \alpha \wedge (i_X \beta)$$

• The pairing between k-form and k vectors is defined via interior prod.

$$\omega [X_1, \dots, X_k] := i_{X_k} \cdots i_{X_1} \omega$$

• Consequently,  $(i_X \omega) [X_1, \dots, X_{k-1}] = \omega [X, X_1, \dots, X_{k-1}]$

• Consequently, for wedge of 1-forms

$$(\alpha^1 \wedge \cdots \wedge \alpha^k) [X_1, \dots, X_k] == \det \begin{bmatrix} \langle \alpha^1 | X_1 \rangle & \cdots & \langle \alpha^1 | X_k \rangle \\ \vdots & \ddots & \vdots \\ \langle \alpha^k | X_1 \rangle & \cdots & \langle \alpha^k | X_k \rangle \end{bmatrix}$$

# In 3D

$$\mathbf{u}_{1\text{-form}} \wedge \mathbf{v}_{1\text{-form}} = (\mathbf{u} \times \mathbf{v})_{2\text{-form}}$$

$$\mathbf{u}_{1\text{-form}} \wedge \mathbf{v}_{2\text{-form}} = (\mathbf{u} \cdot \mathbf{v})_{3\text{-form}}$$

$$i_{\mathbf{u}_{\text{vec}}} \mathbf{v}_{1\text{-form}} = (\mathbf{u} \cdot \mathbf{v})_{0\text{-form}}$$

$$i_{\mathbf{u}_{\text{vec}}} \mathbf{w}_{2\text{-form}} = (\mathbf{w} \times \mathbf{u})_{1\text{-form}}$$

$$i_{\mathbf{u}_{\text{vec}}} f_{3\text{-form}} = (f \mathbf{u})_{2\text{-form}}$$

# Pullback, wedge, and interior prod

- Pullback is a homomorphism on the whole exterior algebra

$$\phi^*(\alpha \wedge \beta) = (\phi^*\alpha) \wedge (\phi^*\beta)$$

- For interior product, interpreting it as insertion

$$(i_X \omega)[X_1, \dots, X_{k-1}] = \omega[X, X_1, \dots, X_{k-1}]$$

we have

$$i_X(\phi^*\omega) = \phi^*(i_{\phi_*X}\omega)$$

# Pullback in coordinates

- We know that

$$\phi^* f_{0\text{-form}} = f \circ \phi$$

$$\mathbf{F} := d\phi$$

$$\phi^*(\mathbf{v}_{1\text{-form}}) = \left( \mathbf{F}^\top(\mathbf{v} \circ \phi) \right)_{1\text{-form}}$$

- What about the pullback formula for 2-forms and 3-forms in 3D?

$$J := \det \mathbf{F}$$

$$\phi^*(\mathbf{w}_{2\text{-form}}) = \left( J \mathbf{F}^{-1}(\mathbf{w} \circ \phi) \right)_{2\text{-form}}$$

$$\phi^*(f_{3\text{-form}}) = \left( J(f \circ \phi) \right)_{3\text{-form}}$$

# Exterior Derivative

- Tensor product spaces
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# Exterior derivative

- Exterior derivative is the extension of taking differential

$$d: \Omega^k(M) \rightarrow \Omega^{k+1}(M)$$

- It is defined axiomatically by the following three rules:

▶ For  $f \in \Omega^0(M)$ ,  $df \in \Omega^1(M)$  is the differential of func.

▶ Nilpotent:  $dd\omega = 0$

▶ Exterior Leibniz rule:

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^{\deg(\alpha)} \alpha \wedge (d\beta)$$

# In 3D

$$df_{0\text{-form}} = (\nabla f)_{1\text{-form}}$$

$$d\mathbf{v}_{1\text{-form}} = (\nabla \times \mathbf{v})_{2\text{-form}}$$

$$d\mathbf{w}_{2\text{-form}} = (\nabla \cdot \mathbf{w})_{3\text{-form}}$$

# Integration and exterior derivative

- Stokes' Theorem

$$\int_S d\omega = \oint_{\partial S} \omega$$

- In 3D

- ▶ 0-form  $\int_a^b \nabla f \cdot d\mathbf{l} = f(b) - f(a)$  (fundamental theorem of calculus)
- ▶ 1-form  $\iint_S (\nabla \times \mathbf{u}) \cdot \mathbf{n} dA = \oint_{\partial S} \mathbf{u} \cdot d\mathbf{l}$  (Kelvin–Stokes curl theorem)
- ▶ 2-form  $\iiint_V \nabla \cdot \mathbf{w} dV = \iint_{\partial V} \mathbf{w} \cdot \mathbf{n} dA$  (Gauss divergence theorem)

# Pullback and exterior derivative

- Theorem

$$\phi^*(d\omega) = d(\phi^*\omega)$$

- Proof using Stokes theorem
- Translating to 3D: some non-obvious identities.