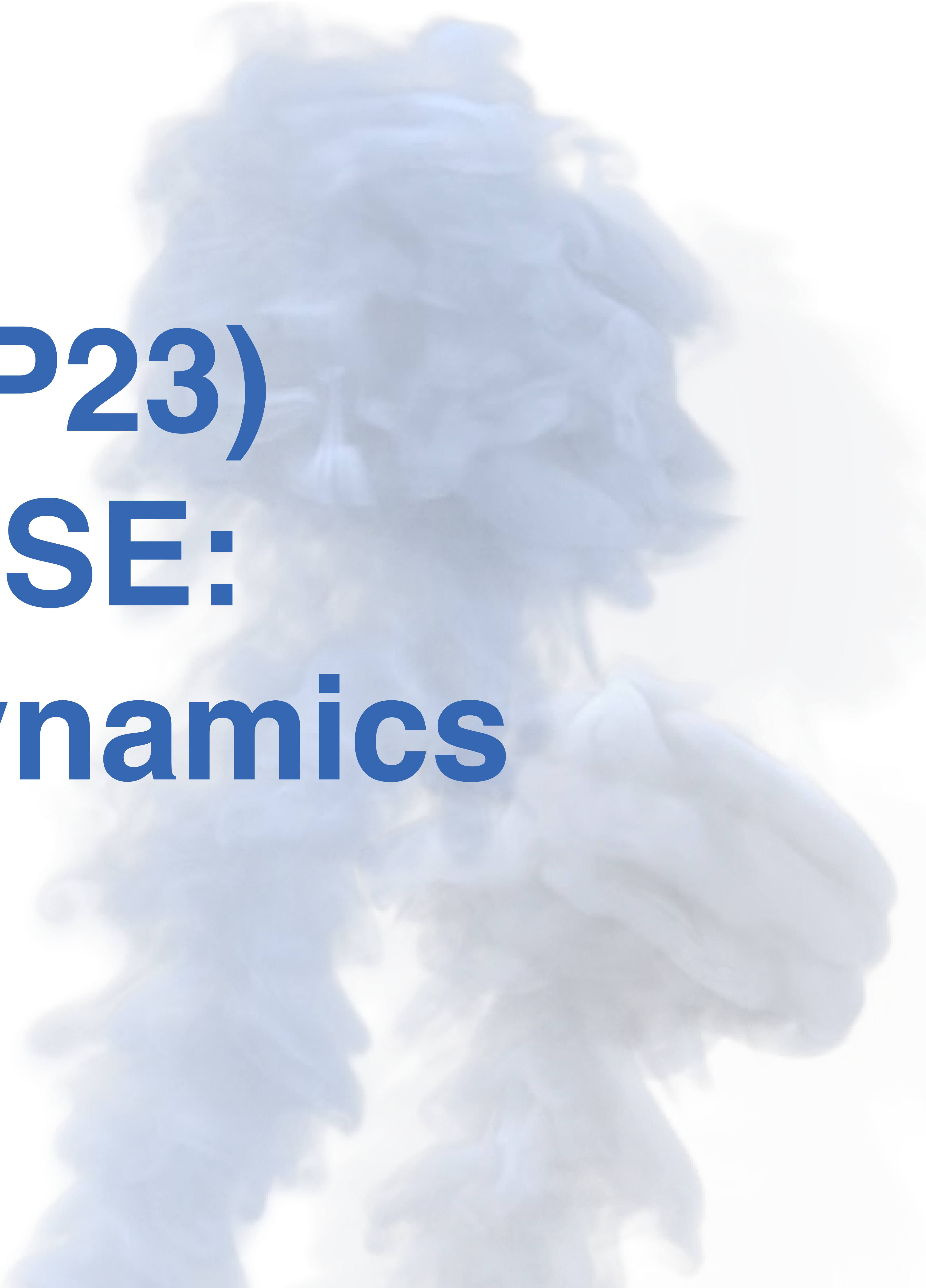

CSE 291 (SP23)
Topics in CSE:
Riemannian Dynamics

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Crash course on Differential Geometry

- Crash course on differential geometry
- Constraints in mechanical systems
- Geodesic equation
- Back to constraints

Crash corse on Differential Geometry

- Differential geometry is a coordinate-free type system for calculus.

Crash corse on Differential Geometry

- Differential geometry is a coordinate-free type system for calculus.
- Starting point:
  - Domain is a set of points M
  - Space of functions is denoted by $C^\infty(M)$ which is a commutative algebra:
    - We can do linear combination $c_1 f_1 + c_2 f_2$
    - We can do pointwise multiplication $f\,g$

Tangent space and space of vector field

- For each point $p \in M$ define

$$T_pM := \left\{ X_p : C^\infty(M) \xrightarrow{\text{linear}} \mathbb{R} \;\middle|\; X_p(fg) = (X_pf)g + f(X_pg)\ \forall f, g \in C^\infty(M) \right\}$$

- Drop the "$p \in M$"

$$\Gamma(TM) := \left\{ X : C^\infty(M) \xrightarrow{\text{linear}} C^\infty(M) \;\middle|\; X(fg) = (Xf)g + f(Xg)\ \forall f, g \in C^\infty(M) \right\}$$

- We can scale VF by scalar func: $\underbrace{(fX)g}_{\text{pointwise scaling then differentiate g}} := \underbrace{f(Xg)}_{\text{differentiate g then scale by f}}$

Cotangent space

- For each point $p \in M$ define

$$T_p^* M := \left\{ \alpha_p : T_p M \xrightarrow{\text{linear}} \mathbb{R} \right\}$$ (dual space of tangent space)

- Drop the "$p \in M$"

$$\Gamma(T^* M) := \left\{ \alpha : T_p M \xrightarrow{\text{linear}} C^\infty(M) \,\middle|\, \alpha(fX) = f\alpha(X) \right\}$$

Differential of function

- Differentiation and evaluate at $p \in M$

$$d_p : C^\infty(M) \xrightarrow{\text{linear}} T_p^* M \qquad (d_p f)(X_p) := X_p f$$

- Drop the “$p \in M$”

$$d : C^\infty(M) \xrightarrow{\text{linear}} \Gamma(T^* M) \qquad (d f)(X) := X f$$

Pullback and pushforward

- A map between point sets

$$\varphi : M \to N$$

should also come with a algebra homomorphism on the functions

$$\varphi^* : C^\infty(N) \xrightarrow{\text{hom}} C^\infty(M)$$

$$\begin{gathered}\varphi^*(c_1 g_1 + c_2 g_2) = c_1 \varphi^* g_1 + c_2 \varphi^* g_2 \\ \varphi^*(g_1 g_2) = (\varphi^* g_1)(\varphi^* g_2)\end{gathered}$$

- This is the formalization of function composition $\varphi^* g = g \circ \varphi$ (when we know what is a function evaluation at a point)

Pullback and pushforward

- Any $\varphi^*: C^\infty(N) \xrightarrow{\text{hom}} C^\infty(M)$ gives rise to a pushfoward for vector (fields)

$$\varphi_*: T_pM \xrightarrow{\text{linear}} T_{\varphi(p)}N \qquad (\varphi_* X_p)(g) := X_p(\varphi^* g)$$

$$\varphi_*: \Gamma(TM) \xrightarrow{\text{linear}} \Gamma(TN) \qquad (\varphi_* X)g := X(\varphi^* g)$$

- Overload pullback for covector (fields)

$$\varphi^*: T^*_{\varphi(p)}N \xrightarrow{\text{linear}} T^*_pM \qquad (\varphi^* \alpha_{\varphi(p)})(X_p) := \alpha_{\varphi(p)}(\varphi_* X_p)$$

$$\varphi^*: \Gamma(T^*N) \xrightarrow{\text{linear}} \Gamma(T^*M) \qquad (\varphi^* \alpha)(X) := \alpha(\varphi_* X)$$

- Easy to check: $d(\varphi^* g) = \varphi^*(dg)$

Velocity of a path

- A path is $\gamma : (a, b) \subset \mathbb{R} \to M$
- Its velocity is $\gamma' = \gamma_*(1)$

Lie algebra

- A Lie algebra $(V, [\cdot,\cdot])$ is a vector space V equipped with a bilinear operator

$$[\cdot,\cdot]\colon V \times V \xrightarrow{\text{bilinear}} V$$

  satisfying

  - Skew symmetry $[u,v] = -[v,u]$
  - Jacobi identity $[u,[v,w]] + [v,[w,u]] + [w,[u,v]] = 0$

- Example: $(\mathbb{R}^3, \times)$ cross product in 3D
- Example: $(\mathbb{R}^{n\times n}, [A,B] := AB - BA)$

Space of vector fields is a Lie algebra

- Define $$[\cdot,\cdot]\colon \Gamma(TM)\times\Gamma(TM)\xrightarrow{\text{bilinear}}\Gamma(TM)$$

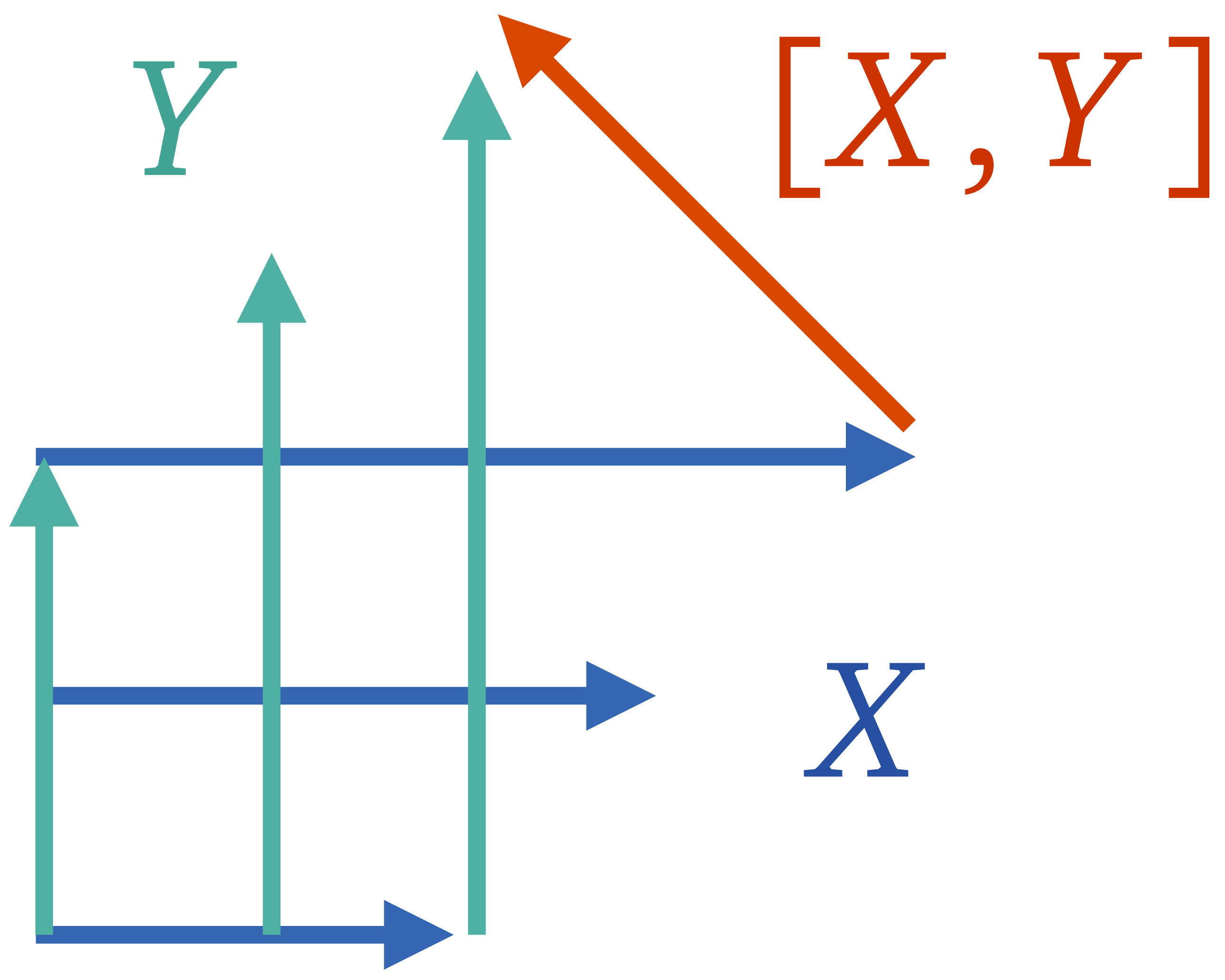
  by
  $$[X,Y]f := XYf - YXf$$

- Check that it is still a vector field (i.e. product rule holds)

- Check that it is a Lie algebra bracket (the same as checking the commutator for matrices is a Lie bracket)

Space of vector fields is a Lie algebra

- Lie bracket of vector fields measure commutability between following each vector field



- Coordinate vector fields must have vanishing Lie bracket

Computing Lie bracket in coordinates

- Suppose $x^1, \ldots, x^n \in C^\infty(M)$ is a coordinate system.
- Covector basis: $dx^1, \ldots, dx^n$
- Coordinate vector basis $e_1 = \frac{\partial}{\partial x^1}, \ldots, e_n = \frac{\partial}{\partial x^n}$
- Suppose we have two vector fields written in coordinates

$$X = X^i \frac{\partial}{\partial x^i} \qquad\qquad Y = Y^i \frac{\partial}{\partial x^i}$$

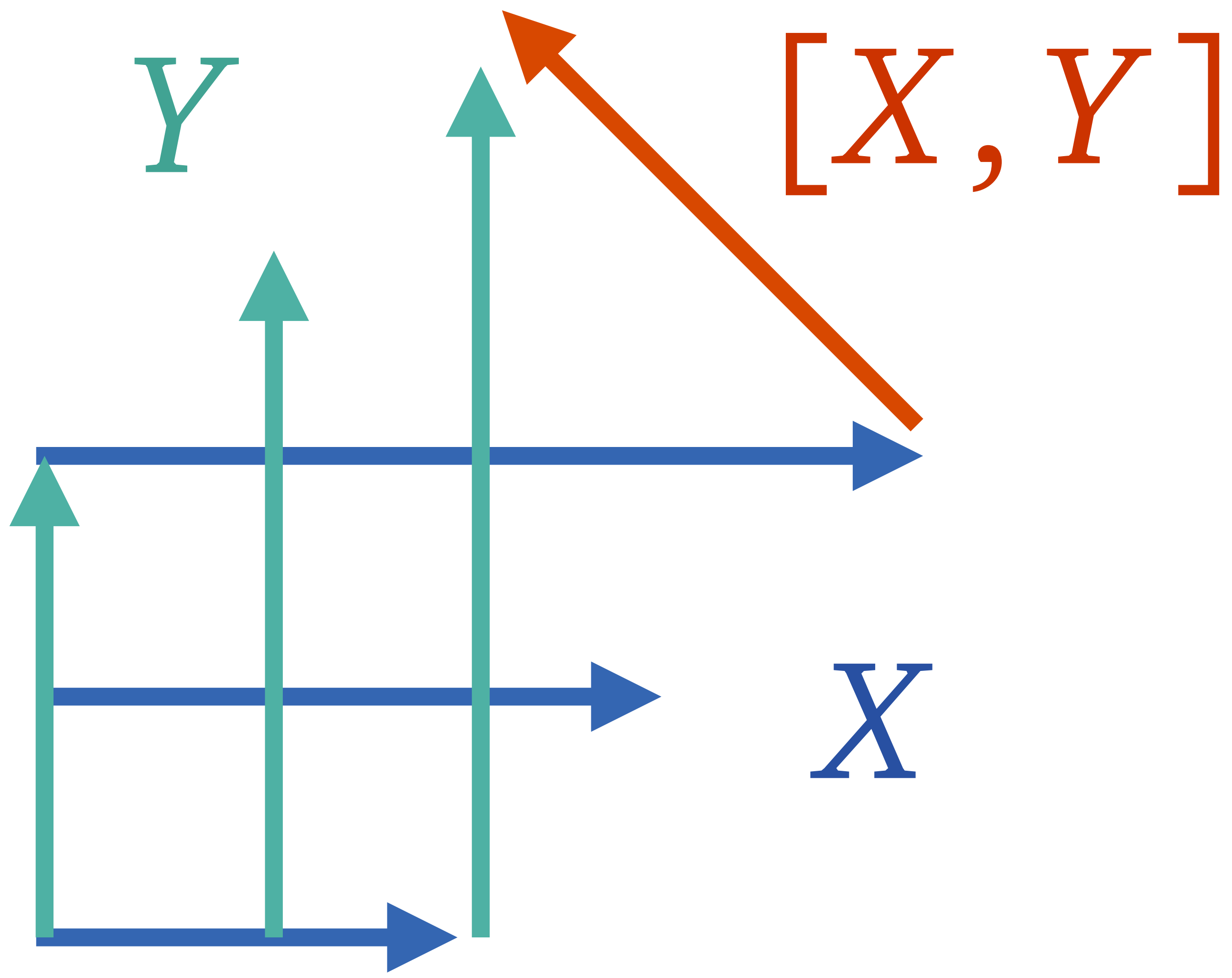
- Then $[X, Y] = Z^i \frac{\partial}{\partial x^i}$

$$Z^i = X^j \frac{\partial Y^i}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j} \qquad\qquad \text{“} [X, Y] = \nabla_X Y - \nabla_Y X \text{”}$$

- (You can derive it by applying to a function)

Computing Lie bracket in coordinates

- Example



$$X = \begin{bmatrix} y \\ 0 \\ 0 \end{bmatrix} \qquad Y = \begin{bmatrix} 0 \\ x \\ 0 \end{bmatrix}$$

$$\nabla_X Y - \nabla_Y X = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ x \\ 0 \end{bmatrix} = \begin{bmatrix} -x \\ y \\ 0 \end{bmatrix}$$

Constrained Mechanical Systems

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Constrained mechanical system

- M : Set of positions
- Dynamics is a path $\gamma : [0, T] \to M$
- A **holonomic constraint** is an equality constraint on the position

$$\{p \in M \mid g(p) = \text{const}\} \qquad g : M \to \mathbb{R}^m$$

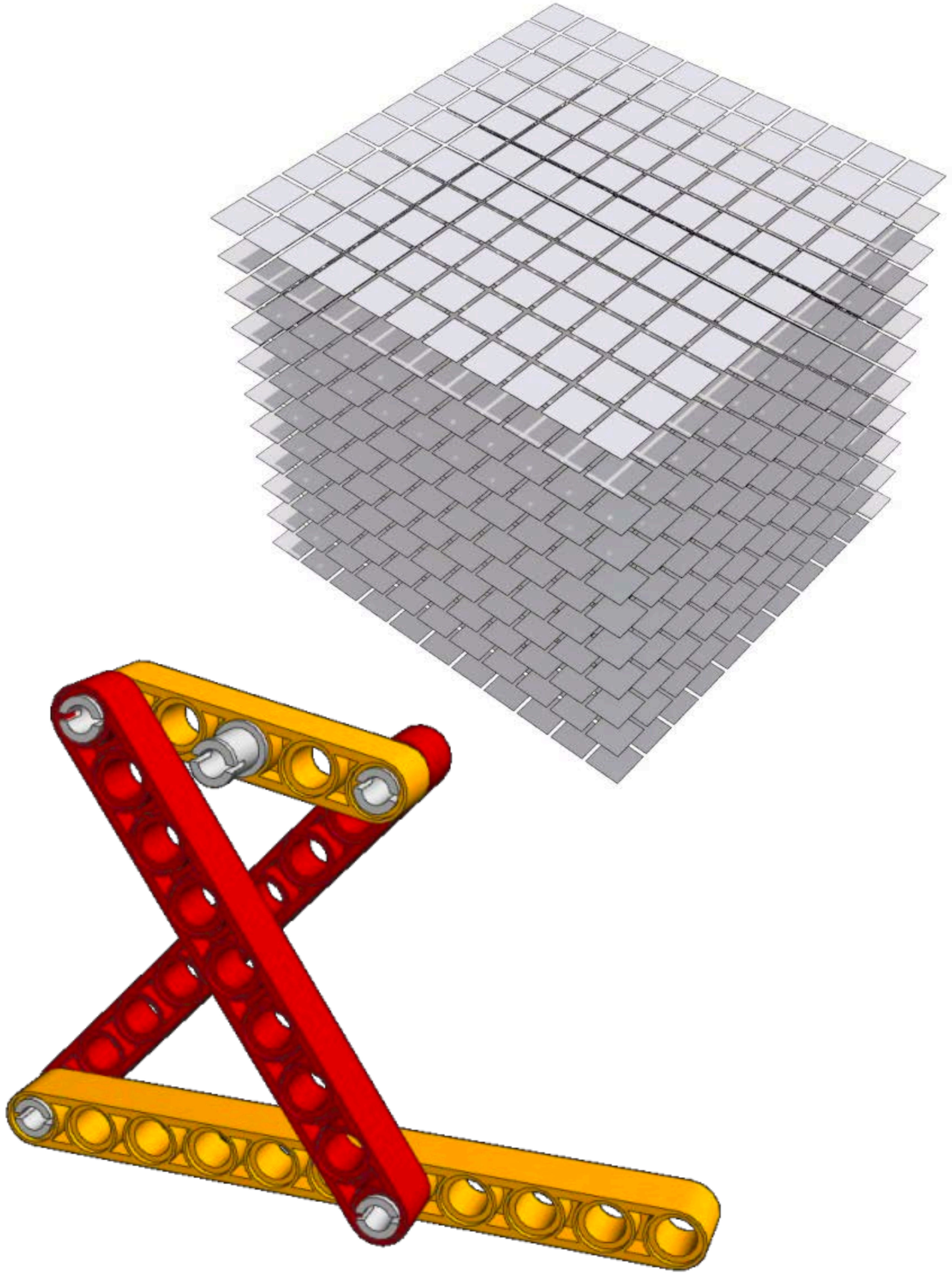
- Holonomic constraints foliate the space M into codimension-m submanifolds
- Holonomic constraints induce a constraint on velocity

$$\dot{\gamma} \in \ker(dg|_\gamma)$$

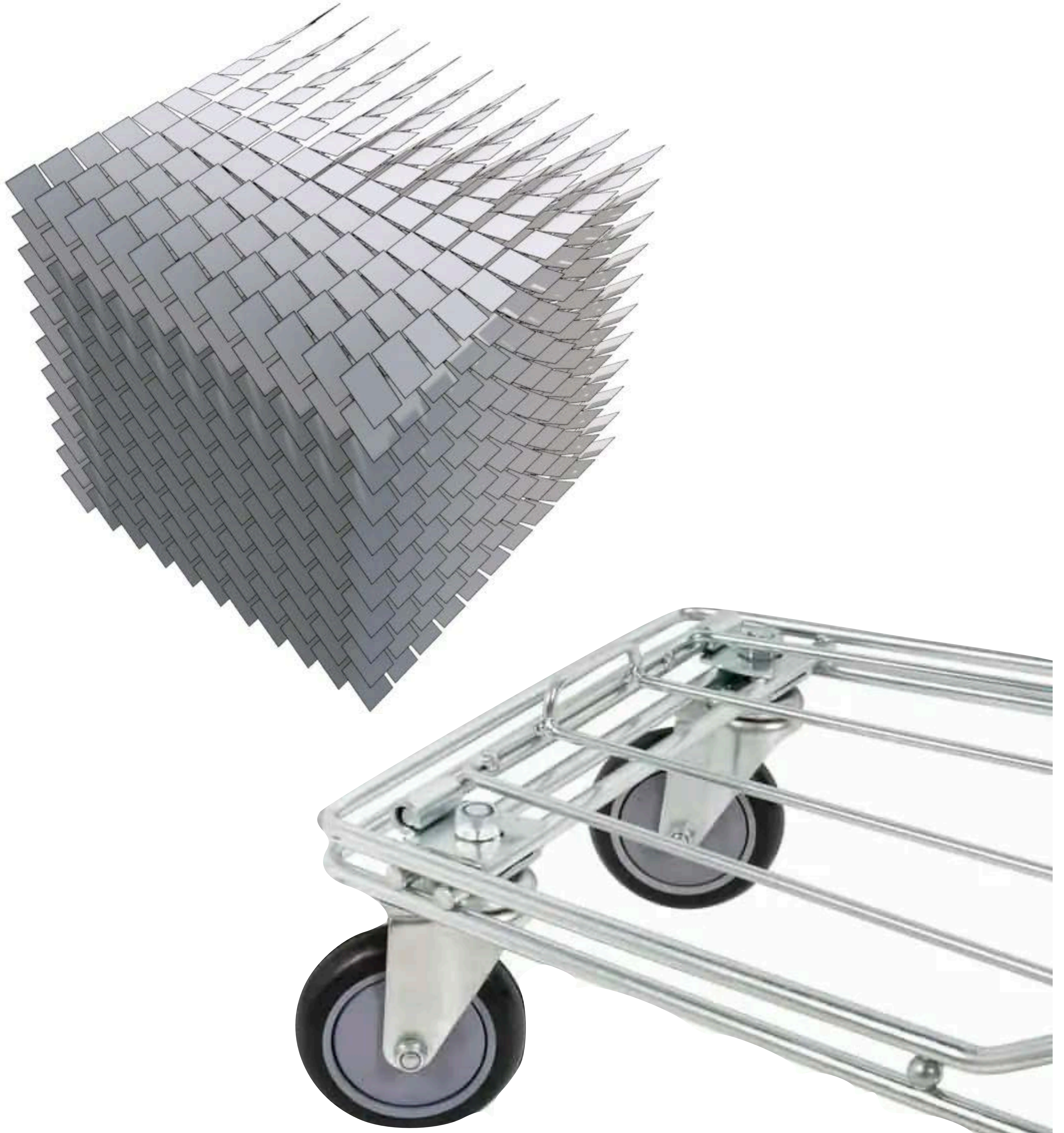
- Many constraints are just a constraint on velocity, not explicitly on positions.

Constrained mechanical system

Holonomic



Non-holonomic



Constrained mechanical system

- Non-holonomic constraints: constraint on velocity but still allow maneuver to all (or at least more dimension of) positions

- A constraint on velocity is an assignment of subspace for each position (as the set of admissible velocities)

$$A_p \subset T_p M$$

- Such a subspace field $(A_p)_{p \in M}$ is called a **distribution**.

- A distribution is **integrable** if it is tangent of a family of submanifolds of its dimension. Integrable distributions are holonomic constrains.

- Non-integrable distributions are non-holonomic constraints.

Constrained mechanical system

- **Theorem** (Frobenius integrability condition)
  A distribution A is (locally) integrable if and only if it is closed under Lie bracket. That is, $\Gamma(A) \subset \Gamma(TM)$ is a Lie subalgebra:

$$[X, Y] \in \Gamma(A) \quad \forall X, Y \in \Gamma(A)$$

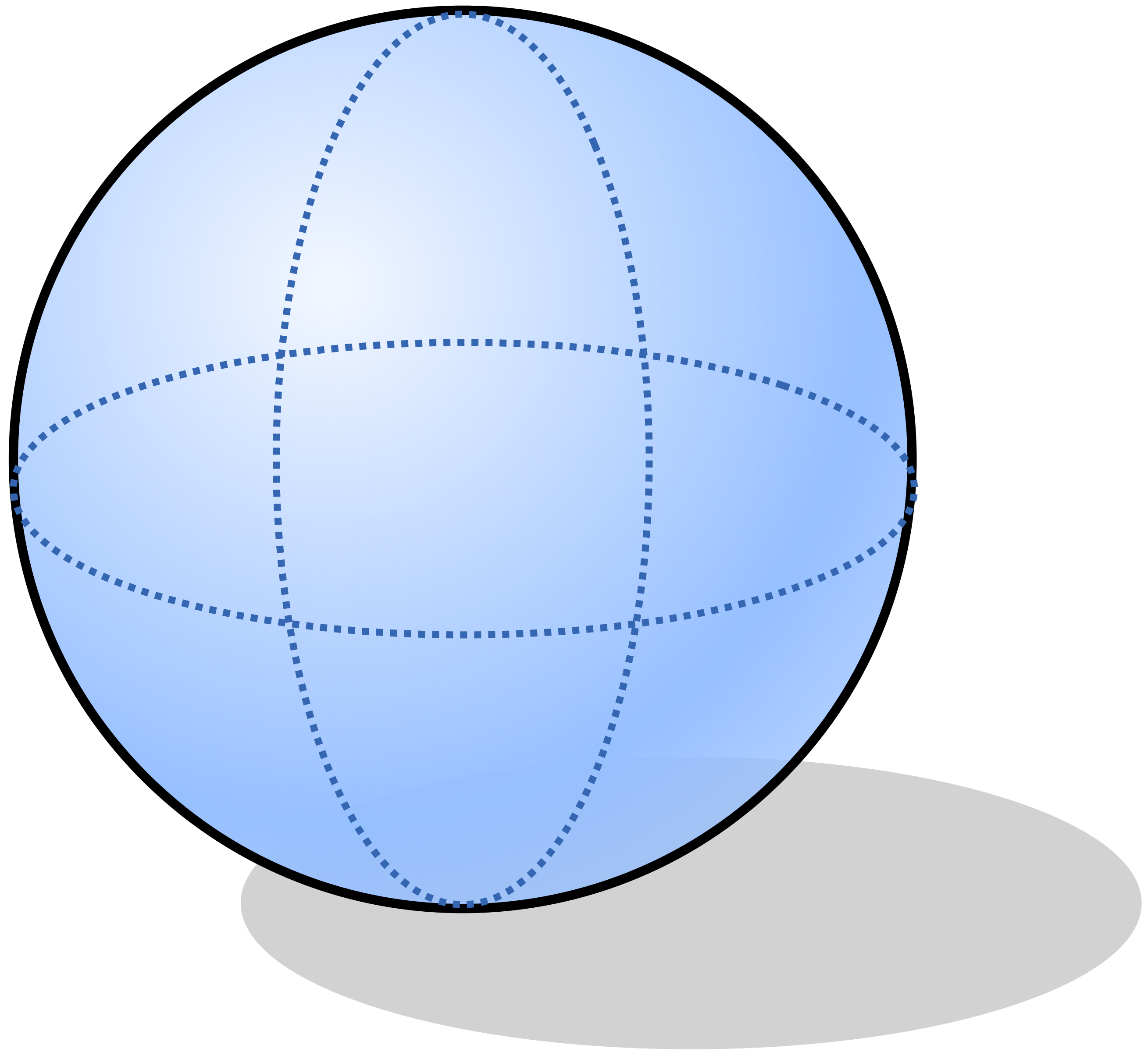
Example: rolling ball

- The state of rotations

$$M = \mathrm{SO}(3) = \{\mathbf{R} \in \mathbb{R}^{3\times 3} \,|\, \mathbf{R}^{\intercal}\mathbf{R} = \mathrm{id}, \det(\mathbf{R}) = 1\}$$

- The tangent space in terms of world-space angular velocity

$$T_{\mathbf{R}}M = \{[\omega\times]\mathbf{R} \,|\, \omega \in \mathbb{R}^3\}$$



- The allowed directions are 2D distribution

$$A_{\mathbf{R}} = \mathrm{span}\{[\mathbf{e}_1\times]\mathbf{R}, [\mathbf{e}_2\times]\mathbf{R}\}$$

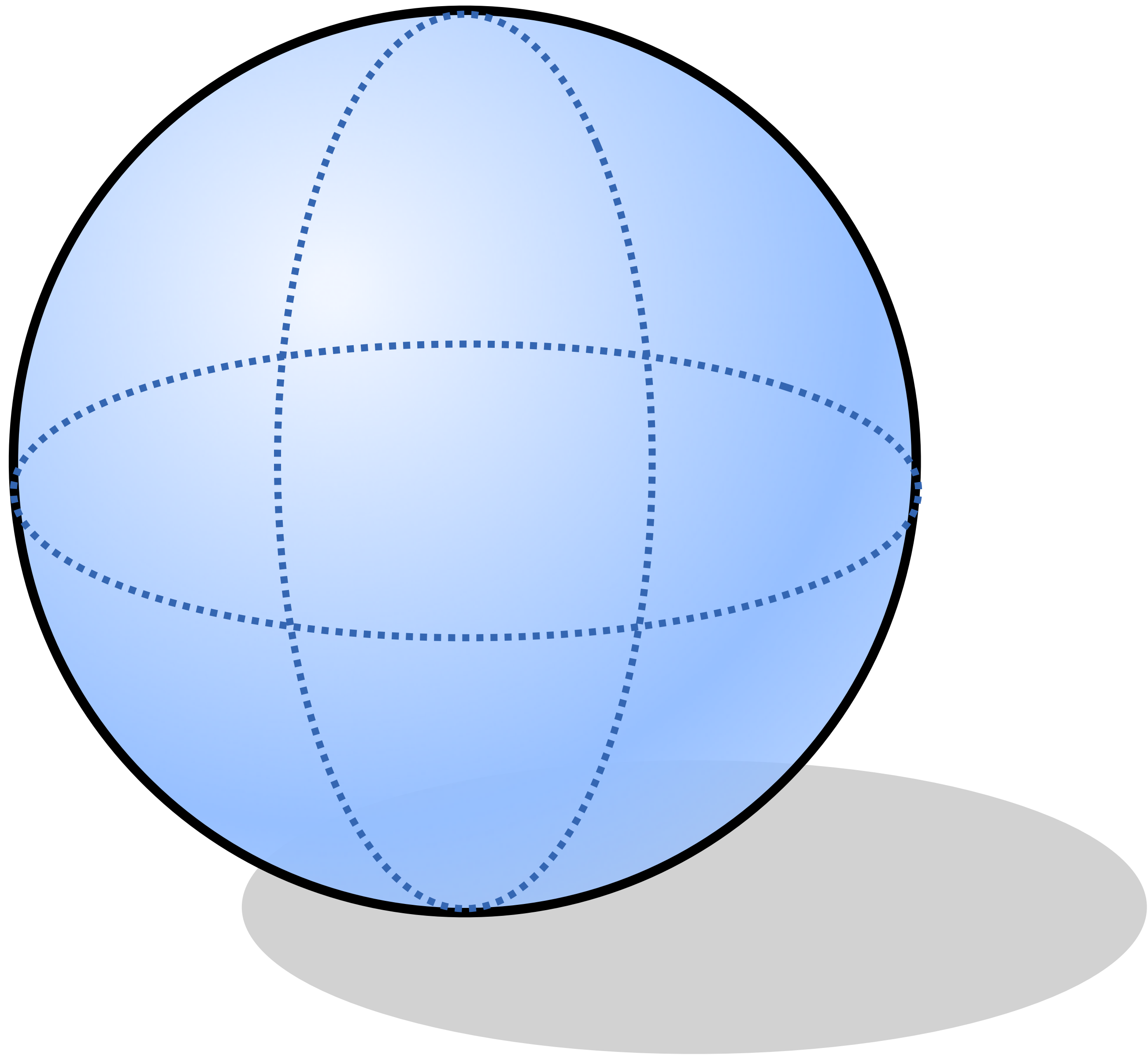
- Check that A is non-holonomic by choosing two vector fields in $\Gamma(A)$ but their Lie bracket is outside of $\Gamma(A)$.

Example: rolling ball

- Check that A is non-holonomic by choosing two vector fields in $\Gamma(A)$ but their Lie bracket is outside of $\Gamma(A)$.

$$X_{\mathbf{R}} = [\mathbf{e}_1 \times]\mathbf{R}$$

$$Y_{\mathbf{R}} = [\mathbf{e}_2 \times]\mathbf{R}$$



$$d_X Y = d_X([\mathbf{e}_2 \times]\mathbf{R}) = [\mathbf{e}_2 \times][\mathbf{e}_1 \times]\mathbf{R}$$

$$d_Y X = d_Y([\mathbf{e}_1 \times]\mathbf{R}) = [\mathbf{e}_1 \times][\mathbf{e}_2 \times]\mathbf{R}$$

$$[X, Y]_{\mathbf{R}} = d_X Y - d_Y X = -[(\mathbf{e}_1 \times \mathbf{e}_2)\times]\mathbf{R} = -[\mathbf{e}_3 \times]\mathbf{R}$$

Example: Driving a car

- The state of cars

$$M = \{(x, y, \theta, \kappa)\} = \mathbb{R}^2 \times \mathbb{S}^1 \times [-a, a]$$

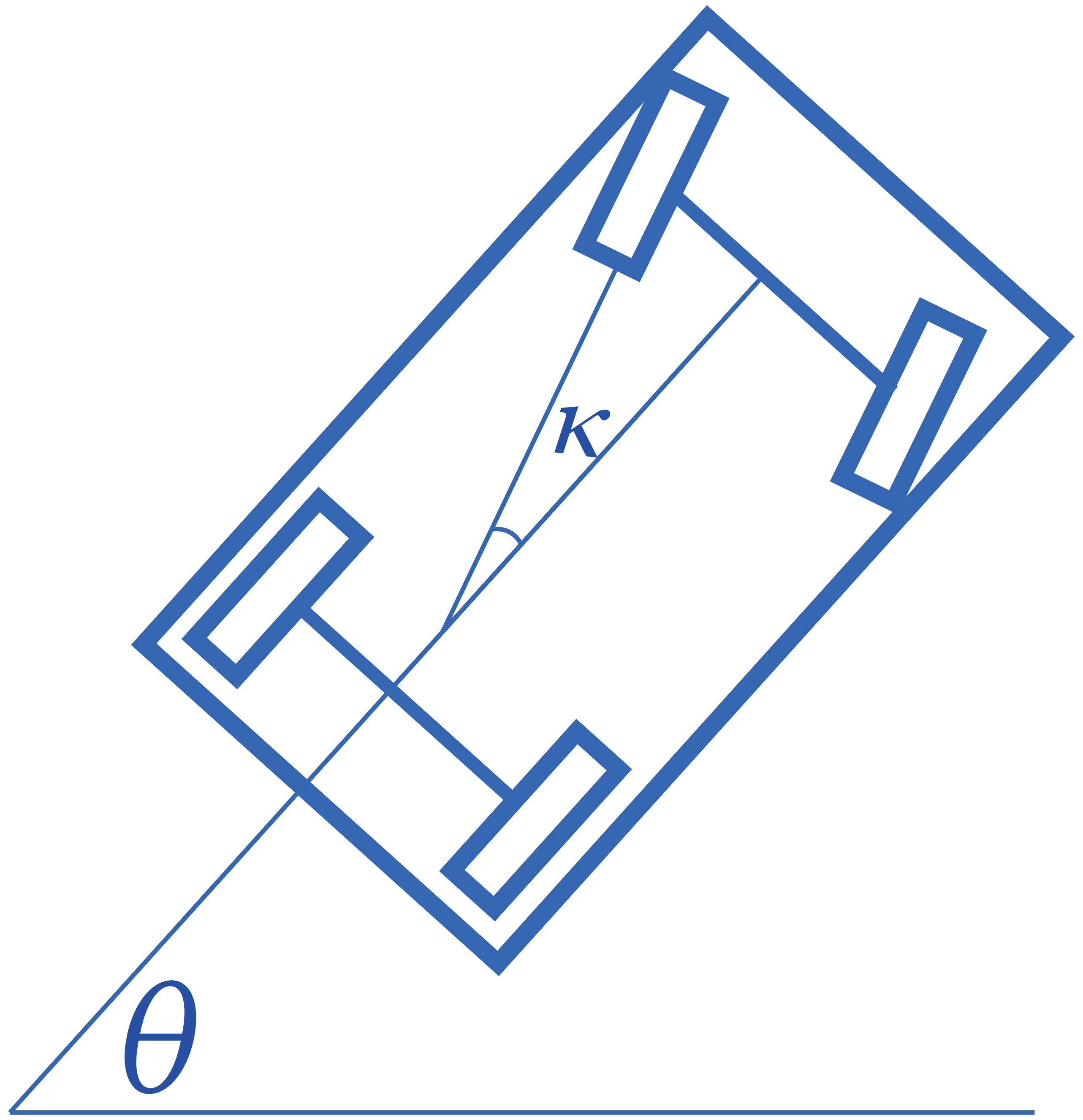
- The tangent space

$$T_{(x,y,\theta,\kappa)}M = \{(\dot{x}, \dot{y}, \dot{\theta}, \dot{\kappa})\} = \mathbb{R}^4$$

- The allowed directions are 2D distribution

$$A_{(x,y,\theta,\kappa)} = \operatorname{span}\left\{\begin{pmatrix}\cos\theta\\ \sin\theta\\ \kappa\\ 0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right\}$$

- Check that A is non-holonomic.



Geodesic Equation

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Riemannian manifold

- A Riemannian manifold is a manifold M equipped with an inner product structure.

$$\flat_p : T_pM \xrightarrow{\text{linear}} T_p^*M$$

$$\flat : \Gamma(TM) \xrightarrow{\text{linear}} \Gamma(T^*M) \qquad \flat(fX) = f\flat X$$

- In a mechanical system, the metric represents the inertia

$$\text{KineticEnergy}(\dot{\gamma}) = \frac{1}{2}\langle \flat\dot{\gamma} | \dot{\gamma} \rangle$$

$$\text{Momentum} = \flat\dot{\gamma}$$

Geodesics

- Consider variational problem: Find $\gamma : [0, T] \rightarrow M$ (with fixed end points) that minimizes

$$S(\gamma) = \int_0^T \frac{1}{2} \langle \flat\dot{\gamma} | \dot{\gamma} \rangle \, dt$$

- Minimizers are shortest paths connecting the end paths, and

$$|\dot{\gamma}|_\flat = \mathrm{const}$$

Geodesics in coordinates

- In a coordinate system, $x^1, \ldots, x^n \in C^\infty(M)$
  - Covector basis: $dx^1, \ldots, dx^n$
  - Coordinate vector basis $e_1 = \frac{\partial}{\partial x^1}, \ldots, e_n = \frac{\partial}{\partial x^n}$

$$g_{ij} = \langle \flat e_i | e_j \rangle$$

- The action written in coordinates:

$$S(\gamma) = \int_0^T \underbrace{\frac{1}{2} g_{ij}(\mathbf{x}) \dot{x}^i \dot{x}^j}_{L(\mathbf{x},\dot{\mathbf{x}})} \, dt$$

Geodesics in coordinates

$$S(\gamma) = \int_0^T \underbrace{\frac{1}{2} g_{ij}(\mathbf{x})\dot{x}^i \dot{x}^j}_{L(\mathbf{x},\dot{\mathbf{x}})} \, dt$$

- Euler–Lagrange equation $\displaystyle \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^k} - \frac{\partial L}{\partial x^k} = 0$

$$\tfrac{d}{dt}\left(g_{kj}\dot{x}^j\right) - \tfrac{1}{2} g_{ij,k}\dot{x}^i \dot{x}^j = 0$$

$$g_{kj}\ddot{x}^j + \underset{\frac{1}{2}(g_{k\ell,j} + g_{kj,\ell})\dot{x}^\ell \dot{x}^j}{\boxed{g_{kj,\ell}\dot{x}^\ell \dot{x}^j}} - \tfrac{1}{2} g_{ij,k}\dot{x}^i \dot{x}^j = 0$$

$$\ddot{x}^m + \tfrac{1}{2} g^{mk}\left(g_{k\ell,j} + g_{kj,\ell} - g_{j\ell,k}\right)\dot{x}^j \dot{x}^\ell = 0$$

Geodesics in coordinates

$$S(\gamma) = \int_0^T \underbrace{\frac{1}{2} g_{ij}(\mathbf{x}) \dot{x}^i \dot{x}^j}_{L(\mathbf{x},\dot{\mathbf{x}})} \, dt$$

- Euler–Lagrange equation

$$\ddot{x}^k + \Gamma^k_{ij} \dot{x}^i \dot{x}^j = 0$$ **(geodesic equation)**

Christoffel symbol: $\Gamma^k_{ij} = \frac{1}{2} g^{k\ell} (g_{\ell i,j} + g_{j\ell,i} - g_{ij,\ell})$

Geodesics in coordinates

$$\ddot{x}^k + \Gamma^k_{ij}\dot{x}^i\dot{x}^j = 0$$ **(geodesic equation)**

Christoffel symbol: $\Gamma^k_{ij} = \frac{1}{2}g^{k\ell}(g_{\ell i,j} + g_{j\ell,i} - g_{ij,\ell})$

The geodesic equation is describing “the acceleration is zero”

where the acceleration is the derivative of velocity using the **covariant derivative**

$$(\nabla_{\dot{x}}v)^k = \dot{v}^k + \Gamma^k_{ij}\dot{x}^i v^j$$

Covariant derivative

- The covariant derivative operator (a.k.a. Levi-Civita connection)

$$\nabla_{(\#1)}(\#2) \colon T_pM \times \Gamma(TM) \to T_pM$$

  - Linear (pointwise) and in the direction

$$\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$$

  - Leibniz rule in the vector field being taken derivative

$$\nabla_X(fY) = (Xf)Y + f\nabla_X Y$$

  - Torsion free (compatible with Lie bracket)

$$[X,Y] = \nabla_X Y - \nabla_Y X$$

  - Compatible with metric:

$$X\langle Y,Z\rangle_\flat = \langle \nabla_X Y, Z\rangle_\flat + \langle Y, \nabla_X Z\rangle_\flat$$

Covariant derivative

- Coordinate-free:

$$S(\gamma) = \int_0^T \frac{1}{2} \langle \flat\dot{\gamma} | \dot{\gamma} \rangle \, dt$$

$$dS|_\gamma[\![\mathring{\gamma}]\!] \overset{\text{comp. w/ metric}}{=} \int_0^T \langle \flat\dot{\gamma} | \overset{\triangledown}{\overset{\circ}{\dot{\gamma}}} \rangle \, dt \overset{\text{torsion-free}}{=} \int_0^T \langle \flat\dot{\gamma} | \overset{\triangledown}{\dot{\mathring{\gamma}}} \rangle \, dt \overset{\text{int.by parts; comp. w/ metric}}{=} -\int_0^T \langle \flat\overset{\triangledown}{\ddot{\gamma}} | \mathring{\gamma} \rangle \, dt$$

- Euler–Lagrange equation

$$\overset{\triangledown}{\ddot{\gamma}} = \frac{\nabla}{dt}\dot{\gamma} = 0 \qquad \ddot{x}^k + \Gamma^k_{ij}\dot{x}^i\dot{x}^j = 0$$

Newton’s Laws of Motion

- The set of static states form a manifold M
- Inertia is a Riemannian structure $\flat$ on M
- **Newton’s 1st law**

  In the absence of external force, the state follows a geodesic path

$$\frac{\nabla}{dt}\dot{\gamma} = 0$$

- **Newton’s 2nd law** (with conservative force)

  Suppose there is a potential $U \in C^{\infty}(M)$ then

$$\flat \frac{\nabla}{dt}\dot{\gamma} = -dU$$

Back to Constraints

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With constraint (non-holonomic form)

- Admissible velocities form a distribution $A_\gamma \subset T_\gamma M$
- Suppose it is described by $A_\gamma = \ker L_\gamma$

  for some linear operator $L_\gamma : T_\gamma M \xrightarrow{\text{linear}} \mathbb{R}^m$
- Then the force law becomes (for some $\lambda \in \mathbb{R}^{m*}$)

$$\flat \frac{\nabla}{dt}\dot{\gamma} = -dU - L_\gamma^* \lambda \qquad\qquad L_\gamma \dot{\gamma} = 0$$

  or as a system:

$$\begin{bmatrix} \flat \frac{\nabla}{dt} & L_\gamma^* \\ L_\gamma & 0 \end{bmatrix} \begin{bmatrix} \dot{\gamma} \\ \lambda \end{bmatrix} = \begin{bmatrix} -dU \\ 0 \end{bmatrix}$$

Kane's method of quasi-velocities

- Admissible velocities form a distribution $A_\gamma \subset T_\gamma M$
- We can also find some basis $A_\gamma = \operatorname{span}\{X_1, \ldots, X_k\}$

  (where vec fields $X_1, \ldots, X_k$ may not be integrable into a coordinate system)

- Express $\dot{\gamma} = \sum_{i=1}^{k} v^i X_i$

  $$\text{KineticEnergy}_\gamma(\mathbf{v}) = \tfrac{1}{2}\mathbf{v}^\intercal \mathbf{M}_\gamma \mathbf{v} \qquad \mathbf{M}_{ij} = \langle X_i, X_j \rangle_\flat$$

- Since $\mathbf{v}$ is not the rate of change of any coordinate we call it quasi-velocity.

Kane's method of quasi-velocities

- Express $\dot{\gamma} = \sum_{i=1}^{k} v^i X_i$

  $\mathrm{KineticEnergy}_\gamma(\mathbf{v}) = \frac{1}{2}\mathbf{v}^\intercal \mathbf{M}_\gamma \mathbf{v} \qquad \mathbf{M}_{ij} = \langle X_i, X_j \rangle_\flat$

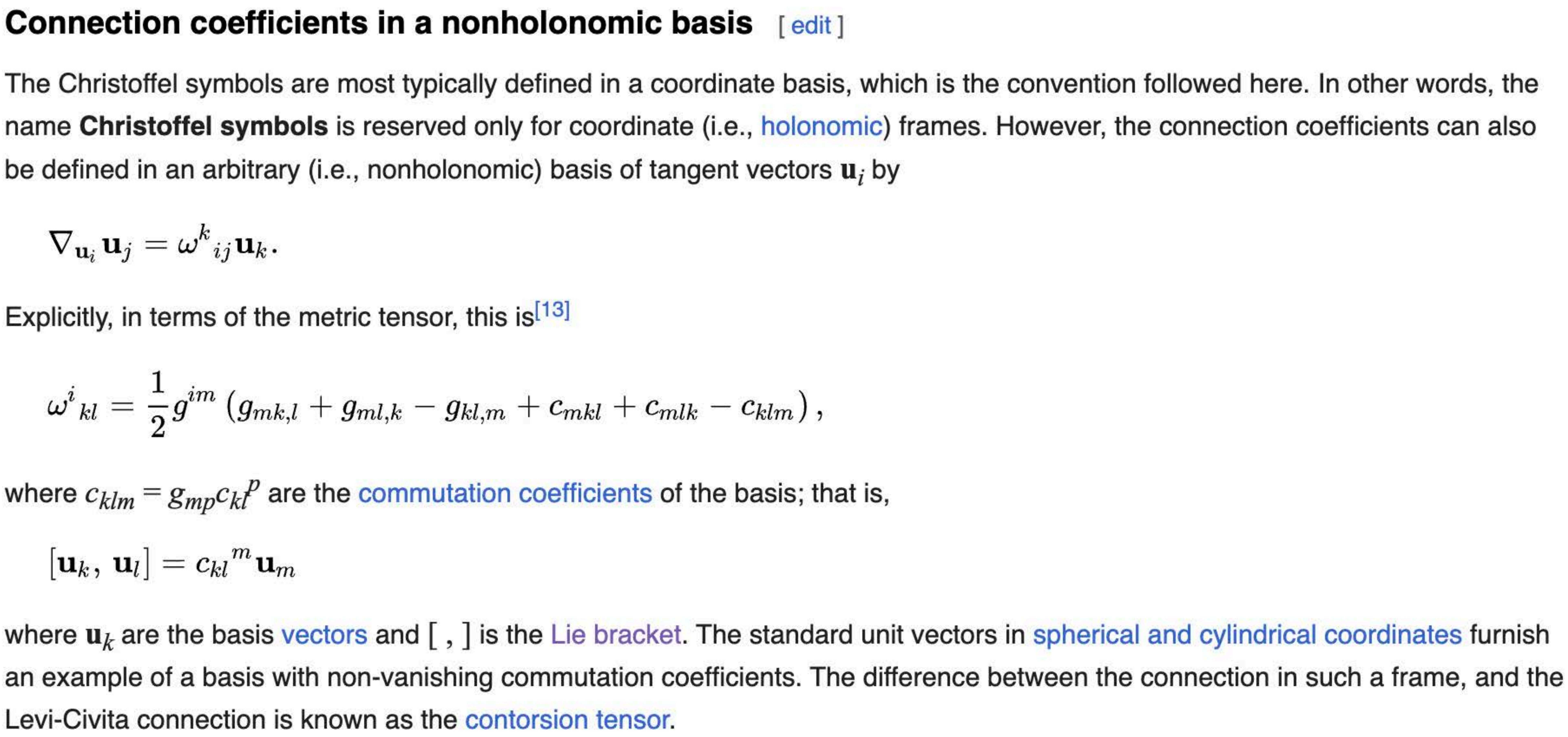
- Use least action principle to derive equation of motion

$$\mathbf{M}\frac{\nabla \mathbf{v}}{dt} = -\begin{bmatrix} X_1 U \\ \vdots \\ X_k U \end{bmatrix}$$

- Here the Christoffel symbol in ∇ is more complicated:

Kane’s method of quasi-velocities

- Here the Christoffel symbol in ∇ is more complicated:



Connection coefficients in a nonholonomic basis [edit]

The Christoffel symbols are most typically defined in a coordinate basis, which is the convention followed here. In other words, the name **Christoffel symbols** is reserved only for coordinate (i.e., holonomic) frames. However, the connection coefficients can also be defined in an arbitrary (i.e., nonholonomic) basis of tangent vectors $\mathbf{u}_i$ by

$$\nabla_{\mathbf{u}_i}\mathbf{u}_j = \omega^k{}_{ij}\mathbf{u}_k.$$

Explicitly, in terms of the metric tensor, this is[13]

$$\omega^i{}_{kl} = \frac{1}{2}g^{im}\left(g_{mk,l} + g_{ml,k} - g_{kl,m} + c_{mkl} + c_{mlk} - c_{klm}\right),$$

where $c_{klm} = g_{mp}c_{kl}{}^p$ are the commutation coefficients of the basis; that is,

$$[\mathbf{u}_k,\, \mathbf{u}_l] = c_{kl}{}^m\mathbf{u}_m$$

where $\mathbf{u}_k$ are the basis vectors and [,] is the Lie bracket. The standard unit vectors in spherical and cylindrical coordinates furnish an example of a basis with non-vanishing commutation coefficients. The difference between the connection in such a frame, and the Levi-Civita connection is known as the contorsion tensor.

Kane's method of quasi-velocities

- Example: rigid body rotation

$$\dot{\mathbf{R}} = [\omega\times]\mathbf{R} = \mathbf{R}[\Omega\times]$$

We used 3D vector ω (or Ω) to parametrize the 3D distribution

$$A_{\mathbf{R}} = \{[\omega\times]\mathbf{R} \,|\, \omega \in \mathbb{R}^3\}$$

within the 9D space of 3x3 matrices.