Crash course on differential geometry.

Differential geometry : Calculus without coordinate.
(i.e. only Types & RULES)

- Manifold M represents a domain
- $C^\infty(M) = \{ f : M \to \mathbb{R} \}$ smooth funtions
  - operations: $\begin{cases} \text{linear combination} & c_1 f_1 + c_2 f_2 \quad c_1, c_2 \in \mathbb{R} \\ \text{pointwise multiplication} & f_1 f_2 \end{cases}$
- For each $p \in M$, define tangent space as a collection of directional derivative operators:

$$T_pM := \left\{ X_p : C^\infty(M) \xrightarrow{\text{lin}} \mathbb{R} \;\middle|\; X_p(fg) = (X_p f) g + f (X_p g) \quad \forall f, g \in C^\infty(M) \right\}$$

- Define space of vector field similarly

$$\Gamma(TM) := \left\{ X : C^\infty(M) \xrightarrow{\text{lin}} C^\infty(M) \;\middle|\; X(fg) = (Xf) g + f(Xg) \quad \forall f, g \in C^\infty(M) \right\}$$

- Define $T_p^*M =$ dual space of T_pM

  $\Gamma(T^*M) =$ dual space of $\Gamma(TM)$.

- Define $d|_p : C^\infty(M) \xrightarrow{\text{lin}} T_p^*M$ as $\langle d_p f \mid X_p \rangle = (Xf)_p$

  $d : C^\infty(M) \to \Gamma(T^*M)$ as $\langle df \mid X \rangle = Xf$.

A vector space V is called a Lie algebra if it is equipped with a bilinear operator

$$[\cdot, \cdot] : V \times V \xrightarrow{\text{bilinear}} V$$

s.t. (1) $[\vec{u}, \vec{v}] = -[\vec{v}, \vec{u}]$, skew symmetry

(2) $[\vec{u}, [\vec{v}, \vec{w}]] + [\vec{v}, [\vec{w}, \vec{u}]] + [\vec{w}, [\vec{u}, \vec{v}]] = \vec{0}$ Jacobi identity

Examples

- $\mathbb{R}^3$ cross product $\begin{cases} \vec{u}\times\vec{v} = -\vec{v}\times\vec{u} \\ \vec{u}\times(\vec{v}\times\vec{w}) + \vec{v}\times(\vec{w}\times\vec{u}) + \vec{w}\times(\vec{u}\times\vec{v}) = 0 \end{cases}$

- Commutators on matrices

$$[A,B] = AB - BA$$

$$[A,[B,C]] = A(BC-CB) - (BC-CB)A$$
$$= ABC - ACB - BCA + CBA$$

(Check Jacobi identity)

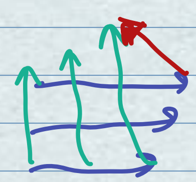
- $\Gamma(TM)$ is a Lie algebra using commutator

$$X, Y \in \Gamma(TM) \rightsquigarrow [X,Y] := XY - YX$$

Check $[X,Y]$ is indeed in $\Gamma(TM)$:

$$[X,Y](fg) \overset{?}{=} ([X,Y]f)g + f([X,Y]g)$$
$$\|$$
$$(XY-YX)(fg)$$
$$= X((Yf)g) + X(f(Yg)) - Y((Xf)g) - Y(f(Xg))$$
$$= (XYf)g + (Yf)(Xg) + (Xf)(Yg) + f(XYg)$$
$$- (YXf)g - (Xf)(Yg) - (Yf)(Xg) - f(YXg)$$
$$= ([X,Y]f)g + f([X,Y]g). \checkmark$$

geometrically $[X,Y]$ is a vector field describing the net drift of $(\text{flow by } Y)\circ(\text{flow by } X) - (\text{flow by } X)\circ(\text{flow by } Y)$



Remark: For any coord vector field $\vec{e}_1, \cdots, \vec{e}_n$

dual to $dx_1, \cdots, dx_n$ that are differentials of a coord system $x_1, \cdots, x_n \in C^\infty(M)$,

$$[\vec{e}_i, \vec{e}_j] = 0.$$

Conversely, if $[X, Y] = 0$ then they "weave" into a coord for a submfd.

If $\gamma : (-\varepsilon, \varepsilon) \to M$, then $\gamma'(0)$ is a tangent vector $\in T_{\gamma(0)} M$ in the sense that

$$[\gamma'(0)] f := \frac{d}{dt} f(\gamma(t)) \Big|_{t=0}.$$

Check $[\gamma'(0)](fg) = ([\gamma'(0)]f) g + f([\gamma'(0)]g)$.

Riemannian mfd is a mfd M where each tangent space T_pM is equipped with a symmetric positive definite bilinear form

$$\langle \cdot , \cdot \rangle : T_pM \times T_pM \to \mathbb{R}$$

equivalently $\flat_p : T_pM \to T_p^*M$.

If $x^1, \cdots, x^n \in C^\infty(M)$ is a coord system

$dx^1, \ldots, dx^n$ covector basis

$\vec{e}_1, \cdots, \vec{e}_n$ coord vectors.

Then $\flat$ is represented by a matrix $[g_{ij}]$

$$g_{ij} = \langle \vec{e}_i , \vec{e}_j \rangle .$$

- Consider variational problem:

Find $\gamma : [0,T] \to M$, $\gamma(0) = p_0$ $\gamma(T) = p_T$ fixed,

that minimizes $S(\gamma) = \int_0^T \frac{1}{2} |\gamma'(t)|_\flat^2 \, dt$

Minimizers are paths of shortest length connecting p_0, p_T

and $|\gamma'| = \text{const}$ (constant speed).

We call these minimizers <u>geodesic paths</u>.

<u>Using coord:</u>

$$S(\gamma) = \int_0^T \underbrace{\frac{1}{2} \dot{x}^k(t) \, \dot{x}^l(t) \, g_{kl}(\vec{x}(t))}_{L(x^1, \cdots, x^n, \dot{x}^1, \cdots, \dot{x}^n)} \, dt$$

$$\left(\sum_j \frac{\partial L}{\partial \dot{x}^j} \dot{x}^j - L = \text{const} \Rightarrow |\gamma'| \text{ const} \right)$$

E-L eq

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} = 0$$

$$\frac{d}{dt}\Big(\underbrace{\delta^k_i \dot{x}^l g_{kl}}_{\dot{x}^l g_{il}}\Big) - \frac{1}{2}\dot{x}^k \dot{x}^l g_{kl,i} = 0$$

polarize $\frac{1}{2}(g_{ik,l} + g_{il,k})\dot{x}^k\dot{x}^l$

$$g_{il}\ddot{x}^l + g_{il,k}\dot{x}^l\dot{x}^k \qquad - \frac{1}{2}\dot{x}^k\dot{x}^l g_{kl,i} = 0$$

apply g^{-1}

$$\ddot{x}^k + \Gamma^k_{ij}\dot{x}^i\dot{x}^j = 0$$

g^{-1}

$$\Gamma^k_{ij} = \frac{1}{2}g^{kl}\left(g_{ki,j} + g_{jk,i} - g_{ij,k}\right)$$ Christoffel symbol

Covariant derivative in coordinate:

Let $\vec{v} \in \Gamma(TM)$, $\vec{v} = v^i \vec{e}_i$ coord vector

and Let $\gamma : (-\varepsilon, \varepsilon) \to M$ $\quad \gamma(t) = (x^1(t), \cdots, x^n(t))$

then the rate of change of $\vec{v}$ along $\dot{\gamma}$ is the vector

$$\left(\frac{\nabla}{dt} v\right)^k \vec{e}_k := \left(\dot{v}^k + \Gamma^k_{ij}\dot{x}^i v^j\right)\vec{e}_k$$

$$\parallel$$

$$(\nabla_{\dot{\gamma}} v)^k \vec{e}_k = \nabla_{\dot{\gamma}} v$$

Back to coord free.

The covariant derivative, or the Levi-Civita connection, is

$$\nabla_{(\#1)}(\#2) : \Gamma(T_pM) \times \Gamma(TM) \to \Gamma(T_pM)$$

with properties:

① $\nabla_{c_1X_1+c_2X_2} Y = c_1\nabla_{X_1}Y + c_2\nabla_{X_2}Y$

② $\nabla_X(c_1Y_1+c_2Y_2) = c_1\nabla_X Y_1 + c_2\nabla_X Y_2$ $\quad c_1, c_2 \in \mathbb{R}$

③ $\nabla_X(fY) = (Xf)Y + f\,\nabla_X Y$ $\quad f\in C^\infty(M)$

④ Torsion-free: $\nabla_X Y - \nabla_Y X = [X,Y]$ when both $X, Y \in \Gamma(TM)$.

⑤ Metric: $Z\langle X,Y\rangle = \langle\nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle$

or $d\langle X,Y\rangle = \langle\nabla X, Y\rangle + \langle X, \nabla Y\rangle$.

$$S = \frac{1}{2}\int_0^T |\dot\gamma|_b^2\, dt$$

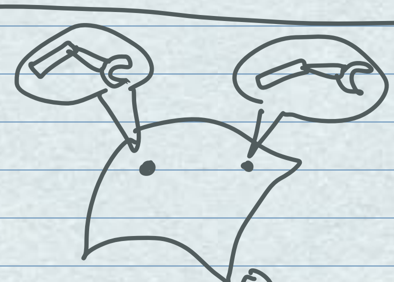
$$dS|_\gamma[\mathring\gamma] = \int_0^T \langle\dot\gamma, \overset{\nabla}{\mathring{\dot\gamma}}\rangle\, dt \overset{\text{torsion free}}{=\!=} \int\langle\dot\gamma, \overset{\nabla}{\dot{\mathring\gamma}}\rangle\, dt$$

$$\overset{\text{metric}}{=} \underbrace{\int_0^T \frac{d}{dt}\langle\dot\gamma, \mathring\gamma\rangle\, dt}_{\langle\dot\gamma,\mathring\gamma\rangle|_0^T} - \int_0^T \langle\overset{\nabla}{\ddot\gamma}, \mathring\gamma\rangle\, dt$$

$\Rightarrow$ EL eq: $b\overset{\nabla}{\ddot\gamma} = 0$ or $\overset{\nabla}{\ddot\gamma} = 0$.

(geodesic equation)

Newton's 1st Law.



Q

Q = Space of static states

Define $\langle\,,\rangle$ Riemannian structure by $\frac{1}{2}\langle\dot q, \dot q\rangle$ = Kinetic energy

Then, in the absence of external forces, $q(t)$ follows a geodesic path.

Newton's 2nd Law (conservative force)

In addition to the setup of the 1st Law, suppose there is a potential $U \in C^\infty(Q)$ then

$$\flat \overset{\nabla}{\dot{q}} = -dU.$$

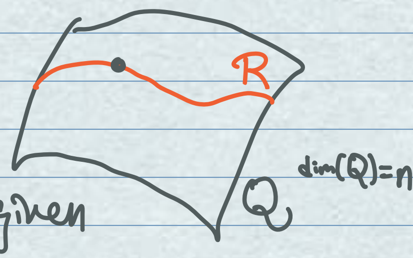
induced ∇ for covector field

Equivalently $\frac{\nabla}{dt}(\flat \dot{q}) = -dU$

$d\langle \alpha | X\rangle = \langle \nabla\alpha | X\rangle + \langle \alpha | \nabla X\rangle$

<u>Constraints</u>

A <u>holonomic constraint</u> is a constraint on the positions q.



If the submfd $R \subset Q$ is given by

$R = \{g = 0\}$ for some $g : Q \to \mathbb{R}^m$ $m < n$

$SO(3) = \{R \mid R^T R = id\}$

then

$$\begin{cases} \flat \overset{\nabla}{\dot{q}} = -dU - dg_q^* \lambda \\ g(q) = 0 \quad \text{or just} \quad dg|_q[\dot{q}] = 0 \end{cases}$$

(The constraint g induces a contraint on $\dot{q}$ as $dg|_q \dot{q} = 0$)

$$\begin{bmatrix} \flat \frac{\nabla}{dt} & dg^* \\ dg & 0 \end{bmatrix} \begin{bmatrix} \dot{q} \\ \lambda \end{bmatrix} = \begin{bmatrix} -dU \\ 0 \end{bmatrix}$$ (non-holonomic form)

A non-holonomic constraint is a constraint only on the velocity $\dot{q}$.

It is described by assigning a subspace $A_q \subset T_q Q$ for each q as the set of admissible velocities.

$A = (A_q \subset T_q Q)_{q \in Q}$ is called a distribution (differential geometry) which is subspace of $\Gamma(TQ)$.

If A cannot be written as $\ker dg$ for some holonomic constraint g, then A is (truly) nonholonomic.

In general $A = \ker(C)$ for some linear map $C_q : T_q Q \xrightarrow{\text{lin}} \mathbb{R}^m$

$C_q \in \mathrm{Hom}(T_q Q; \mathbb{R}^m)$

$$\begin{bmatrix} b\frac{\nabla}{dt} & C^* \\ C & 0 \end{bmatrix} \begin{bmatrix} \dot{q} \\ \lambda \end{bmatrix} = \begin{bmatrix} -dU \\ 0 \end{bmatrix}.$$

Example

Ball rolling on a table

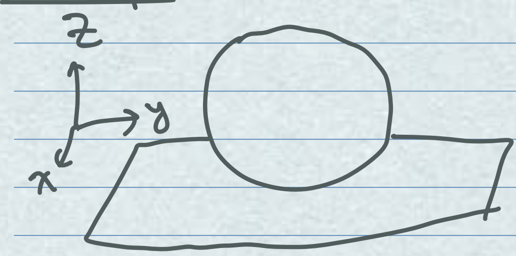
Thm (Frobenius integrability)

A distribution A is holonomic (i.e. A is tangent to a family of non-intersecting submfds) if and only if

$[X, Y] \in A$ for all $X, Y \in \Gamma(A)$

i.e. $A \subset \Gamma(TM)$ is not only a linear subspace but also a Lie subalgebra

Example



$$T_R SO(3) = \{ [\omega\times] R \mid \omega \in \mathbb{R}^3 \}$$

$$A_R = \left\{ [\omega\times] R \;\middle|\; \omega = \begin{bmatrix} \omega_1 \\ \omega_2 \\ 0 \end{bmatrix} \right\}$$

Consider vector fields $X \in \Gamma(A)$ $\quad X_R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} R \quad (\omega_X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix})$

$Y \in \Gamma(A)$ $\quad Y_R = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} R \quad (\omega_Y = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix})$

(These are called right-invariant vector fields)

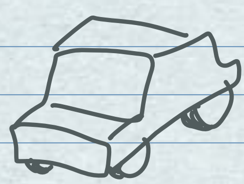
$[X, Y]_R$

$$= -\left(\begin{bmatrix} & & \\ & & -1 \\ & 1 & \end{bmatrix}\begin{bmatrix} & & 1 \\ & & \\ -1 & & \end{bmatrix} - \begin{bmatrix} & & 1 \\ & & \\ -1 & & \end{bmatrix}\begin{bmatrix} & & \\ & & -1 \\ & 1 & \end{bmatrix} \right) = -\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) = -\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

it would be $+$ if it is a left invariant VF.

leaving the distribution A.

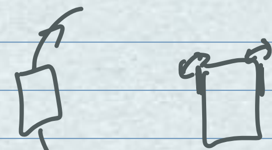
Example



center of mass, orientation, front wheel config

$$Q = \mathbb{R}^2 \times S^1 \times [-a, a]$$

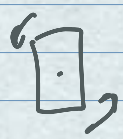
$$(\begin{pmatrix} x \\ y \end{pmatrix}, \theta, \kappa)$$



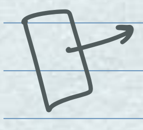
We can only move

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\kappa} \end{pmatrix} = \operatorname{span}\left\{ \underbrace{\begin{pmatrix} \cos\theta \\ \sin\theta \\ \kappa \\ 0 \end{pmatrix}}_{F}, \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{T} \right\}$$

$$[F, T] \notin \operatorname{span}\{F, T\}$$



$$S = \begin{pmatrix} 0 \\ 0 \\ \omega \\ 0 \end{pmatrix} \in \operatorname{span}\{F, T, [F, T]\}$$



$$P = \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \\ 0 \end{pmatrix} \in \operatorname{span}\{F, T, S, [S, F]\}.$$

$$\operatorname{span}\{F, T, S, P\} = T_q Q \Rightarrow \text{Car can go everywhere.}$$