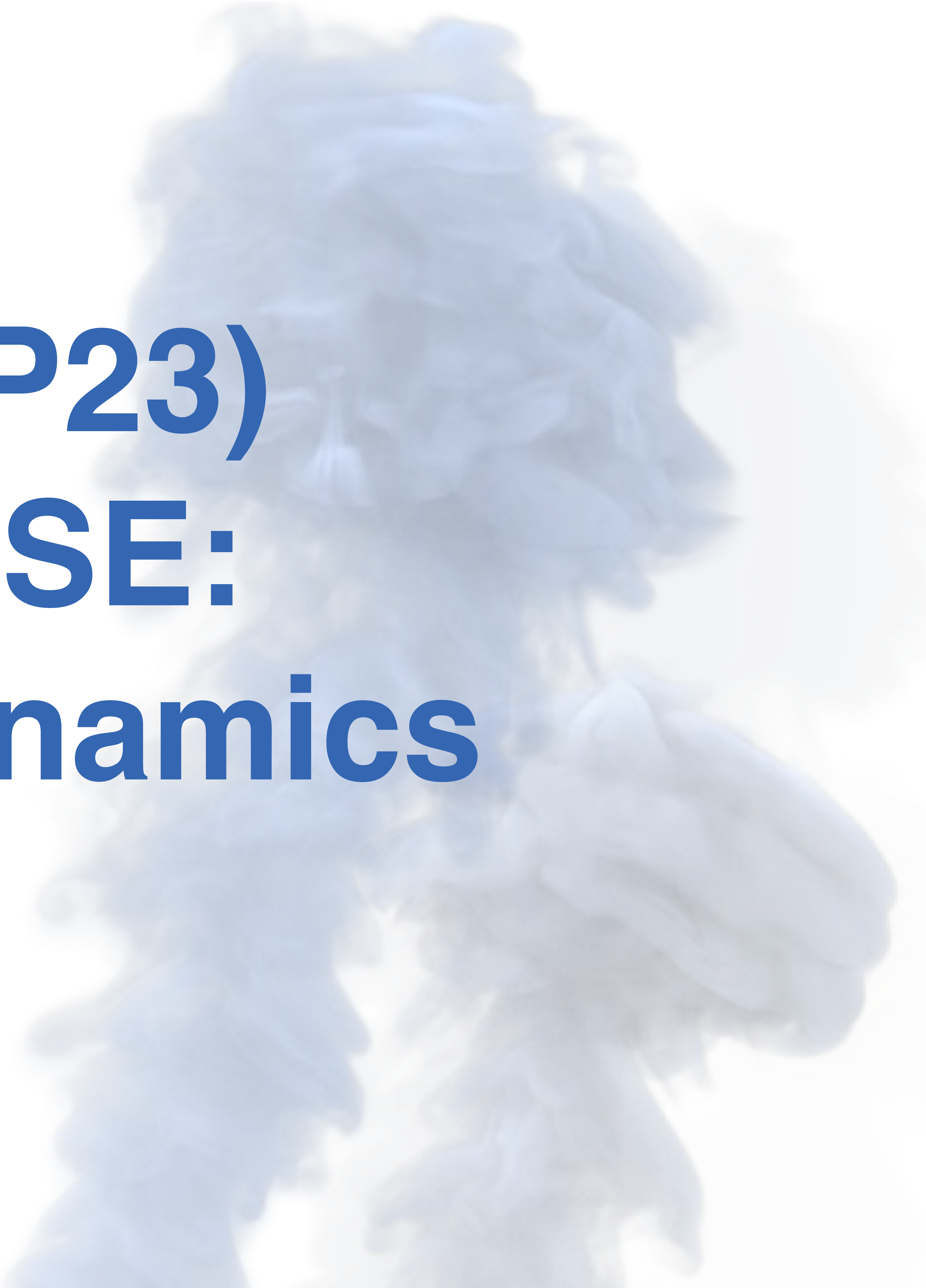

CSE 291 (SP23)
Topics in CSE:
Rigid Body Dynamics

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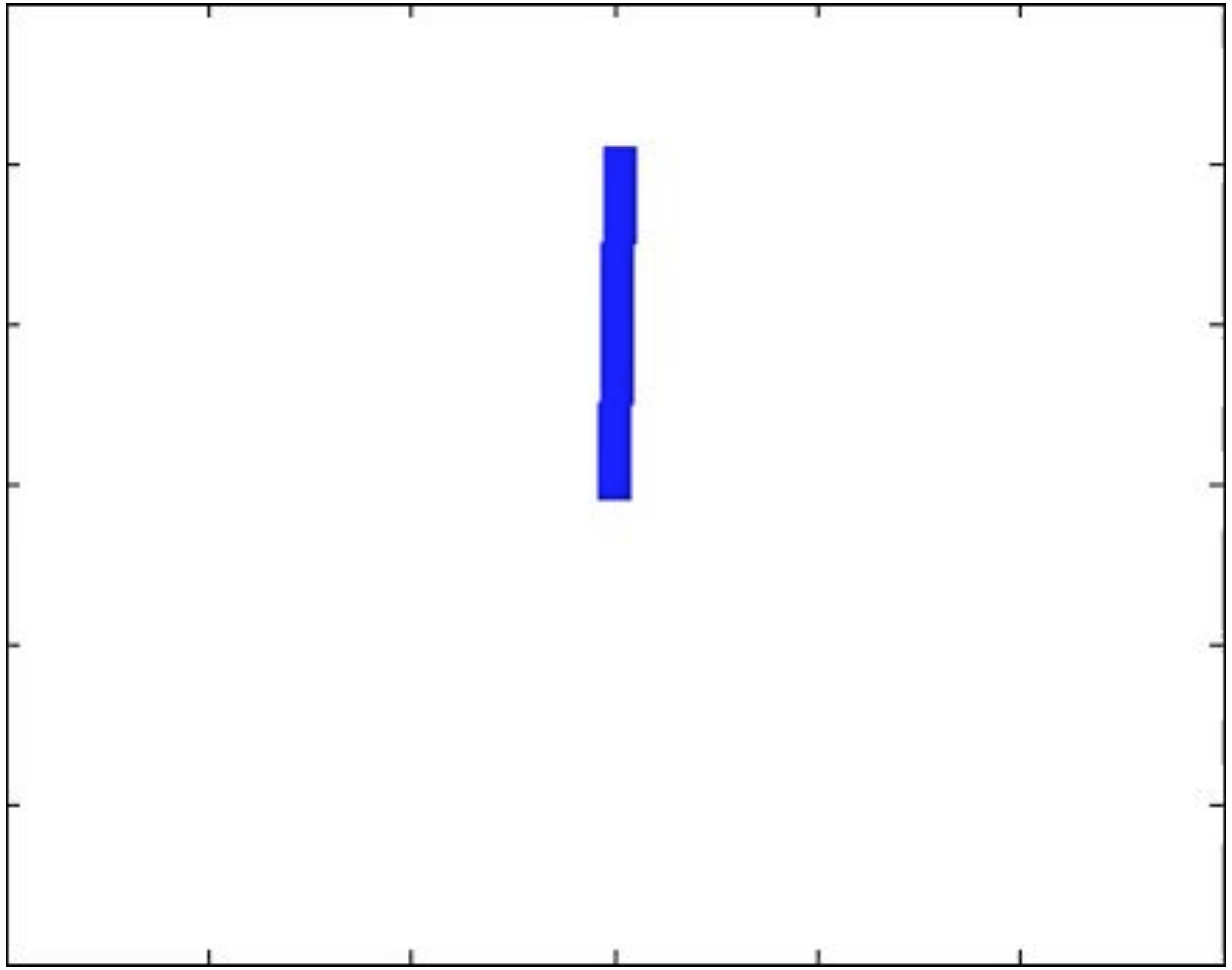
Rigid Body



Generalized Coordinates

- Generalized coordinates
- Rigid body dynamics
- Numerical methods

Least action principle

- Identify the variable describing movable parts

$$\mathbf{q} = (q_1, \ldots, q_m)$$

- Define the kinetic energy $K(\mathbf{q}, \dot{\mathbf{q}})$ and potential energy $U(\mathbf{q})$ for each static state $\mathbf{q}$ and dynamic state $(\mathbf{q}, \dot{\mathbf{q}})$
- The following are derived quantities
  - Momentum $\mathbf{p} = \frac{\partial K}{\partial \dot{\mathbf{q}}}$ $\qquad p_i = \frac{\partial K}{\partial \dot{q}_i}$
  - Lagrangian $L(\mathbf{q}, \dot{\mathbf{q}}) = K(\mathbf{q}, \dot{\mathbf{q}}) - U(\mathbf{q})$
  - Action $S(\mathbf{q}) = \int_0^T L(\mathbf{q}(t), \dot{\mathbf{q}}(t))\, dt$
  - Optimality for least action: $\frac{\partial L}{\partial \mathbf{q}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) = \mathbf{0}$

Least action principle

- Lagrangian $L(\mathbf{q}, \dot{\mathbf{q}}) = K(\mathbf{q}, \dot{\mathbf{q}}) - U(\mathbf{q})$
- Optimality for least action: $\frac{\partial L}{\partial \mathbf{q}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) = \mathbf{0}$

$$\boxed{\frac{d}{dt}\frac{\partial K}{\partial \dot{\mathbf{q}}}} = \boxed{\frac{\partial K}{\partial \mathbf{q}}}\boxed{-\frac{\partial U}{\partial \mathbf{q}}}$$

Change of momentum

Fictitious force

Force

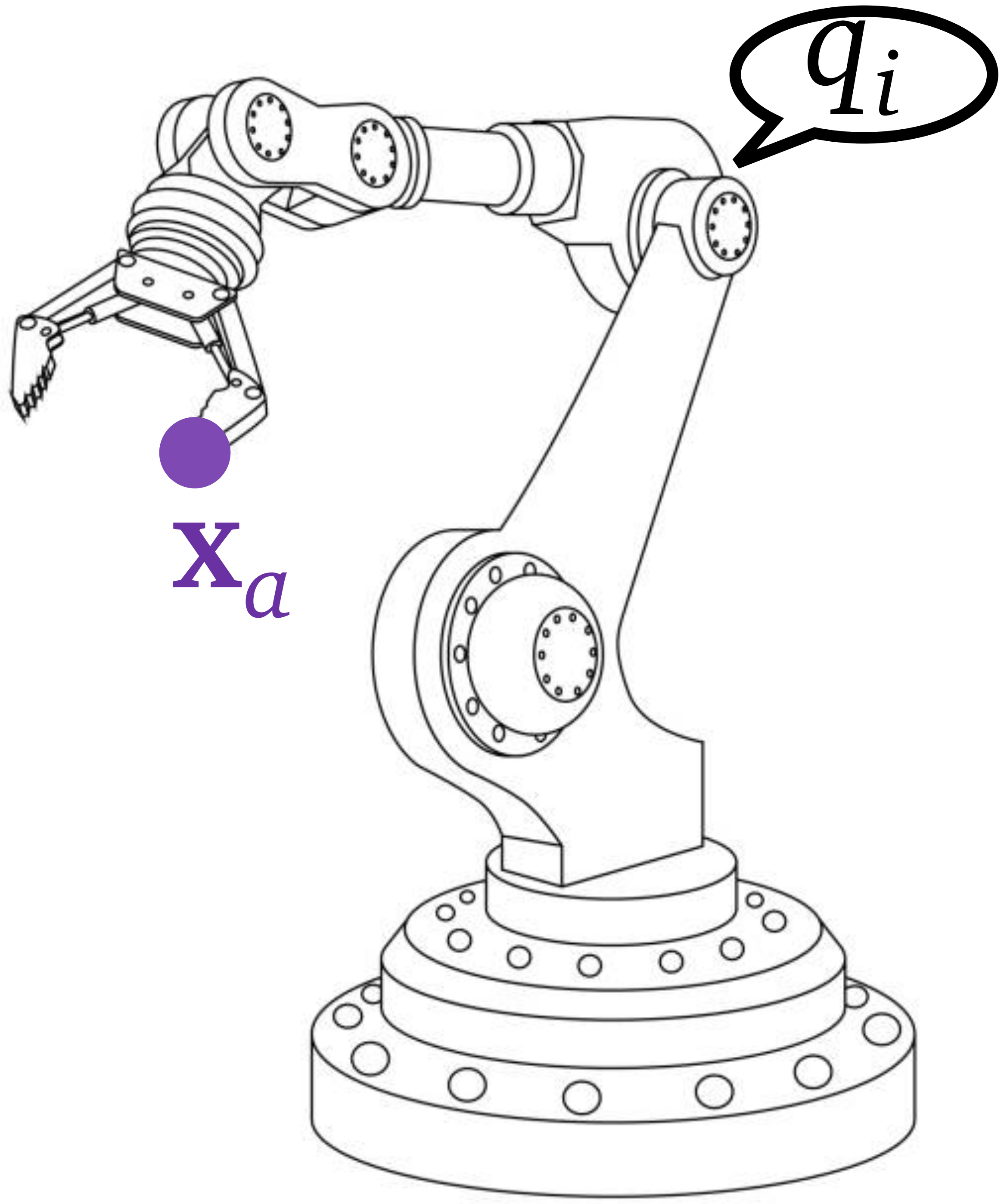
- If there is only K and $U = 0$, the resulting motion is called pure inertial motion.

Lagrangian in Newtonian mechanics

- The generalized coordinate $\mathbf{q}$ is a low dimensional parametrization of infinitely many atoms $a \in \mathcal{A}$
- There is a smooth function describing the physical 3D position of each atom as a function of $\mathbf{q}$

$$\mathbb{R}^m \xrightarrow{\mathbf{f}_a} \mathbb{R}^3$$

$$\mathbf{x}_a = \mathbf{f}_a(\mathbf{q}) \in \mathbb{R}^3$$



- The kinetic and potential energy on $(\mathbf{q}, \dot{\mathbf{q}})$ are given by pulling back the energies from 3D space

$$K(\mathbf{q}, \dot{\mathbf{q}}) = \int_{a \in \mathcal{A}} \frac{1}{2} \left| d\mathbf{f}_a|_{\mathbf{q}} [\![\dot{\mathbf{q}}]\!] \right|^2_{\mathbb{R}^3} da \qquad U(\mathbf{q}, \dot{\mathbf{q}}) = \int_{a \in \mathcal{A}} u(\mathbf{f}_a(\mathbf{q}))\, da$$

Lagrangian in Newtonian mechanics

$$K(\mathbf{q},\dot{\mathbf{q}}) = \int_{a\in\mathcal{A}} \frac{1}{2}\left| d\mathbf{f}_a|_{\mathbf{q}}[\![\dot{\mathbf{q}}]\!]\right|^2_{\mathbb{R}^3} da \qquad U(\mathbf{q},\dot{\mathbf{q}}) = \int_{a\in\mathcal{A}} u(\mathbf{f}_a(\mathbf{q}))\, da$$

- In general

  $K(\mathbf{q},\dot{\mathbf{q}})$ is a positive definite quadratic function of $\dot{\mathbf{q}}$

  This defines an inner product

$$\begin{aligned} K(\mathbf{q},\dot{\mathbf{q}}) &= \tfrac{1}{2}\flat_{\mathbf{q}}(\dot{\mathbf{q}})(\dot{\mathbf{q}}) \\ &= \tfrac{1}{2}\dot{\mathbf{q}}^\intercal \mathbf{M}_{\mathbf{q}}\dot{\mathbf{q}} \end{aligned}$$

- Momentum from velocity is given by taking flat

$$\mathbf{p} = \frac{\partial K}{\partial \dot{\mathbf{q}}} = \flat_{\mathbf{q}}\dot{\mathbf{q}} = \mathbf{M}_{\mathbf{q}}\dot{\mathbf{q}}$$

Riemannian geometric description

- Newtonian physics can be described as follows.
- There is an abstract manifold Q (**optional:** and a function

$$\mathbf{f}\colon Q \to (\mathcal{A} \to \mathbb{R}^3)$$

  describing the physical positioning of each atom in an index set $\mathcal{A}$
- Each tangent space $T_{\mathbf{q}}Q$ is equipped with a metric $\flat_{\mathbf{q}}$ called the **inertia**.
- There is also a function $U : Q \longrightarrow \mathbb{R}$ called potential.
  - Kinetic energy $K_{\mathbf{q}}(\dot{\mathbf{q}}) = \frac{1}{2}|\dot{\mathbf{q}}|_{\flat}^{2}$
  - Momentum $\mathbf{p} = \flat\dot{\mathbf{q}}$
  - Force $-dU$

Riemannian geometric description

- Each tangent space $T_{\mathbf{q}}Q$ is equipped with a metric $\flat_{\mathbf{q}}$ called the **inertia**.
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    - Momentum $\mathbf{p} = \flat\dot{\mathbf{q}}$
    - Force $-dU$
- To write down the general EL equation, we need more Riemannian geometry calculus (we'll discuss about it next time)

$$\xcancel{\frac{d}{dt}\frac{\partial K}{\partial \dot{\mathbf{q}}} = \frac{\partial K}{\partial \mathbf{q}} - \frac{\partial U}{\partial \mathbf{q}}} \qquad \flat\frac{\nabla\dot{\mathbf{q}}}{dt} = -dU$$

Rigid Body Dynamics

- Generalized coordinates
- Rigid body dynamics
- Numerical methods

Rigid body kinematics

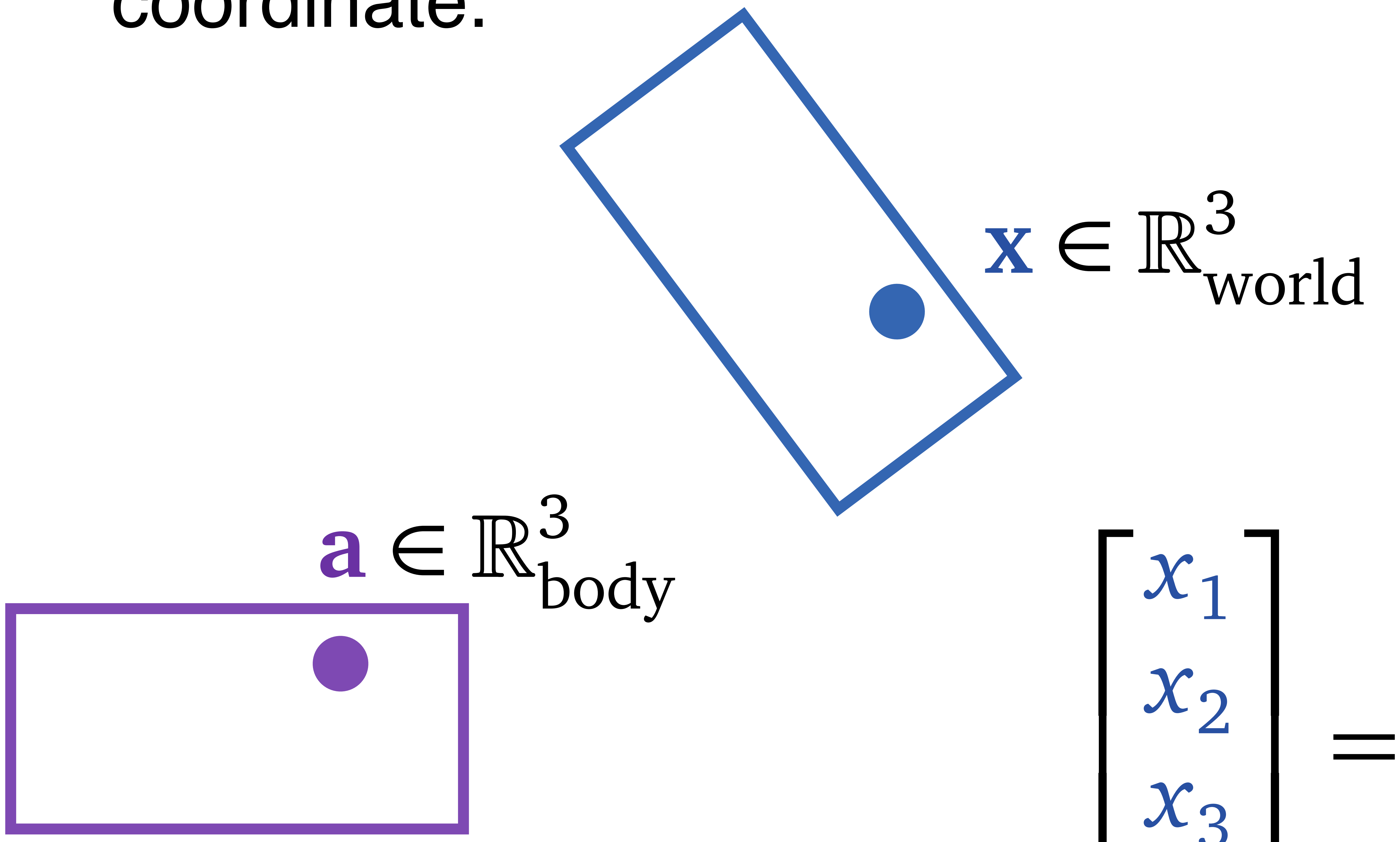
- For a rigid body, the degree of freedom is given by the space of Euclidean transformations

$$Q = \mathrm{SE}(3) = \left\{ \begin{bmatrix} & \mathbf{R}_{3\times 3} & & \begin{matrix} | \\ \mathbf{c} \\ | \end{matrix} \\ 0 & 0 & 0 & 1 \end{bmatrix} \middle| \begin{matrix} \mathbf{R}^{\intercal}\mathbf{R} = \mathrm{id}, \\ \det(\mathbf{R}) = 1, \\ \mathbf{c} \in \mathbb{R}^3 \end{matrix} \right\}$$

$$= \{(\mathbf{R}, \mathbf{c})\}$$

Rigid body kinematics

- Each element $\mathbf{q} = (\mathbf{R}, \mathbf{c}) \in Q$ is a transformation that maps a position in the body coordinate to a position in the world coordinate.



$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 1 \end{bmatrix} = \begin{bmatrix} & \mathbf{R}_{3\times3} & & \begin{matrix} | \\ \mathbf{c} \\ | \end{matrix} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ 1 \end{bmatrix}$$

Rigid body kinematics

- How do we describe a tangent vector $\dot{\mathbf{q}} \in T_{\mathbf{q}}Q$?

  $(\dot{\mathbf{R}}, \dot{\mathbf{c}})$ but we need to account for the constraint $\mathbf{R}^\intercal \mathbf{R} = \mathrm{id}$

Rigid body kinematics

- Take variation of the constraint $\mathrm{R}^{\intercal}\mathrm{R} = \mathrm{id}$

$$\dot{\mathrm{R}}^{\intercal}\mathrm{R} + \mathrm{R}^{\intercal}\dot{\mathrm{R}} = \mathbf{0}$$

$$(\mathrm{R}^{\intercal}\dot{\mathrm{R}})^{\intercal} + \mathrm{R}^{\intercal}\dot{\mathrm{R}} = \mathbf{0}$$

- Therefore $\mathrm{R}^{\intercal}\dot{\mathrm{R}}$ must be skew symmetric.
- Similarly, from $\mathrm{R}\mathrm{R}^{\intercal} = \mathrm{id}$, $\dot{\mathrm{R}}\mathrm{R}^{\intercal}$ must be skew symmetric
- Define $\mathrm{A} = \mathrm{R}^{\intercal}\dot{\mathrm{R}}$, $\mathrm{W} = \dot{\mathrm{R}}\mathrm{R}^{\intercal}$

$$\dot{\mathrm{R}} = \mathrm{R}\mathrm{A} = \mathrm{W}\mathrm{R}$$

- Every 3x3 skew sym mat is a cross product with a vector

$$\dot{\mathrm{R}} = \mathrm{R}[\Omega\times] = [\omega\times]\mathrm{R}$$

Rigid body kinematics

$$\dot{\mathbf{R}} = \mathbf{R}[\Omega \times] = [\omega \times]\mathbf{R}$$

- We call $\Omega \in \mathbb{R}^3_{\text{body}}$ the body coordinate angular velocity
- We call $\omega \in \mathbb{R}^3_{\text{world}}$ the world coordinate angular velocity
- One can check that $\omega = \mathbf{R}\Omega$
- A tangent vector $\dot{\mathbf{q}} \in T_{\mathbf{q}}Q$ is represented by a 6D vector

$$(\omega \in \mathbb{R}^3_{\text{world}}, \dot{\mathbf{c}} \in \mathbb{R}^3_{\text{world}}) \quad \text{or} \quad (\Omega \in \mathbb{R}^3_{\text{body}}, \dot{\mathbf{c}} \in \mathbb{R}^3_{\text{world}})$$

Inertia

- The kinetic energy is a quadratic form on

$$(\omega \in \mathbb{R}^3_{\text{world}}, \dot{\mathbf{c}} \in \mathbb{R}^3_{\text{world}}) \quad \text{or} \quad (\Omega \in \mathbb{R}^3_{\text{body}}, \dot{\mathbf{c}} \in \mathbb{R}^3_{\text{world}})$$

- If the origin for the body coordinate is the center of mass, then

$$\begin{aligned} K(\mathbf{q}, \dot{\mathbf{q}}) &= \frac{m}{2}|\dot{\mathbf{c}}|^2 + \frac{1}{2}\omega^\intercal \mathbf{I}_{\text{world}}\,\omega \\ &= \frac{m}{2}|\dot{\mathbf{c}}|^2 + \frac{1}{2}\Omega^\intercal \mathbf{I}_{\text{body}}\Omega \end{aligned}$$

moment of inertia

- That is, the translation and rotation decouple.
- Check that the 3x3 symm pos. def. matrices satisfy

$$\mathbf{I}_{\text{world}} = \mathbf{R}\mathbf{I}_{\text{body}}\mathbf{R}^\intercal$$

Inertia

- For a rigid body, $\mathbf{I}_{\text{body}}$ is time-independent;
  $\mathbf{I}_{\text{world}} = \mathbf{R}\mathbf{I}_{\text{body}}\mathbf{R}^{\mathsf{T}}$ co-rotates.

Inertia

- For a rigid body, $\mathbf{I}_{\text{body}}$ is time-independent;

  $\mathbf{I}_{\text{world}} = \mathbf{R}\mathbf{I}_{\text{body}}\mathbf{R}^{\mathsf{T}}$ co-rotates.

- This is equivalent to frame indifference:

$$K(\mathbf{R}, \mathbf{c}, \dot{\mathbf{R}}, \dot{\mathbf{c}}) = K(\mathbf{Q}\mathbf{R}, \mathbf{Q}\mathbf{c} + \mathbf{d}, \mathbf{Q}\dot{\mathbf{R}}, \mathbf{Q}\dot{\mathbf{c}}) \qquad \forall (\mathbf{Q}, \mathbf{d}) \in \mathrm{SE}(3)$$

  - We call this inertia a left-invariant metric
  - In general it is not right-invariant

$$K(\mathbf{R}, \mathbf{c}, \dot{\mathbf{R}}, \dot{\mathbf{c}}) \neq K(\mathbf{R}\mathbf{Q}, \mathbf{c} + \mathbf{R}\mathbf{d}, \dot{\mathbf{R}}\mathbf{Q}, \dot{\mathbf{c}})$$

unless $\mathbf{Q}\mathbf{I}_{\text{body}}\mathbf{Q}^{\mathsf{T}} = \mathbf{I}_{\text{body}}$ (e.g. spinning top)

Euler Lagrange equation

$$K(\mathbf{q}, \dot{\mathbf{q}}) = \frac{m}{2}|\dot{\mathbf{c}}|^2 + \frac{1}{2}\omega^\intercal \mathbf{I}_{\text{world}}\,\omega$$

- If the potential energy $U(\mathbf{R}, \mathbf{c})$ depends only on $\mathbf{c}$

$$\frac{d}{dt}\frac{\partial K}{\partial \dot{\mathbf{c}}} = -\frac{\partial U}{\partial \mathbf{c}} \implies m\ddot{\mathbf{c}} = -\frac{\partial U}{\partial \mathbf{c}}$$

$$\frac{d}{dt}\frac{\partial K}{\partial \omega} = 0 \implies \text{Let's derive it}$$

(It is not immediately clear why there is no $\frac{\partial K}{\partial \mathbf{R}}$ on the RHS)

Euler Lagrange equation

$$\frac{d}{dt}\frac{\partial K}{\partial \omega} = 0 \qquad K(\mathbf{q}, \dot{\mathbf{q}}) = \frac{m}{2}|\dot{\mathbf{c}}|^2 + \frac{1}{2}\omega^{\mathsf{T}}\mathbf{I}_{\text{world}}\,\omega$$

$$\implies \quad \tfrac{d}{dt}\left(\mathbf{I}_{\text{world}}\,\omega\right) = 0$$

$$\implies \quad \tfrac{d}{dt}\left(\mathbf{R}\mathbf{I}_{\text{body}}\mathbf{R}^{\mathsf{T}}\omega\right) = 0$$

$$\implies \quad \dot{\mathbf{R}}\mathbf{I}_{\text{body}}\mathbf{R}^{\mathsf{T}}\omega + \mathbf{R}\mathbf{I}_{\text{body}}\dot{\mathbf{R}}^{\mathsf{T}}\omega + \mathbf{R}\mathbf{I}_{\text{body}}\mathbf{R}^{\mathsf{T}}\dot{\omega} = 0$$

$$\implies \quad [\omega\times]\mathbf{R}\mathbf{I}_{\text{body}}\mathbf{R}^{\mathsf{T}}\omega + \mathbf{R}\mathbf{I}_{\text{body}}\mathbf{R}^{\mathsf{T}}[\omega\times]^{\mathsf{T}}\omega + \mathbf{R}\mathbf{I}_{\text{body}}\mathbf{R}^{\mathsf{T}}\dot{\omega} = 0$$

$$\implies \quad \omega \times (\mathbf{I}_{\text{world}}\,\omega) + \mathbf{I}_{\text{world}}\,\dot{\omega} = 0$$

Euler equation of rigid motion

Euler Lagrange equation

$$\begin{cases} \dot{\mathbf{R}} = [\omega \times]\mathbf{R} \\ \mathbf{I}_{\text{world}}\dot{\omega} = -\omega \times (\mathbf{I}_{\text{world}}\omega) \\ \mathbf{I}_{\text{world}} = \mathbf{R}\mathbf{I}_{\text{body}}\mathbf{R}^{\mathsf{T}} \\ m\ddot{\mathbf{c}} = -\dfrac{\partial U}{\partial \mathbf{c}} \end{cases}$$

Euler Lagrange equation

- But if we just take Euler Lagrange equation in the body coordinate:

$$\frac{d}{dt}\frac{\partial K}{\partial \Omega} = 0$$

it's incorrect

- Why?

Euler Lagrange equation

- Take variation properly

$$\int_0^T \frac{1}{2}\boldsymbol{\Omega}(t)\mathbf{I}_{\mathrm{body}}\boldsymbol{\Omega}(t)\,dt$$

- (blackboard)

Euler Lagrange equation

$$\begin{cases} \dot{\mathbf{R}} = [\omega \times]\mathbf{R} \\ \mathbf{I}_{\text{world}}\dot{\omega} = -\omega \times (\mathbf{I}_{\text{world}}\omega) \\ \mathbf{I}_{\text{world}} = \mathbf{R}\mathbf{I}_{\text{body}}\mathbf{R}^{\mathsf{T}} \\ m\ddot{\mathbf{c}} = -\dfrac{\partial U}{\partial \mathbf{c}} \end{cases} \iff \frac{d}{dt}\overset{\mathbf{l}}{\boxed{\frac{\partial K}{\partial \omega}}} = 0$$

world coord angular momentum

body coord angular momentum

$$\frac{\partial K}{\partial \Omega} = \mathbf{L} = \mathbf{R}^{\mathsf{T}}\mathbf{l}$$

is not conserved in general

Ponsoit ellipsoids

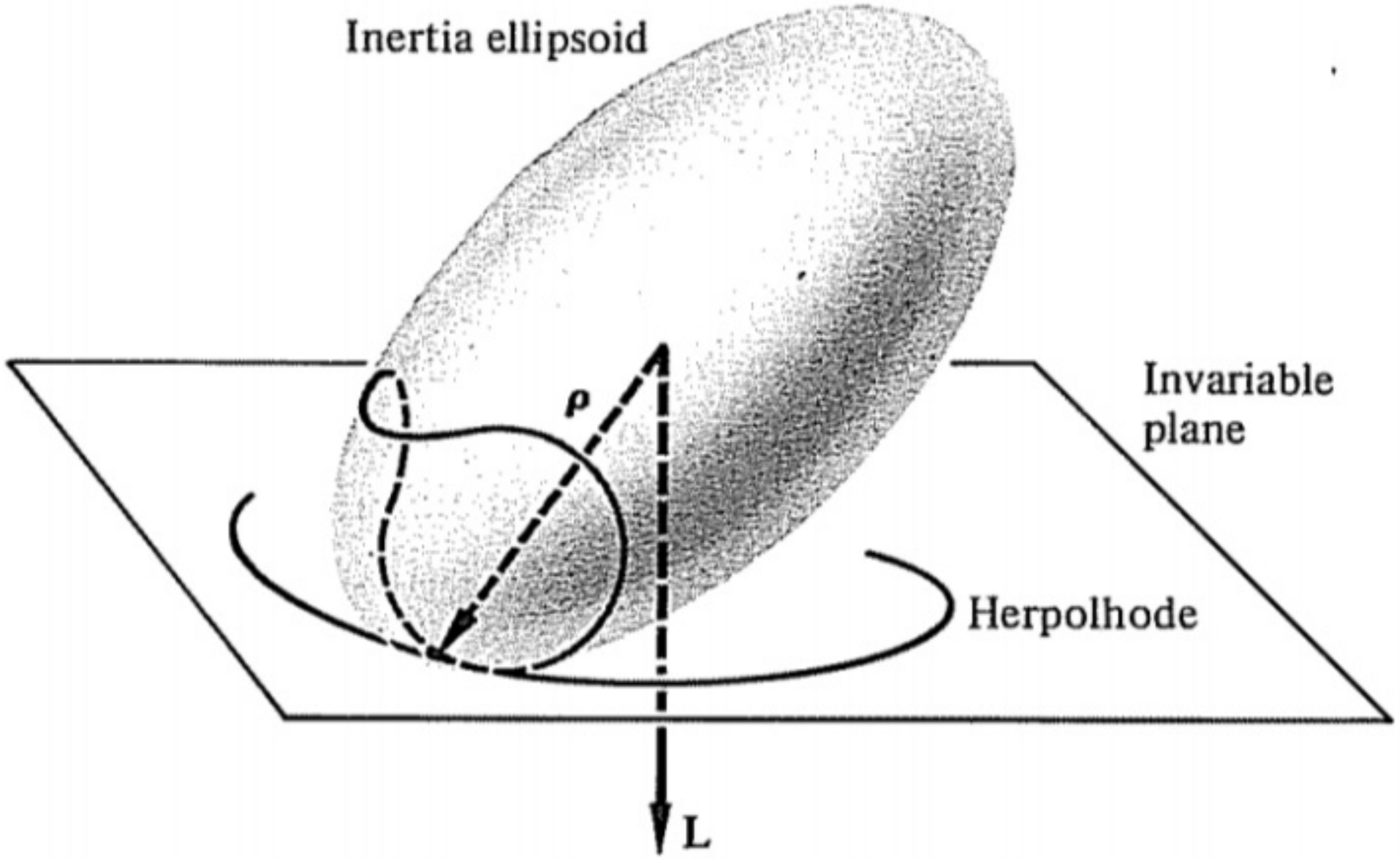
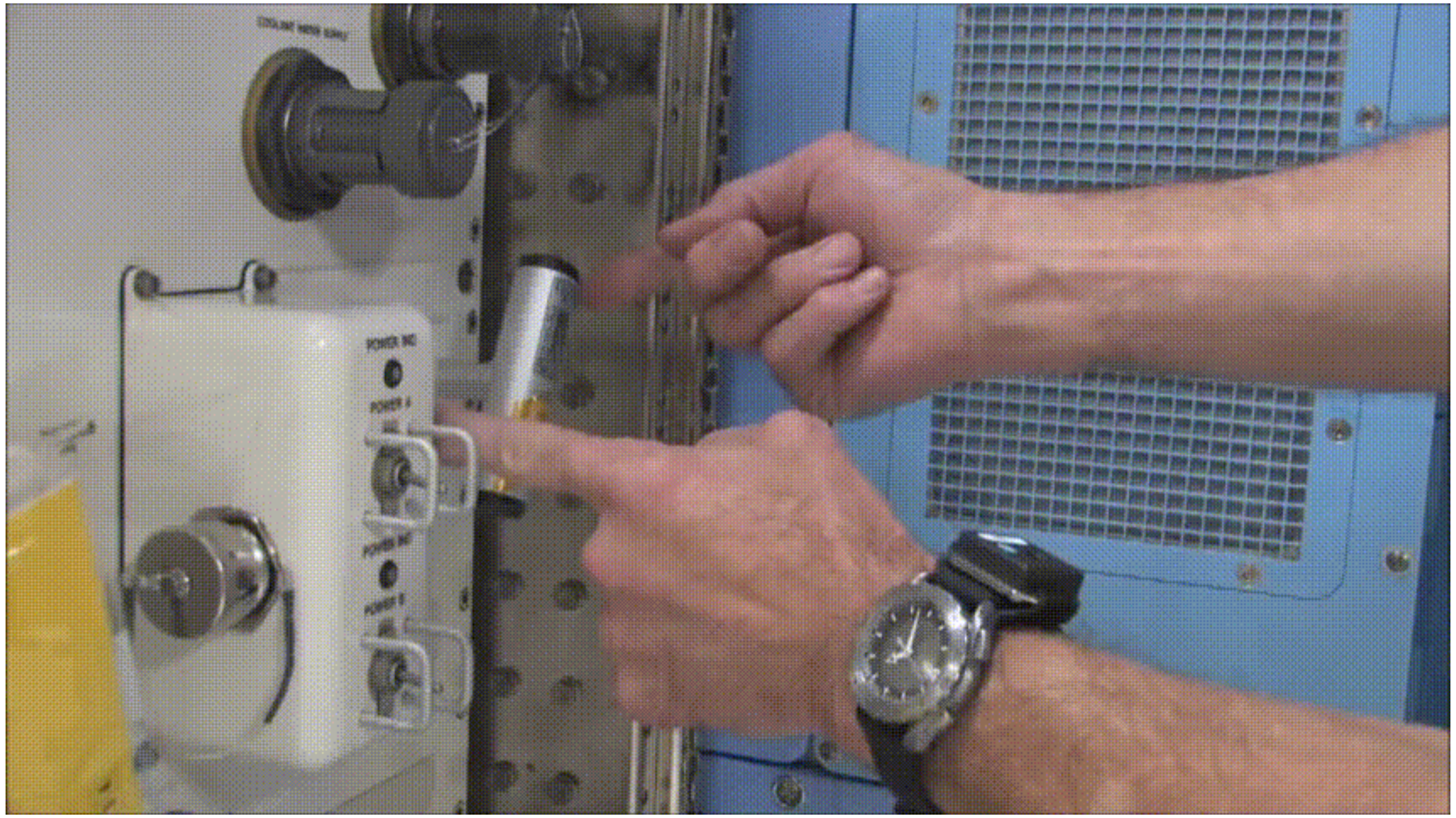


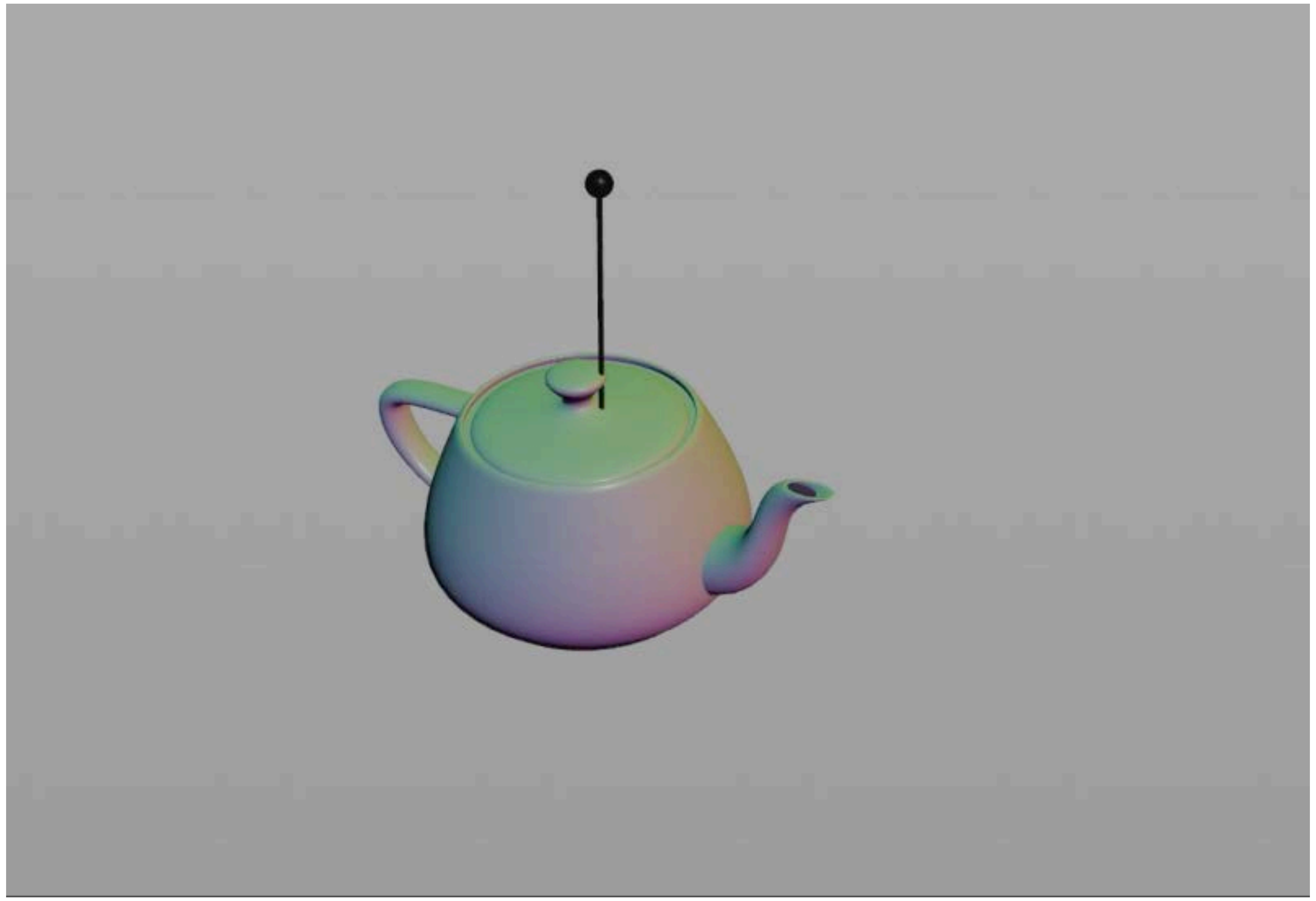
FIGURE 5–4
The motion of the inertia ellipsoid relative to the invariable plane.

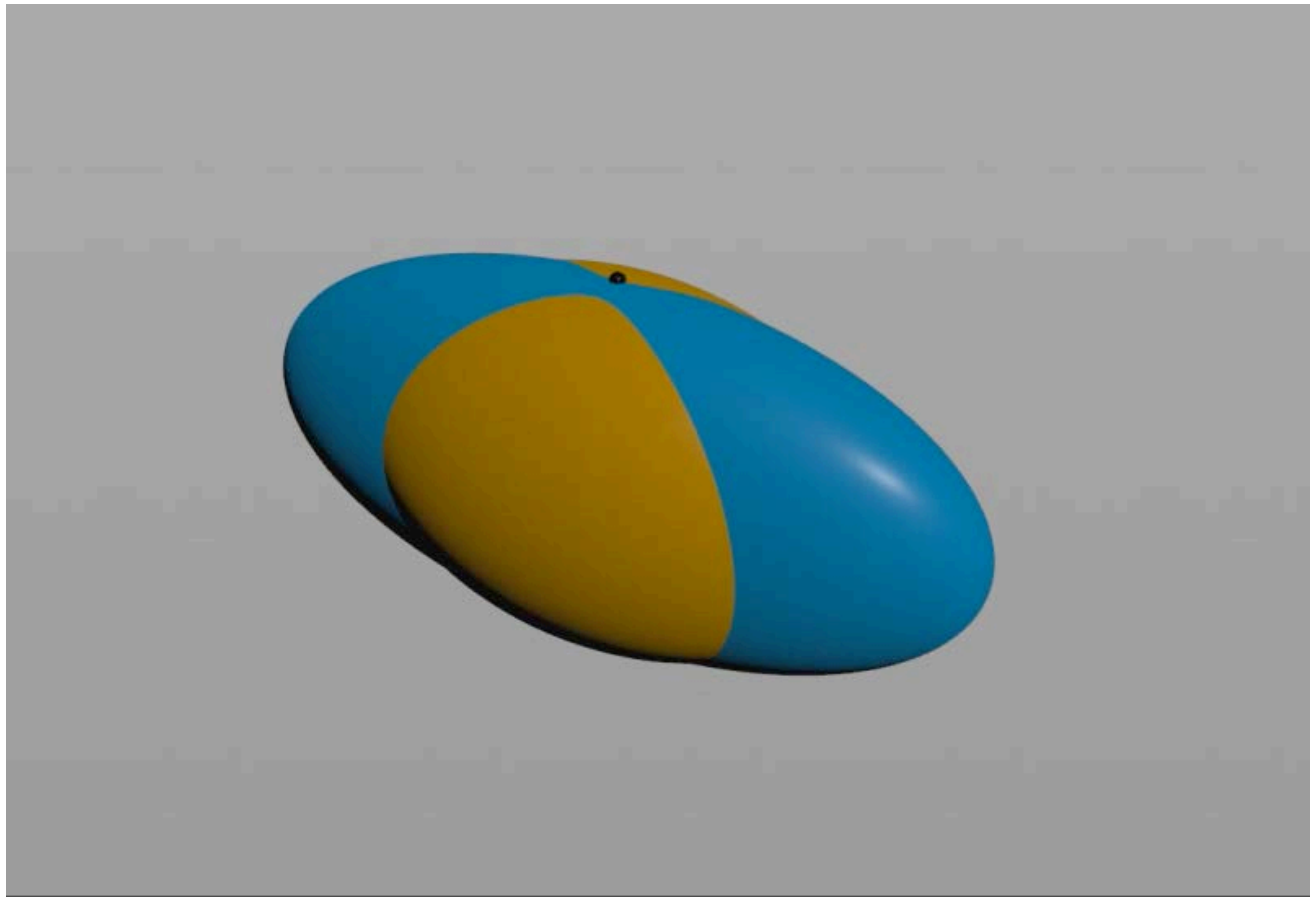
Dzhanibekov effect



Dzhanibekov effect

Dzhanibekov effect





Poinsot ellipsoids: energy and norm of angular momentum in body coord.

Numerical Method

- Generalized coordinates
- Rigid body dynamics
- Numerical methods

Numerical method

$$\dot{\mathbf{R}} = [\omega \times]\mathbf{R}$$

$$\mathbf{I}_{\text{world}}\dot{\omega} = -\omega \times (\mathbf{I}_{\text{world}}\omega)$$

$$\mathbf{I}_{\text{world}} = \mathbf{R}\mathbf{I}_{\text{body}}\mathbf{R}^{\mathsf{T}}$$

- Forward/backward Euler or RK4: Adds ODE RHS to variable (does not respect rotation matrices)

- It’s better to use rotation to update the variable (Lie group integrator)

Numerical method

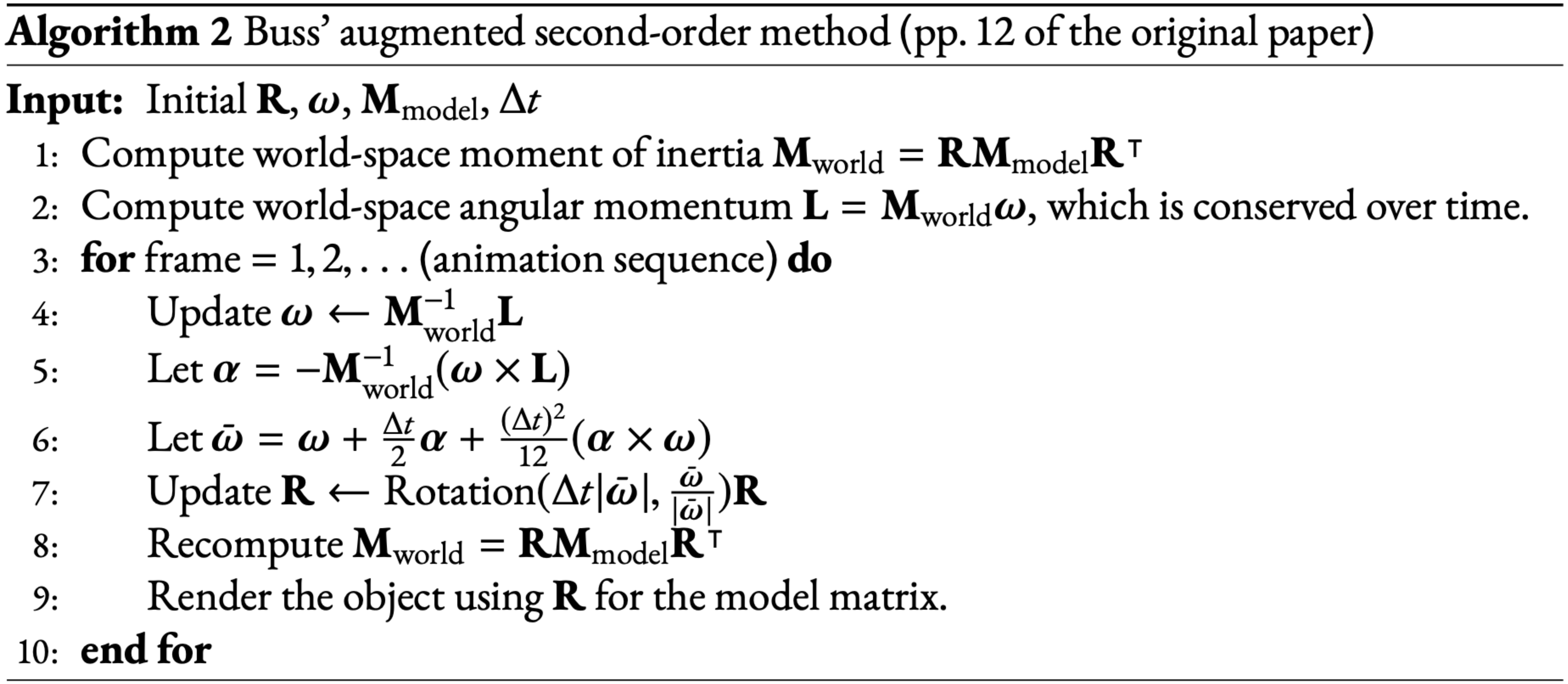
- Simple first-order method based on conservation of angular momentum

(here **L** denote the world-space angular momentum)

```
**Input:** Initial $\mathbf{R}$, $\boldsymbol{\omega}$, $\mathbf{M}_{\text{model}}$, $\Delta t$
Compute world-space moment of inertia $\mathbf{M}_{\text{world}} = \mathbf{R}\mathbf{M}_{\text{model}}\mathbf{R}^{\top}$
Compute world-space angular momentum $\mathbf{L} = \mathbf{M}_{\text{world}}\boldsymbol{\omega}$, which is conserved over time.
**for** frame = $1, 2, \ldots$ (animation sequence) **do**
    Update $\boldsymbol{\omega} \leftarrow \mathbf{M}_{\text{world}}^{-1}\mathbf{L}$
                                   angle–axis
    Update $\mathbf{R} \leftarrow \text{Rotation}(\Delta t|\boldsymbol{\omega}|, \frac{\boldsymbol{\omega}}{|\boldsymbol{\omega}|})\mathbf{R}$
    Recompute $\mathbf{M}_{\text{world}} = \mathbf{R}\mathbf{M}_{\text{model}}\mathbf{R}^{\top}$
    Render the object using $\mathbf{R}$ for the model matrix.
**end for**
```

Numerical method

- Samual Buss 2001 “Accurate and Efficient Simulation of Rigid Body Rotations”



Algorithm 2 Buss’ augmented second-order method (pp. 12 of the original paper)

```
Input: Initial $\mathbf{R}$, $\omega$, $\mathbf{M}_{\text{model}}$, $\Delta t$
Compute world-space moment of inertia $\mathbf{M}_{\text{world}} = \mathbf{R}\mathbf{M}_{\text{model}}\mathbf{R}^\top$
Compute world-space angular momentum $\mathbf{L} = \mathbf{M}_{\text{world}}\omega$, which is conserved over time.
for frame = 1, 2, . . . (animation sequence) do
    Update $\omega \leftarrow \mathbf{M}_{\text{world}}^{-1}\mathbf{L}$
    Let $\alpha = -\mathbf{M}_{\text{world}}^{-1}(\omega \times \mathbf{L})$
    Let $\bar{\omega} = \omega + \frac{\Delta t}{2}\alpha + \frac{(\Delta t)^2}{12}(\alpha \times \omega)$
    Update $\mathbf{R} \leftarrow \text{Rotation}(\Delta t|\bar{\omega}|, \frac{\bar{\omega}}{|\bar{\omega}|})\mathbf{R}$
    Recompute $\mathbf{M}_{\text{world}} = \mathbf{R}\mathbf{M}_{\text{model}}\mathbf{R}^\top$
    Render the object using $\mathbf{R}$ for the model matrix.
end for
```