

CSE 291 (SP23)

Topics in CSE:

Physics Simulation

Albert Chern

Course information

Physics Simulation

- **Instructor:** Albert Chern
- **TA:** Shiyang Jia
- **Course website:** https://cseweb.ucsd.edu/~alchern/teaching/cse291_sp23/
- We use **Piazza** and **Gradescope**

Course information

Physics Simulation

- **Goal:** Mathematical principles behind simulation tasks
Hands-on experience with physics-based animations
- **Applications:** Computer animation, scientific computing
classical mechanics, theory abstraction
- **Grade:** HW0–4 (written and mini-project)
- **Collaboration:** Final submissions should be individual work, but we encourage you to study the math and solve the problems together!

Course information

Physics Simulation

- **Prerequisites:**

- ▶ Linear algebra, multivariable calculus, elementary physics
- ▶ Using one programming platform with visualization that is capable of using/importing sparse matrix library
 - e.g. graphics software: Houdini, Blender, Unity
 - e.g. C++, Python, MATLAB, Javascript+WebGL

- **What tools can you use:**

- ▶ Build your own solver from lower level (you can use built-in geometry processing functions) Don't use a full-blown built-in simulation solver.

Syllabus

Week	Tuesday	Thursday
1	4/4: Introduction	4/6: Dimensional analysis
2	4/11: Vector duality <ul style="list-style-type: none">• HW0 due (miniproject)	4/13: Differentials and gradients
3	4/18: Calculus of variations <ul style="list-style-type: none">• HW1 due (written)	4/20: Least action principle
4	4/25: Rigid body motion	4/27: Geodesic equation
5	5/2: Incremental potential <ul style="list-style-type: none">• HW2 due (written+miniproject)	5/4: Constrained systems
6	5/9: Tensors	5/11: Tensors
7	5/16: Elasticity	5/18: Elasticity
8	5/23: Fluids	5/25: Fluids
9	5/30: <ul style="list-style-type: none">• HW3 due (miniproject)	6/1:
10	6/6:	6/8:
Final	6/13: No class <ul style="list-style-type: none">• HW4 due (miniproject)	6/15: No class

Simulation, Physics, Math

- Simulation, Physics, Math
- Getting started: $F = ma$
- Solve ODEs numerically

Physics simulation

- In computational physics, engineering, computer graphics,...
- Generate computer-generated data that mimic that we would observe in the physical world.
- Why?
 - ▶ Make predictions, conduct virtual experiments
 - ▶ Believable visual effects
- How?

Physics simulation

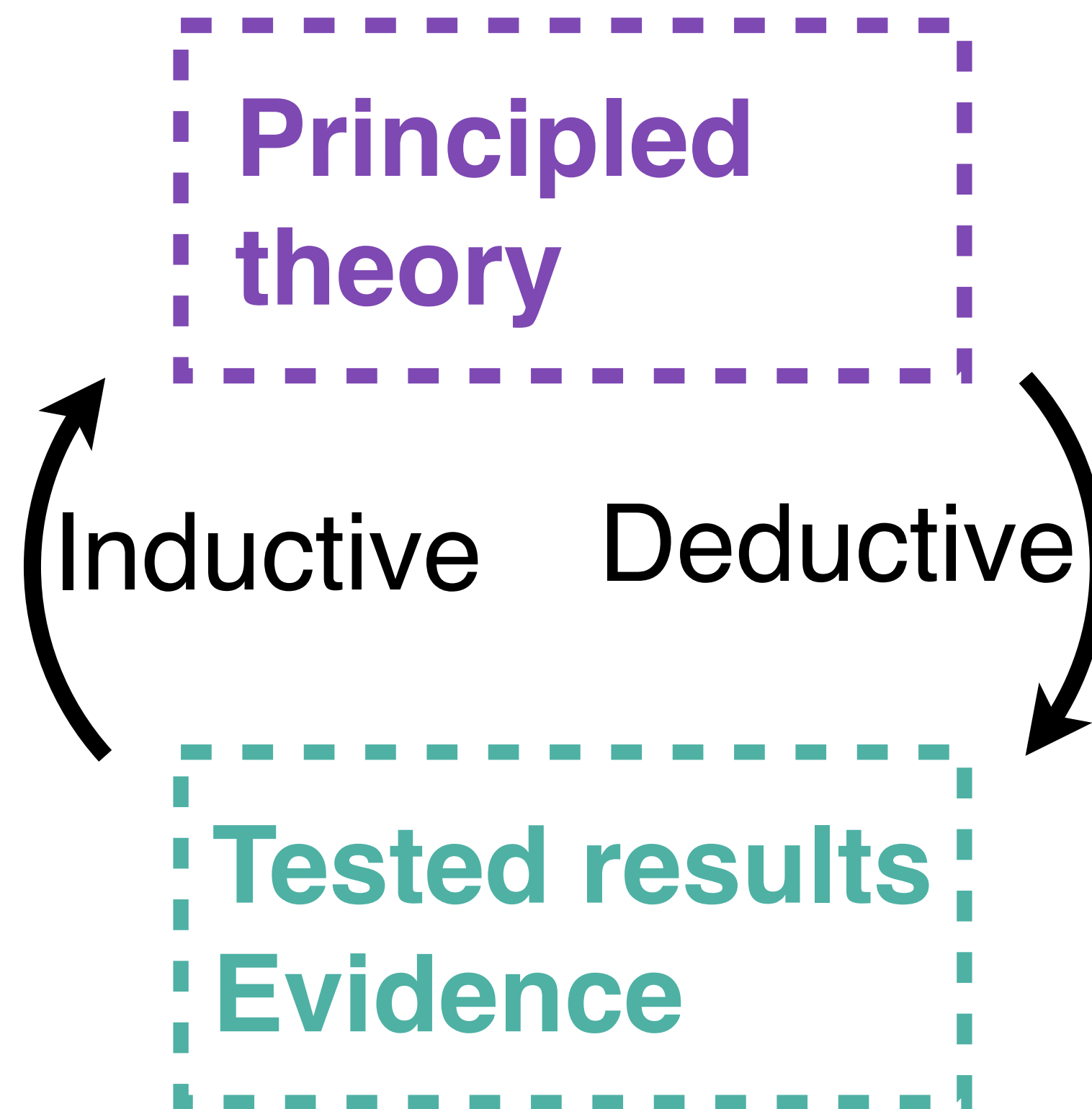
- How?
 - ▶ **Mathematical modeling**
*Turn physical phenomena into mathematical equations.
(What are the variables? What are the laws of physics)*
 - ▶ **Analysis**
*Get a general idea of how the solution should behave.
(Is the problem well-posed?)*
 - ▶ **Computation**
Solve (approximate) solutions analytically or numerically.

Math and physics

Scientific methods

- Deductive science (Testing theory)
- Inductive science (Developing theory)

- Recognizing patterns in specific observations
- Generalize them to a unifying theory
- Test consistency by math



- Formulate hypotheses
- Collect data
- Accept/reject the hypotheses

Math and physics

The Unreasonable Effectiveness of Mathematics in the Natural Sciences

Richard Courant Lecture in Mathematical Sciences delivered at New York University,
May 11, 1959

EUGENE P. WIGNER

Princeton University

*"and it is probable that there is some secret here
which remains to be discovered." (C. S. Peirce)*

There is a story about two friends, who were classmates in high school, talking about their jobs. One of them became a statistician and was working on population trends. He showed a reprint to his former classmate. The reprint started, as usual, with the Gaussian distribution and the statistician explained to his former classmate the meaning of the symbols for the actual population, for the average population, and so on. His classmate was a bit incredulous and was not quite sure whether the statistician was pulling his leg. "How can you know that?" was his query. "And what is this symbol here?" "Oh," said the statistician, "this is π ." "What is that?" "The ratio of the circumference of the circle to its diameter." "Well, now you are pushing your joke too far," said the classmate, "surely the population has nothing to do with the circumference of the circle."

Naturally, we are inclined to smile about the simplicity of the classmate's approach. Nevertheless, when I heard this story, I had to admit to an eerie feeling because, surely, the reaction of the classmate betrayed only plain common sense. I was even more confused when, not many days later, someone came to me and expressed his bewilderment¹ with the fact that we make a rather narrow selection when choosing the data on which we test our theories. "How do we know that, if we made a theory which focusses its attention on phenomena we disregard and disregards some of the phenomena now commanding our attention, that we could not build another theory which has little in common with the present one but which, nevertheless, explains just as many phenomena as the present theory." It has to be admitted that we have not definite evidence that there is no such theory.

The preceding two stories illustrate the two main points which are the

¹The remark to be quoted was made by F. Werner when he was a student in Princeton.

Physics simulation

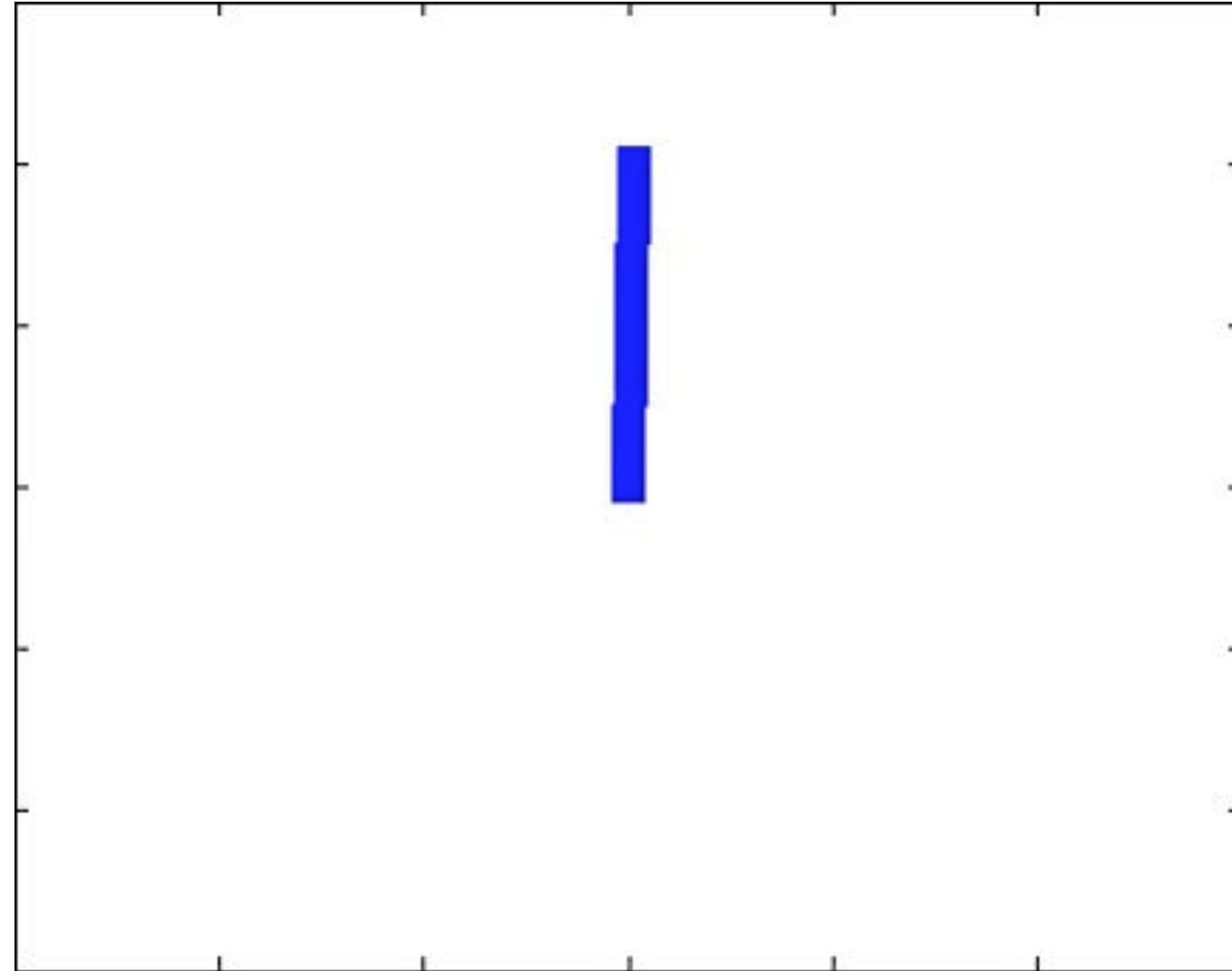
We will focus on **general principles**

- Dimensional analysis
- Least action principle
- Incremental potential formulation
- Constitutive modeling in continuum mechanics

Systems we will cover

- Small mechanical systems
- Rigid body
- Constrained system (linkage, robotics, collision and contact)
- Elastic body
- Fluids

Physics simulation



Physics simulation



Youtube "Soft Body Tetris [01]"
by ImbaPixel

<https://youtu.be/rm44SV8xUDo>



Nabizadeh, Wang, Ramamoorthi, C.
Covector Fluids
2022

Getting started: $F=ma$

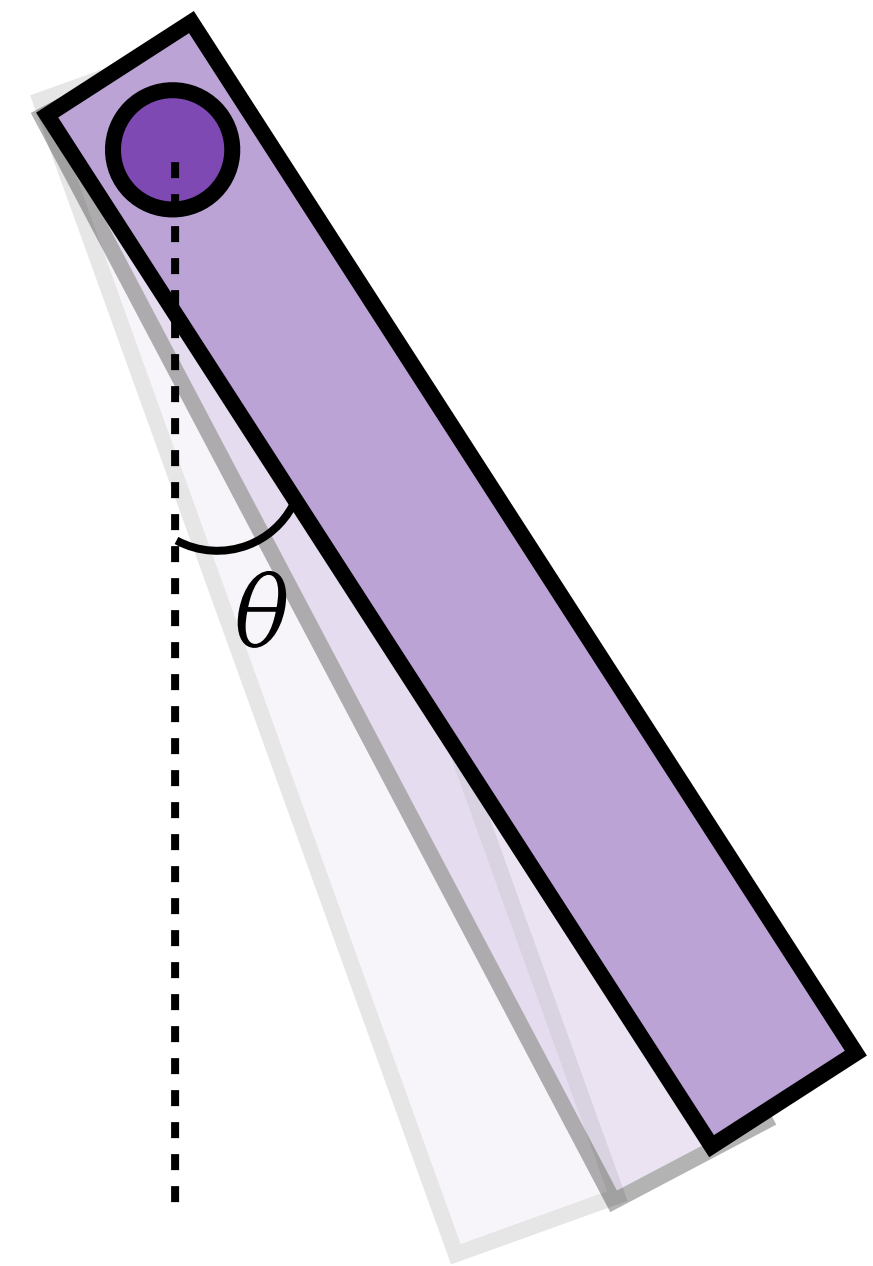
- Simulation, Physics, Math
- Getting started: $F = ma$
- Solve ODEs numerically

Physics based motion

CSE 291 E00 (SP 2023) Homework 0

Exercise 0.1 — 5pt. Using your favorite program to produce an animation of a simple physical system. It could be a pendulum motion like demonstrated in the lecture, or other system.

- (a) Upload a video of your result.
- (b) Upload a written document that briefly explains the system. (Include the equation of motion, an explanation of what each variable in the equation means, and what the time stepping algorithm looks like.)
- (c) Upload the source file(s) (for example .zip).



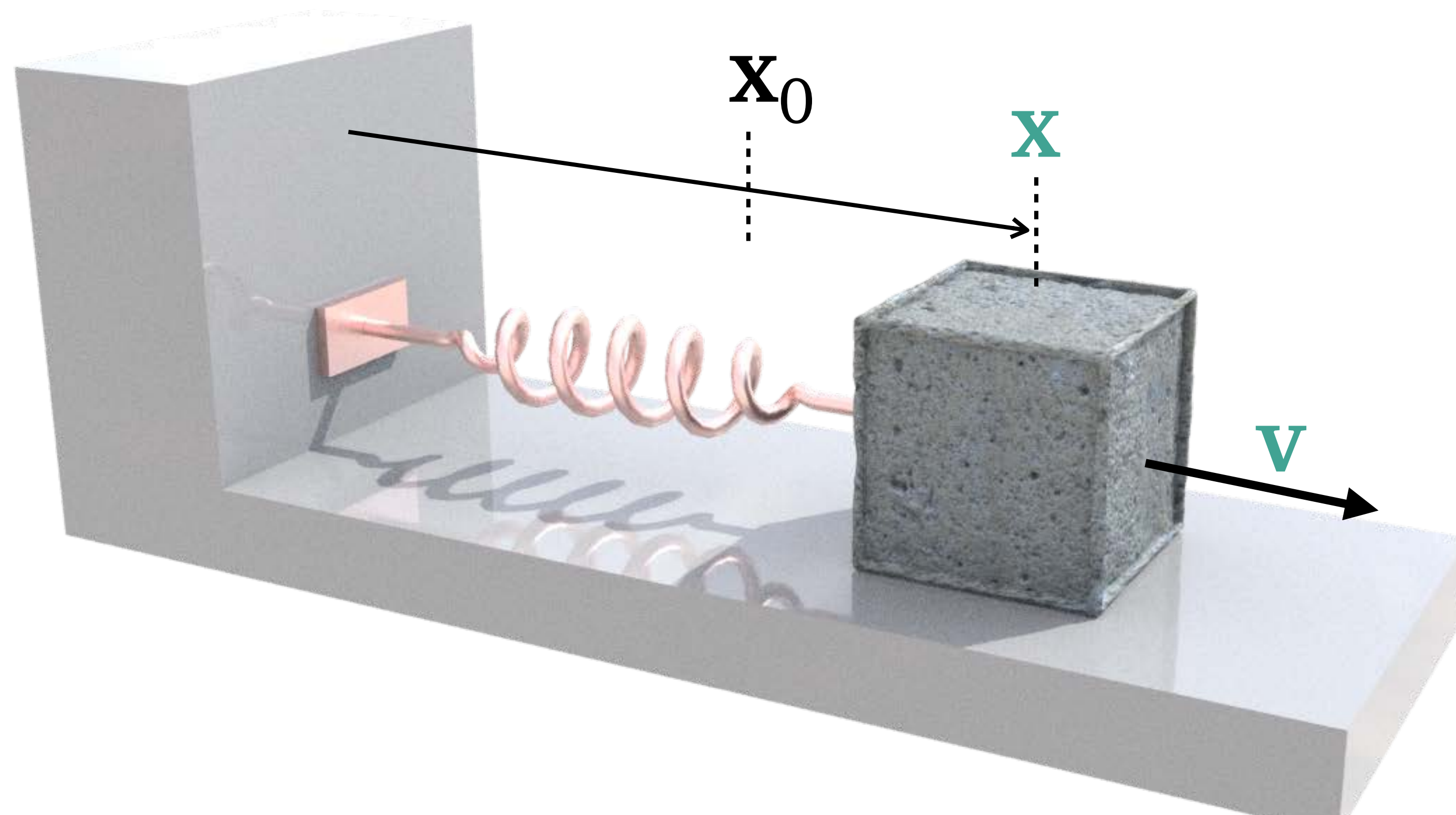
Physics based motion

Rough idea:

- Position of each object is governed by Newton's law of motion
- Rate of change of position is called velocity
- Rate of change of velocity is called acceleration
- Model "force" as a function of position and velocity
- Newton's law of motion: $\text{Mass} \times \text{acceleration} = \text{force}$

Example

- Animate an object attached to a spring
- Identify the moving position: x
- Associated velocity v

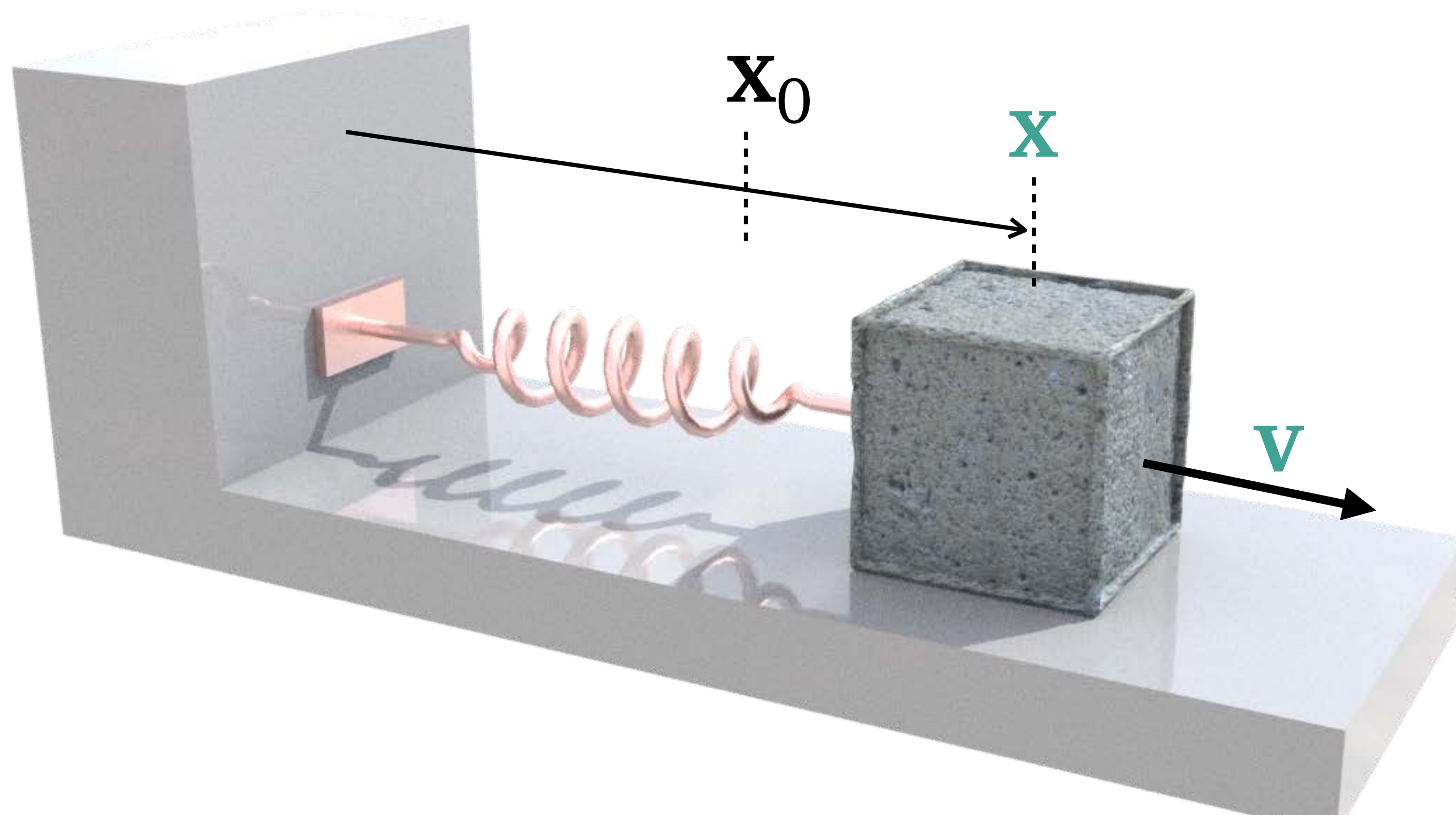


Example

- Force

$$\mathbf{f}(\mathbf{x}, \mathbf{v}) = -k(\mathbf{x} - \mathbf{x}_0) - \mu\mathbf{v}$$

rest position

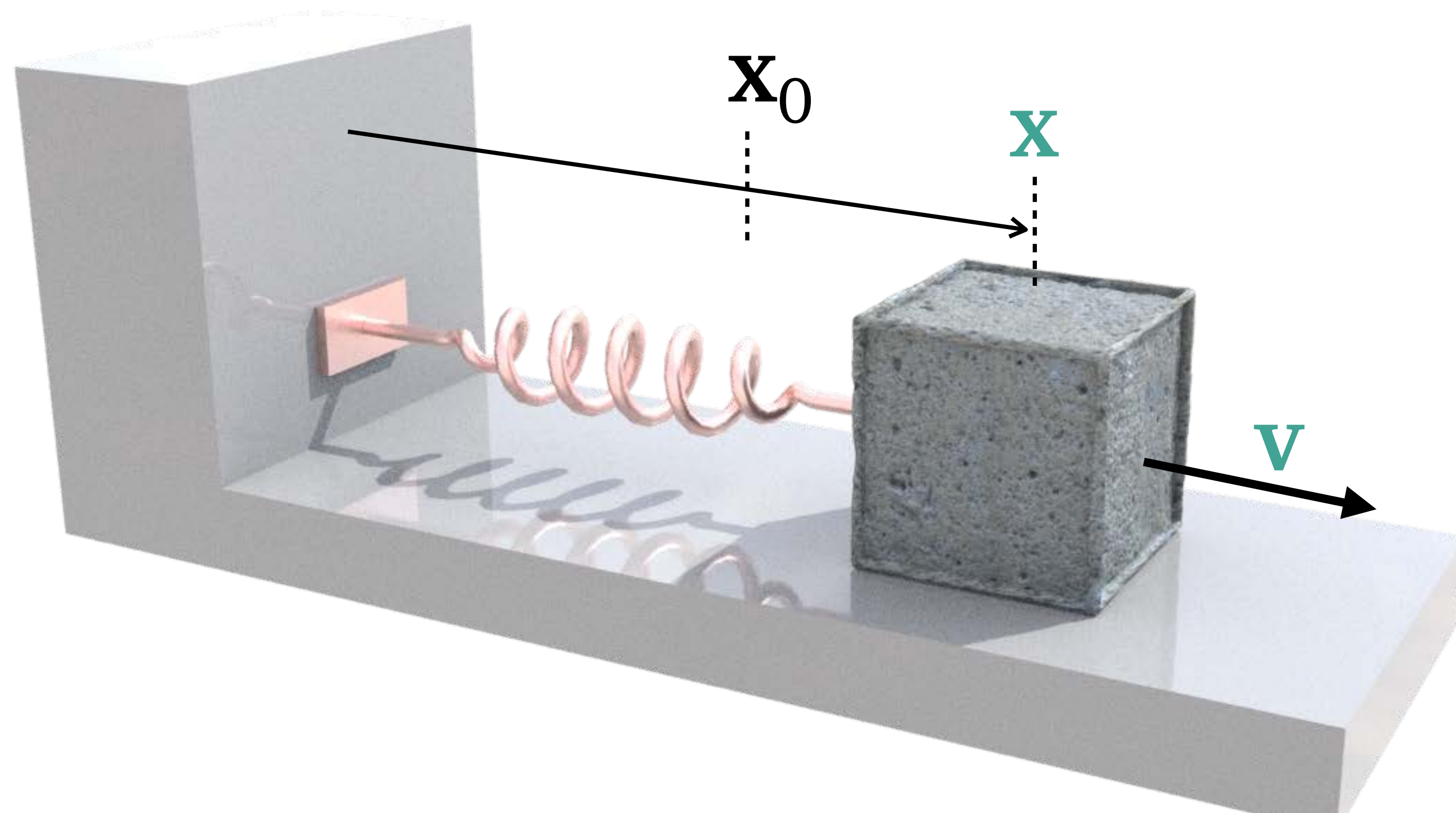


Example

- Force

$$\mathbf{f}(\mathbf{x}, \mathbf{v}) = -k(\mathbf{x} - \mathbf{x}_0) - \mu\mathbf{v}$$

stiffness of spring

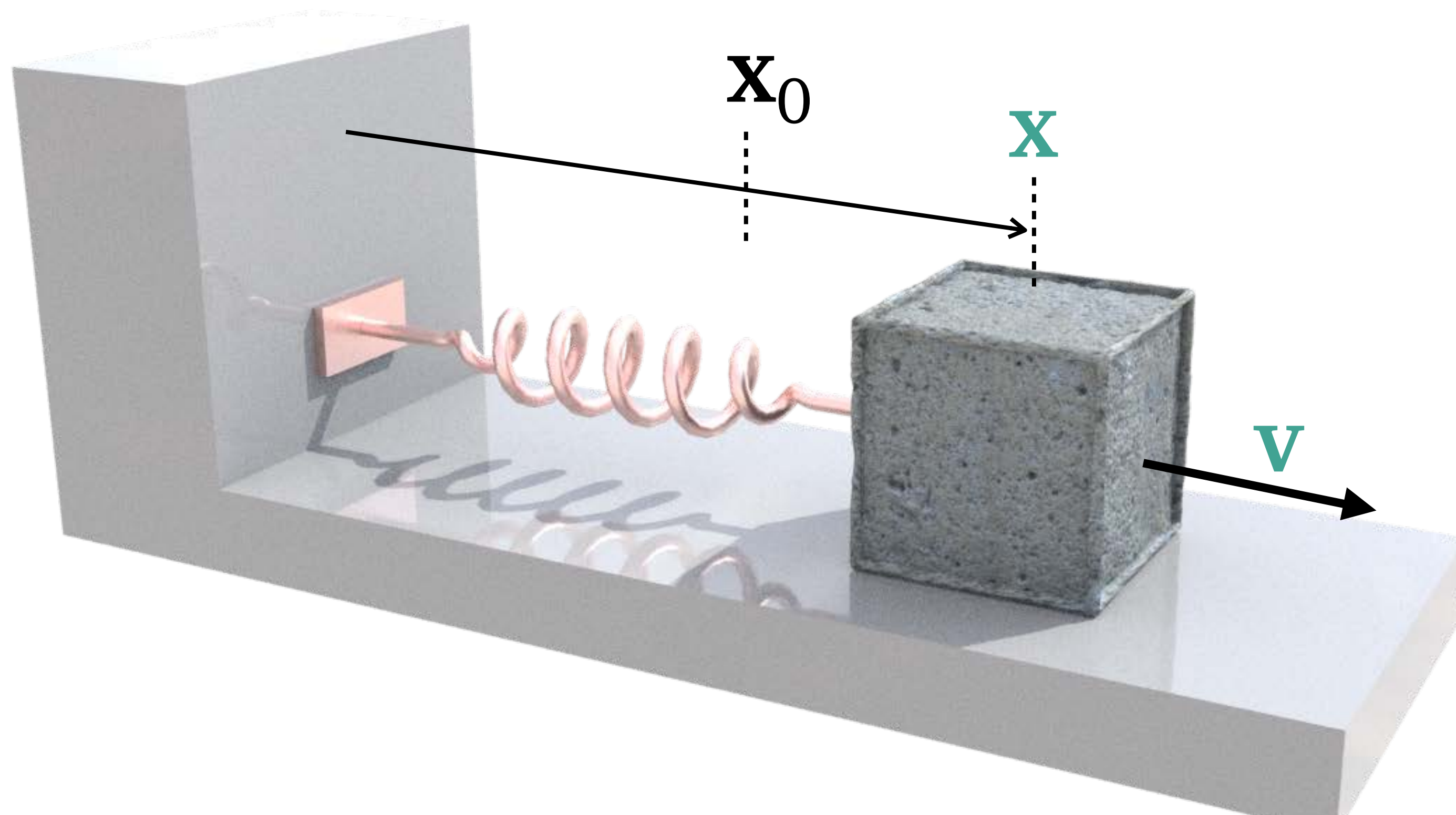


Example

- Force

$$\mathbf{f}(\mathbf{x}, \mathbf{v}) = -k(\mathbf{x} - \mathbf{x}_0) - \boxed{\mu} \mathbf{v}$$

friction



Example

- Force

$$\mathbf{f}(\mathbf{x}, \mathbf{v}) = -k(\mathbf{x} - \mathbf{x}_0) - \mu\mathbf{v}$$

- Equations of motion

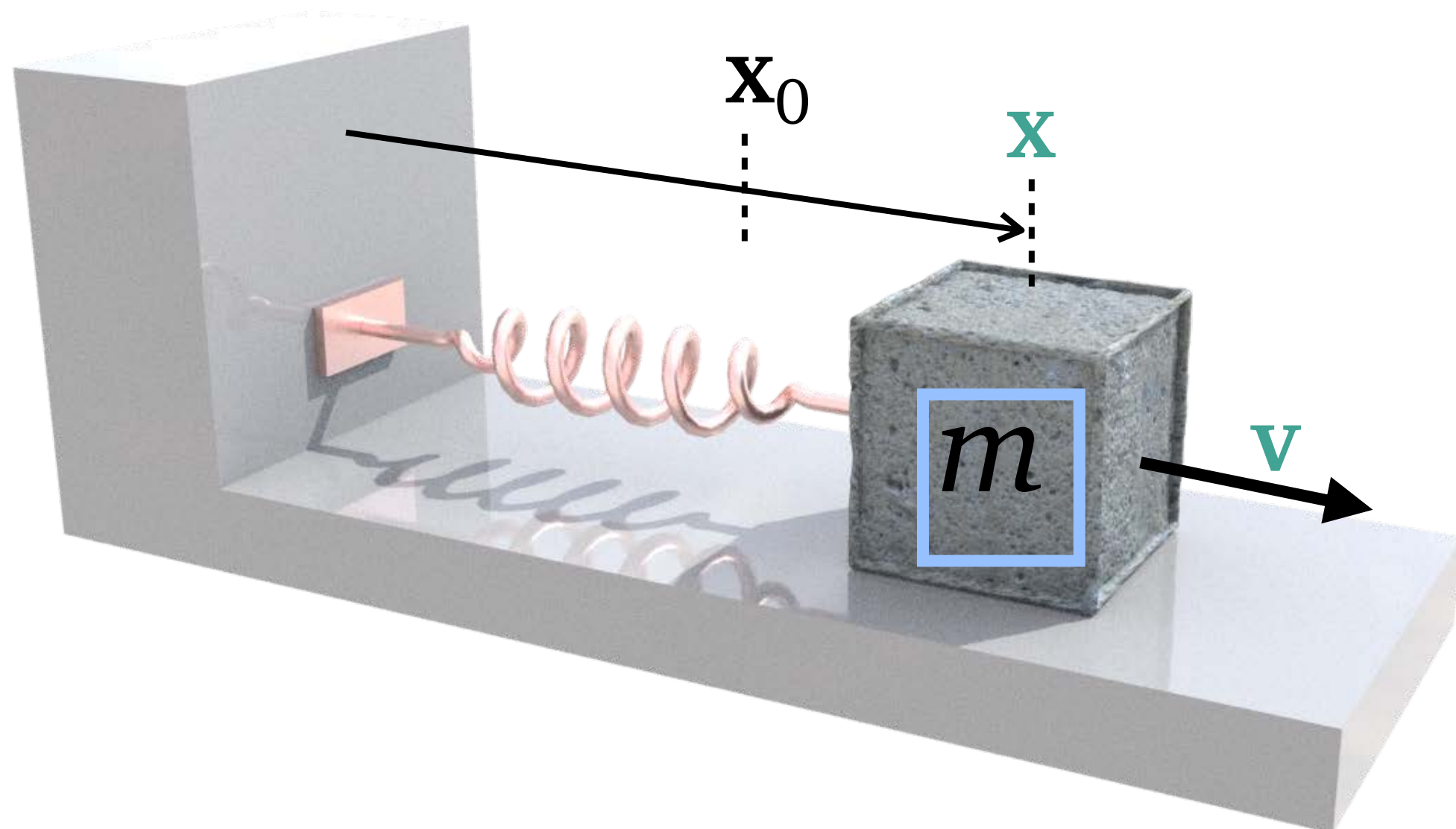
$$\frac{d\mathbf{x}}{dt} = \mathbf{v}$$

relationship between
position and velocity

$$\frac{d\mathbf{v}}{dt} = \frac{1}{m} \mathbf{f}(\mathbf{x}, \mathbf{v})$$

relationship between
acceleration and force

mass



Example

- Force

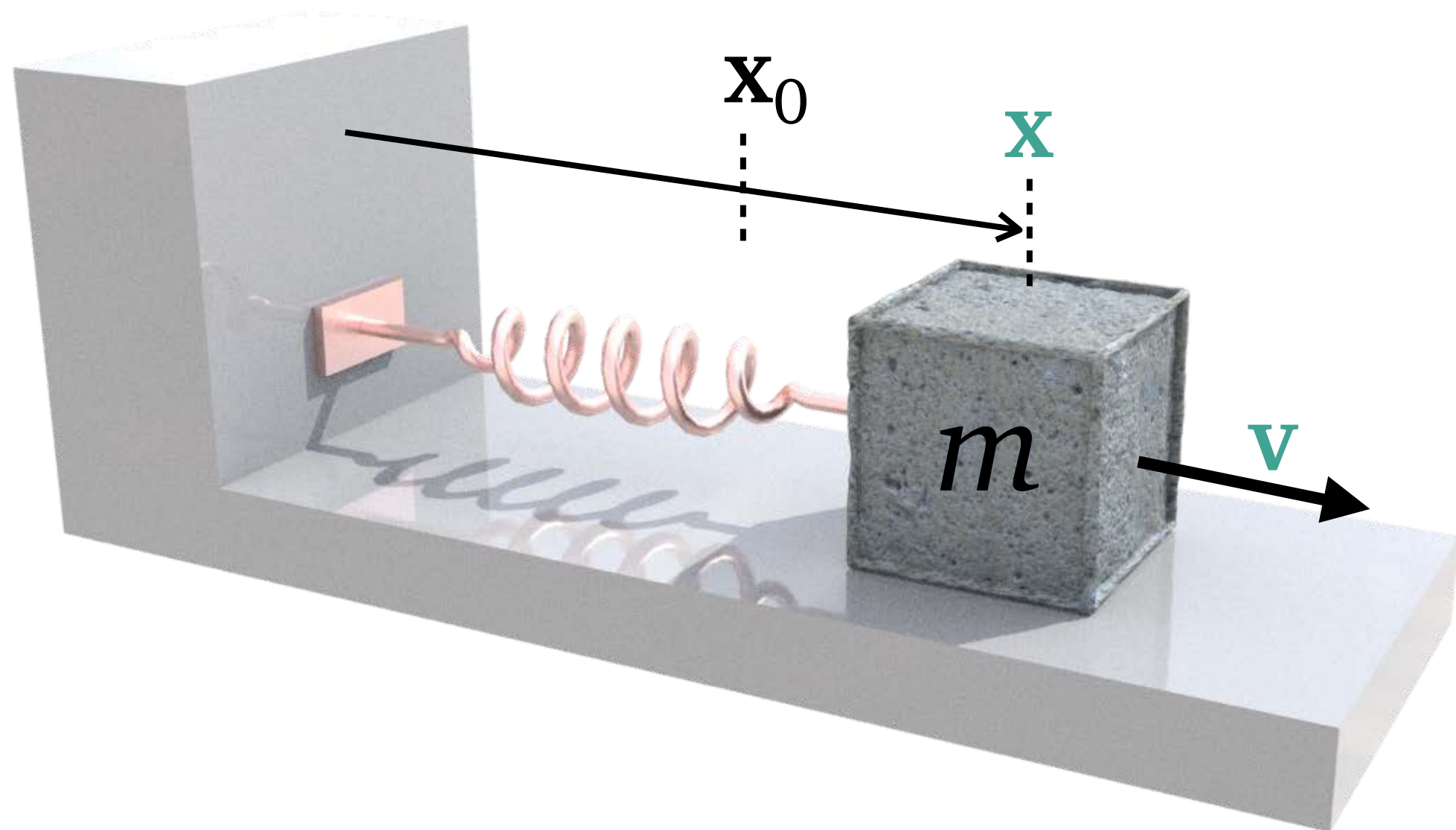
$$\mathbf{f}(\mathbf{x}, \mathbf{v}) = -k(\mathbf{x} - \mathbf{x}_0) - \mu\mathbf{v}$$

- Equations of motion

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}$$

$$\frac{d\mathbf{v}}{dt} = \frac{1}{m}\mathbf{f}(\mathbf{x}, \mathbf{v})$$

$$= -\frac{k}{m}(\mathbf{x} - \mathbf{x}_0) - \frac{\mu}{m}\mathbf{v}$$



Example

- We use the overhead dots to indicate time derivatives

$$\frac{d\mathbf{x}(t)}{dt} = \dot{\mathbf{x}}(t)$$

$$\frac{d^2\mathbf{x}(t)}{dt^2} = \ddot{\mathbf{x}}(t)$$

Example

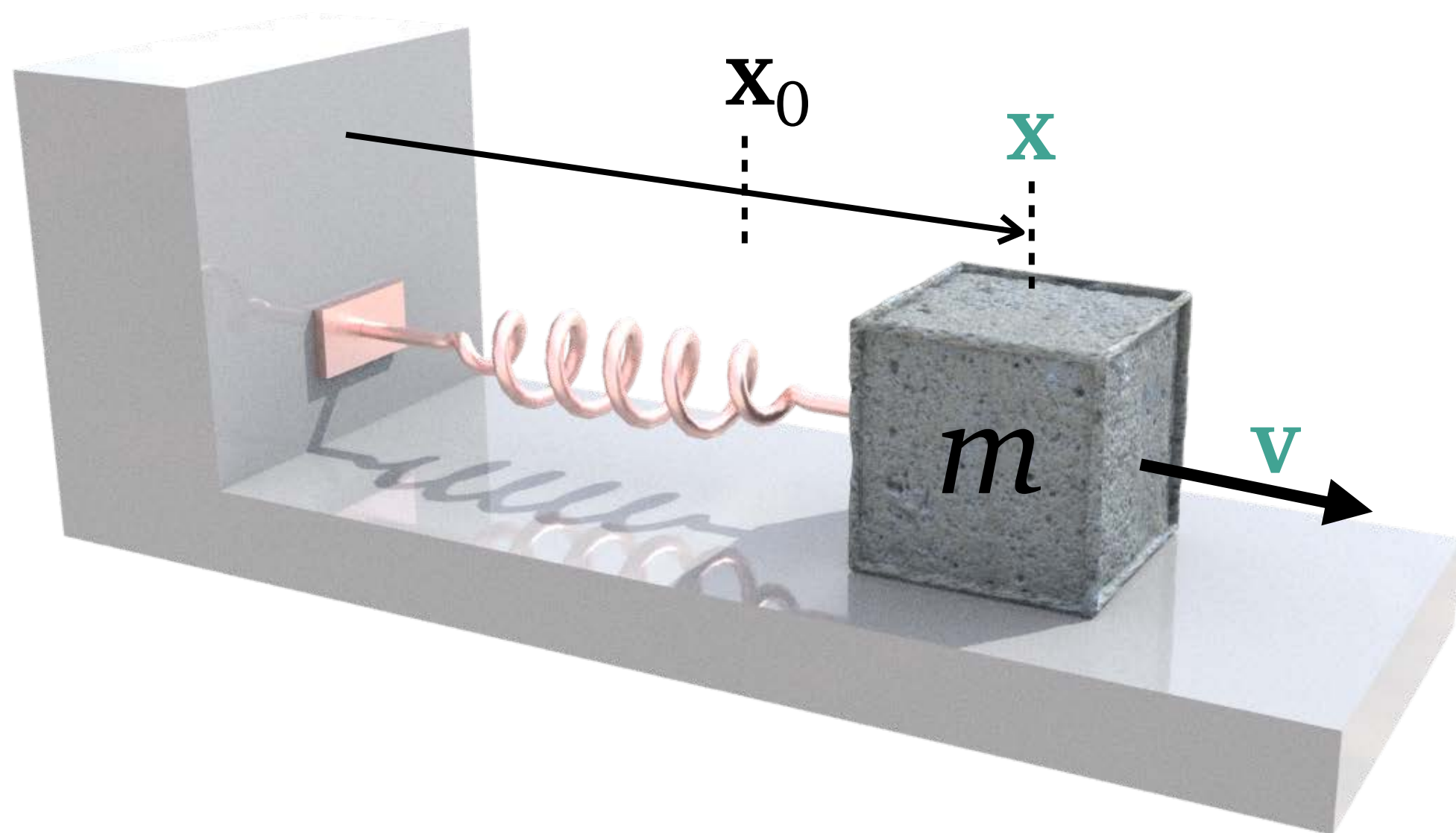
- Equations of motion

$$\dot{\mathbf{x}} = \mathbf{v}$$

$$\dot{\mathbf{v}} = -\frac{k}{m}(\mathbf{x} - \mathbf{x}_0) - \frac{\mu}{m}\mathbf{v}$$

- Substitute \mathbf{v} :

$$\ddot{\mathbf{x}} = -\frac{k}{m}(\mathbf{x} - \mathbf{x}_0) - \frac{\mu}{m}\dot{\mathbf{x}}$$



- ▶ Equation involving \mathbf{x} and its derivatives
- ▶ This is called an *ordinary differential equation (ODE)*

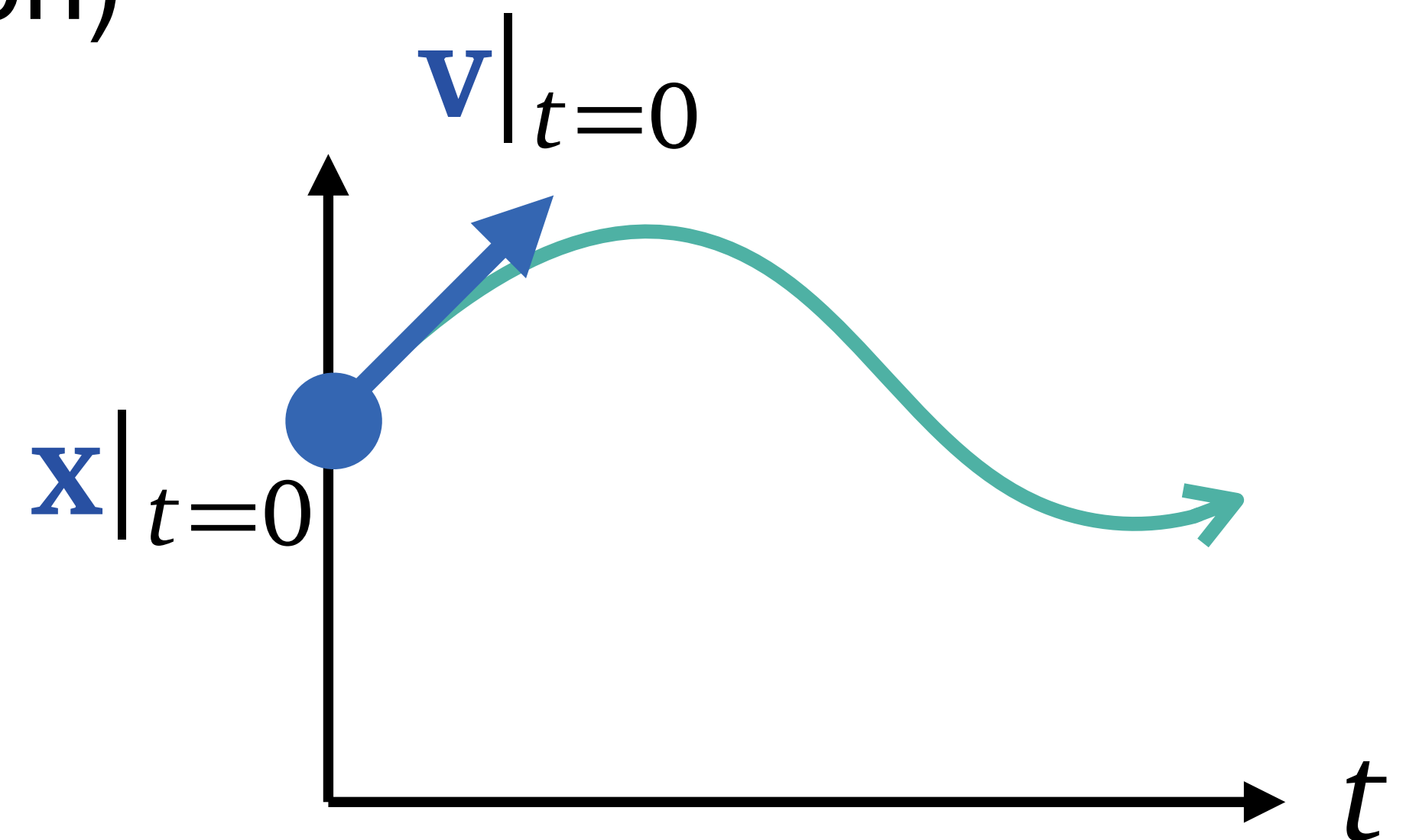
Types of problems

- Equations of motion $\ddot{\mathbf{x}} = -\frac{k}{m}(\mathbf{x} - \mathbf{x}_0) - \frac{\mu}{m}\dot{\mathbf{x}}$
 - ▶ This is called an *ordinary differential equation (ODE)*

- Initial value problem (forward simulation)

- ▶ Given initial conditions
i.e. the values of $\mathbf{x}|_{t=0}$, $\mathbf{v}|_{t=0}$
Extend them into function of time

$$\mathbf{x}(t), \mathbf{v}(t) \quad t \geq 0$$

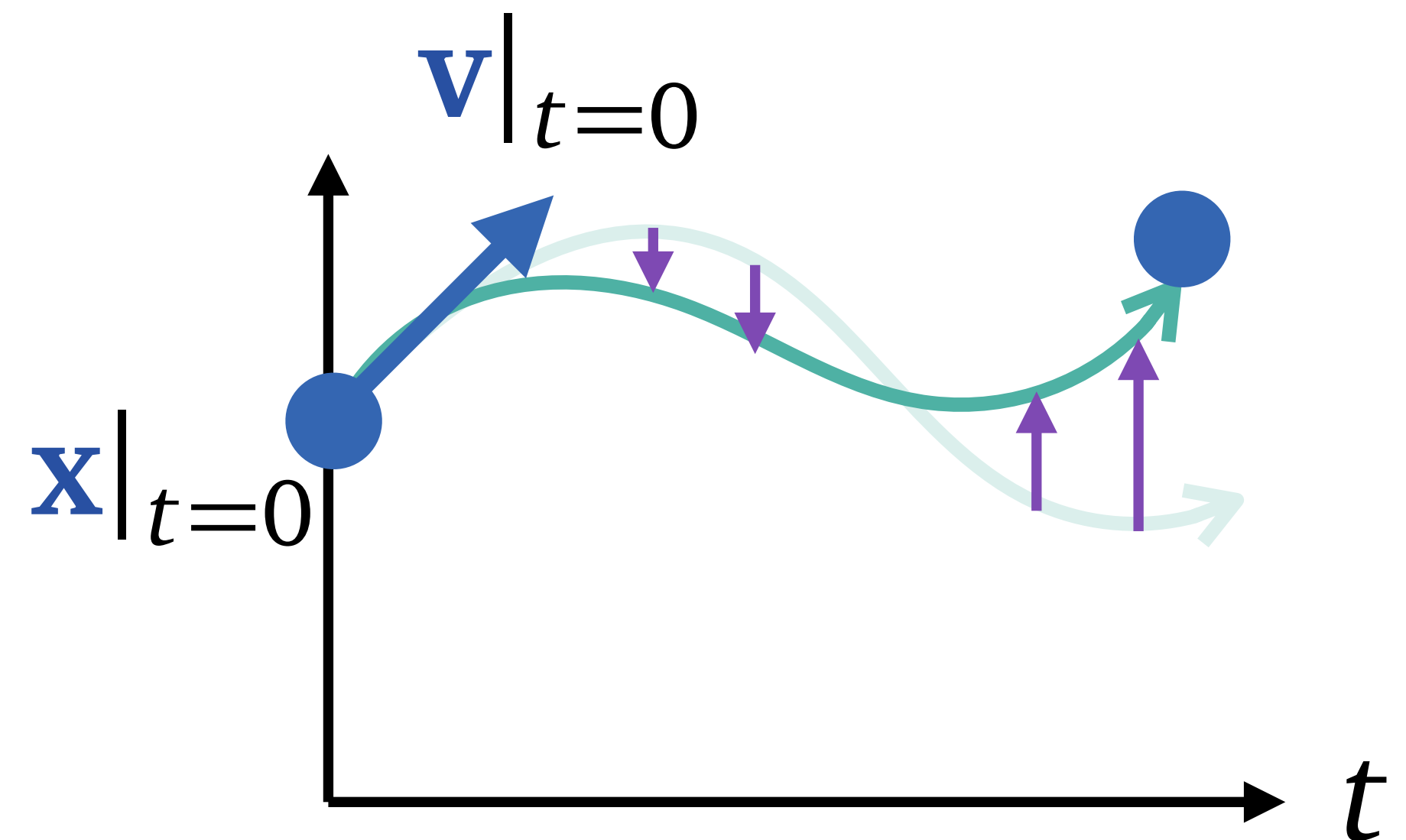


Types of problems

- Equations of motion $\ddot{\mathbf{x}} = -\frac{k}{m}(\mathbf{x} - \mathbf{x}_0) - \frac{\mu}{m}\dot{\mathbf{x}}$
 - ▶ This is called an *ordinary differential equation (ODE)*

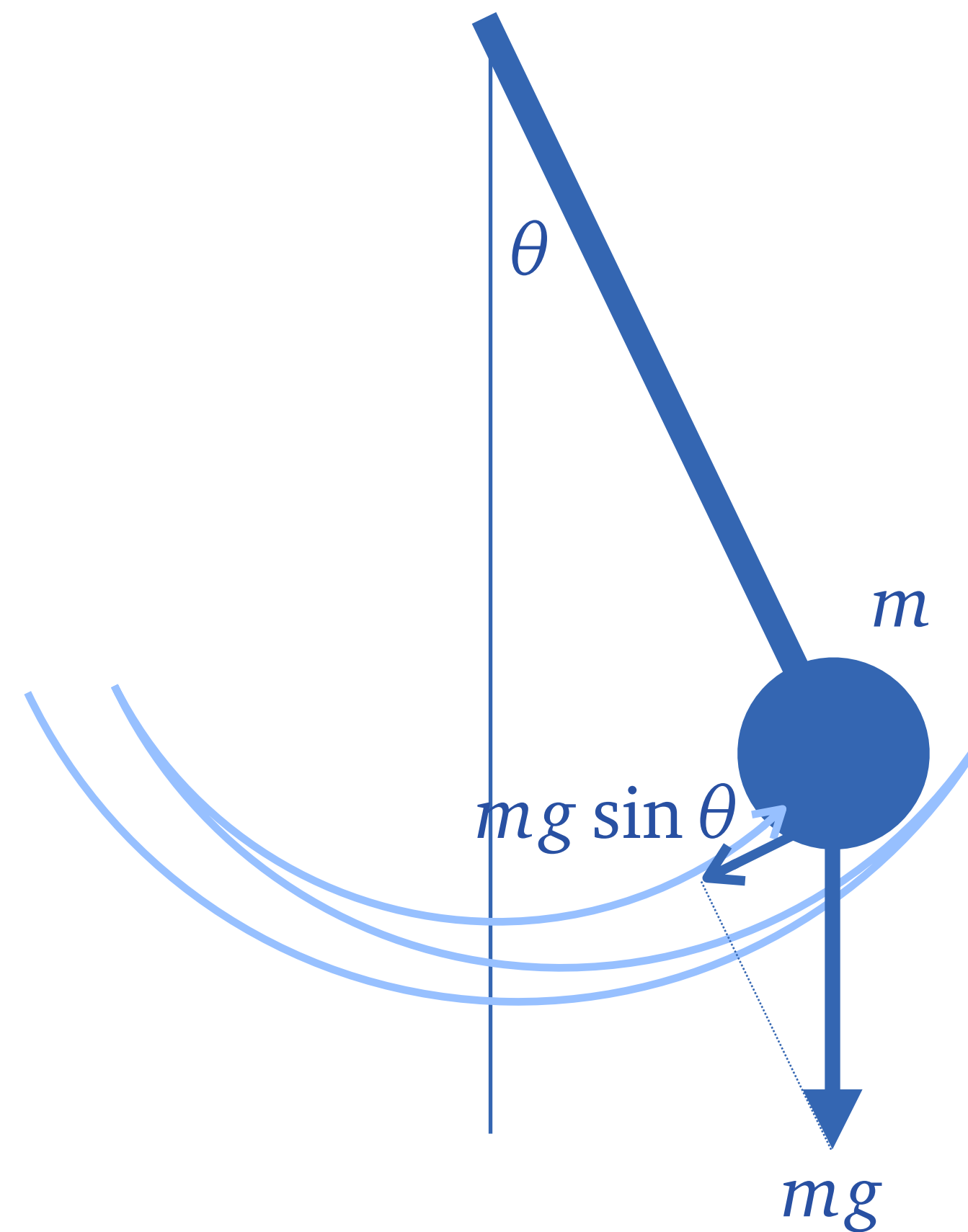
- Control problem

- ▶ Given desired location to arrive at some future time, find minimal correction force to achieve the goal.
- ▶ Robotics, control systems, physics-based keyframe animations



Another example

- Pendulum



$$m\ddot{\theta} = -mg \sin \theta$$

Solving ODE Numerically

- Simulation, Physics, Math
- Getting started: $F = ma$
- Solve ODEs numerically

ODE

- Derive the differential equation (ODE) from physical laws
- Solve the differential equation (ODE)
 - ▶ Hard to solve it by hand most of the time
 - ▶ Numerical method is needed

ODE

- Given any differential equation, for example,

$$\ddot{x} + \dot{x}\ddot{x} + \sin(\ddot{x}) = 1$$

- Convert it into a 1st order system of ODEs (involving at most first derivative)

- ▶ Give each derivative a separate name (except for the highest order derivative) $\mathbf{v} = \dot{\mathbf{x}}$ $\mathbf{a} = \ddot{\mathbf{x}}$

- ▶ Then
$$\begin{cases} \dot{\mathbf{x}} = \mathbf{v} \\ \dot{\mathbf{v}} = \mathbf{a} \\ \dot{\mathbf{a}} = 1 - \mathbf{a}\mathbf{v} - \sin(\mathbf{a}) \end{cases}$$

ODE

- Given any differential equation, for example,

$$\ddot{x} + \dot{x}\ddot{x} + \sin(\ddot{x}) = 1$$

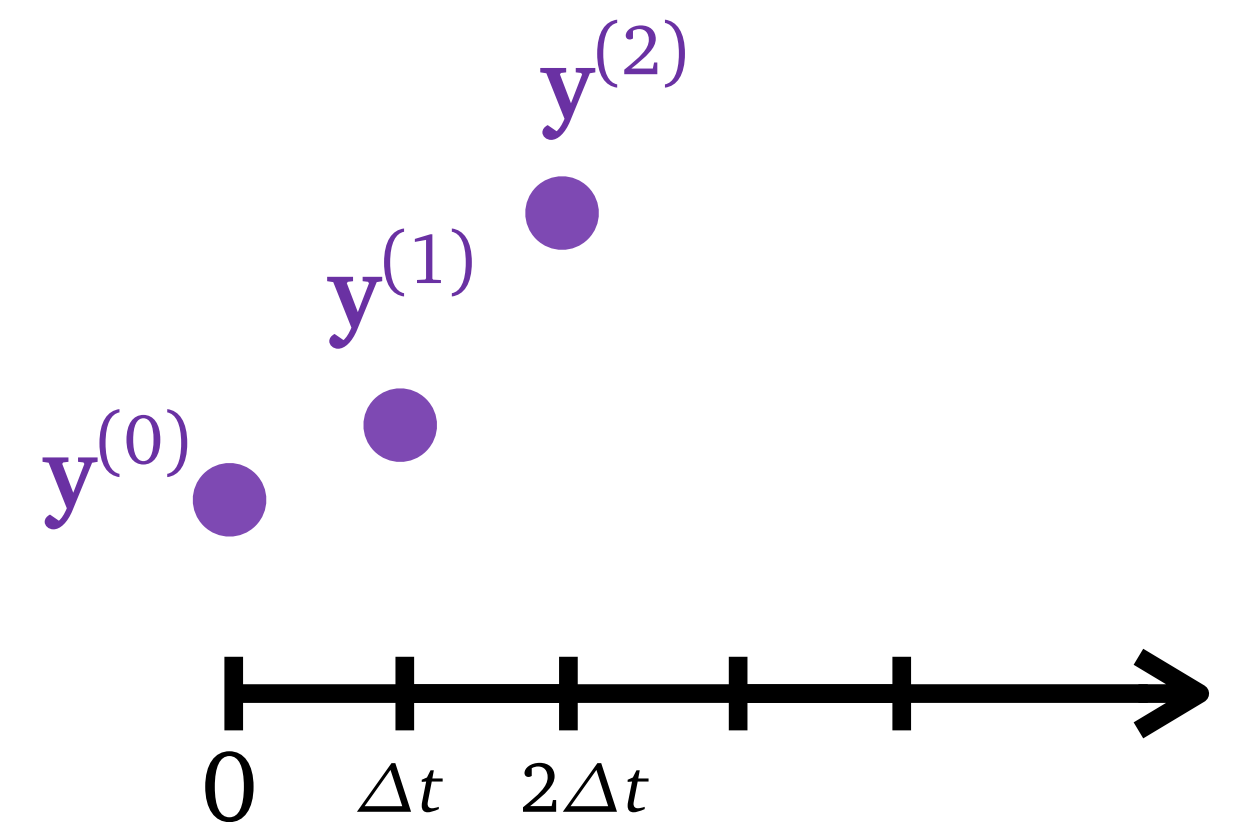
▶ Then
$$\begin{cases} \dot{x} = v \\ \dot{v} = a \\ \dot{a} = 1 - av - \sin(a) \end{cases}$$

▶ Let
$$y = \begin{bmatrix} x \\ v \\ a \end{bmatrix}$$
 ODE becomes $\dot{y} = f(y)$

Numerical ODE

- Generic ODE $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$
- Discretize time into time-frames $\mathbf{y}^{(n)} = \mathbf{y}(n\Delta t)$
- **(Forward) Euler method** $\frac{\mathbf{y}^{(n+1)} - \mathbf{y}^{(n)}}{\Delta t} \approx \mathbf{f}(\mathbf{y}^{(n)})$

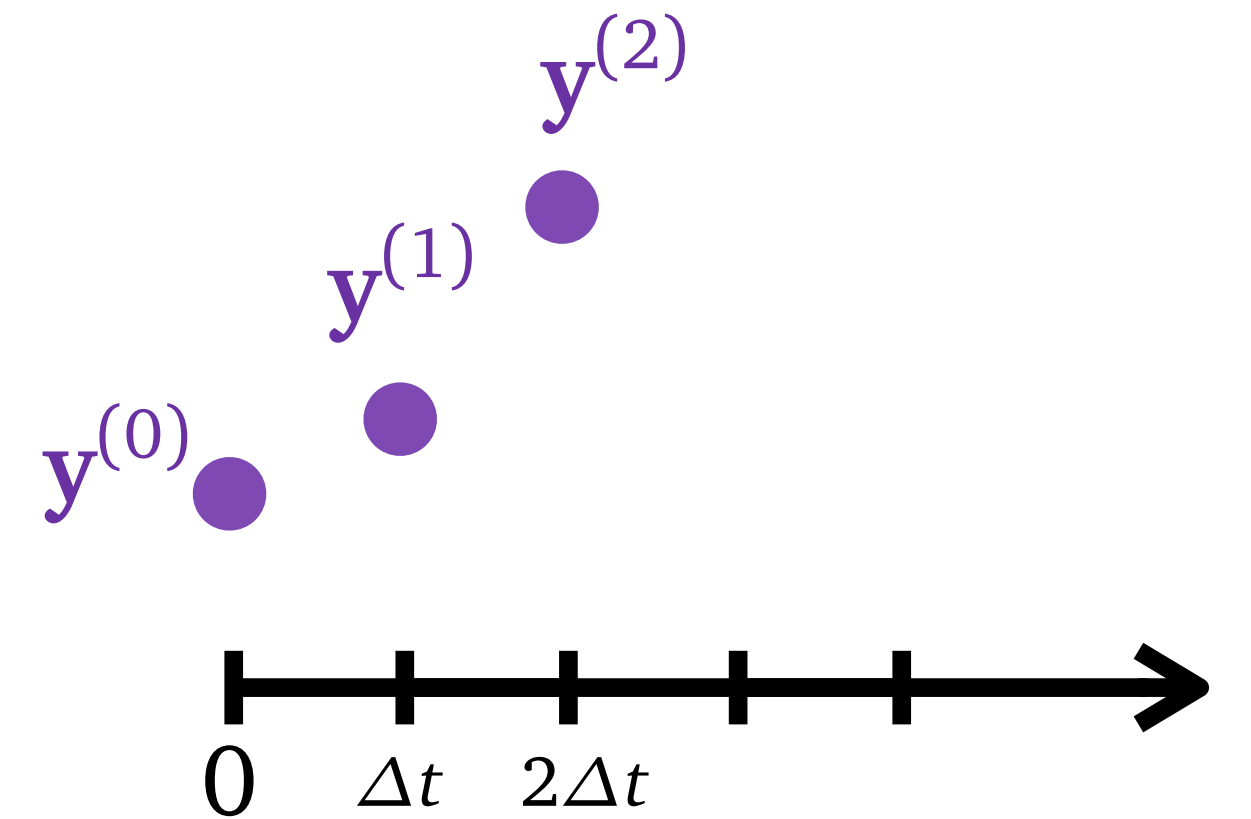
$$\mathbf{y}^{(n+1)} \approx \mathbf{y}^{(n)} + \Delta t \cdot \mathbf{f}(\mathbf{y}^{(n)})$$



Numerical ODE

- **(Forward) Euler method** $\frac{y^{(n+1)} - y^{(n)}}{\Delta t} \approx \mathbf{f}(y^{(n)})$

$$y^{(n+1)} \approx y^{(n)} + \Delta t \cdot \mathbf{f}(y^{(n)})$$

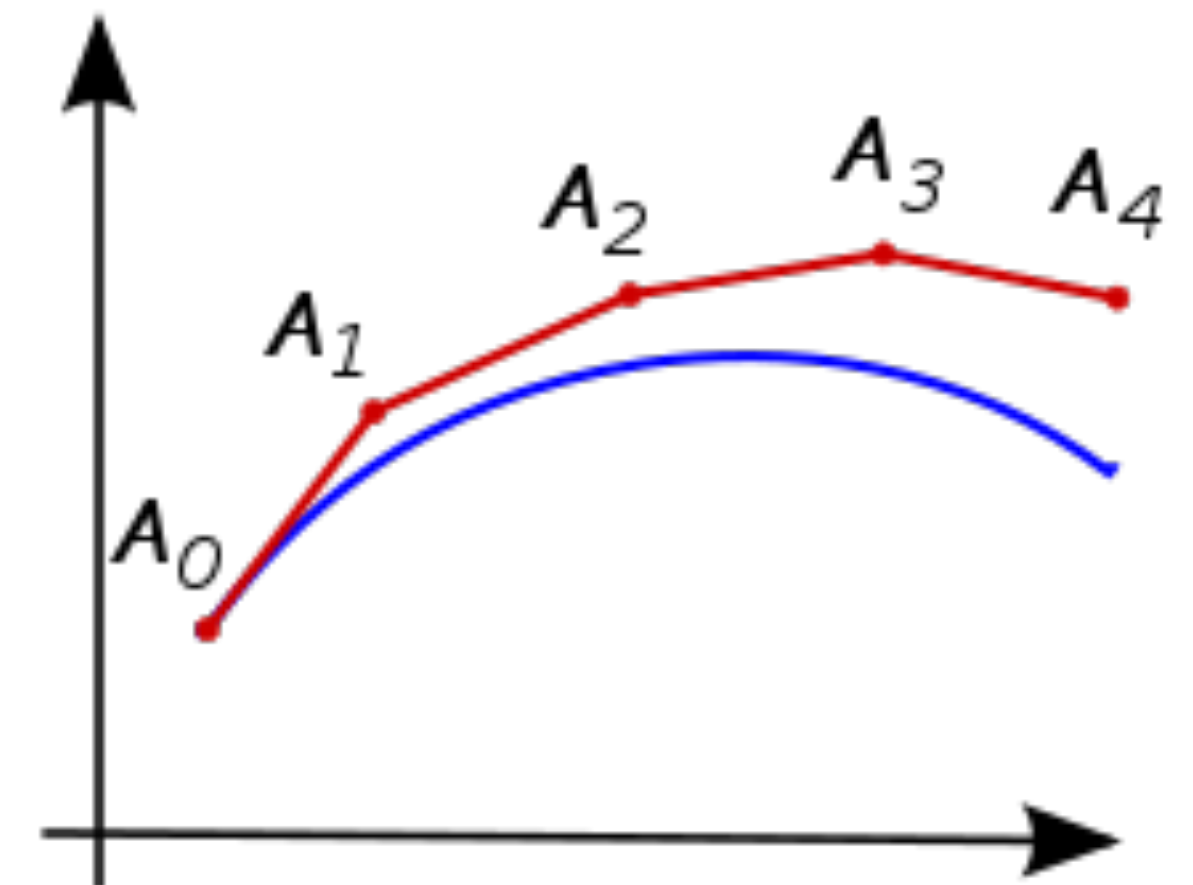


- Advantages

- ▶ There is an **explicit** formula to plug in old state to get new state
- ▶ Fast and simple

- Limitations

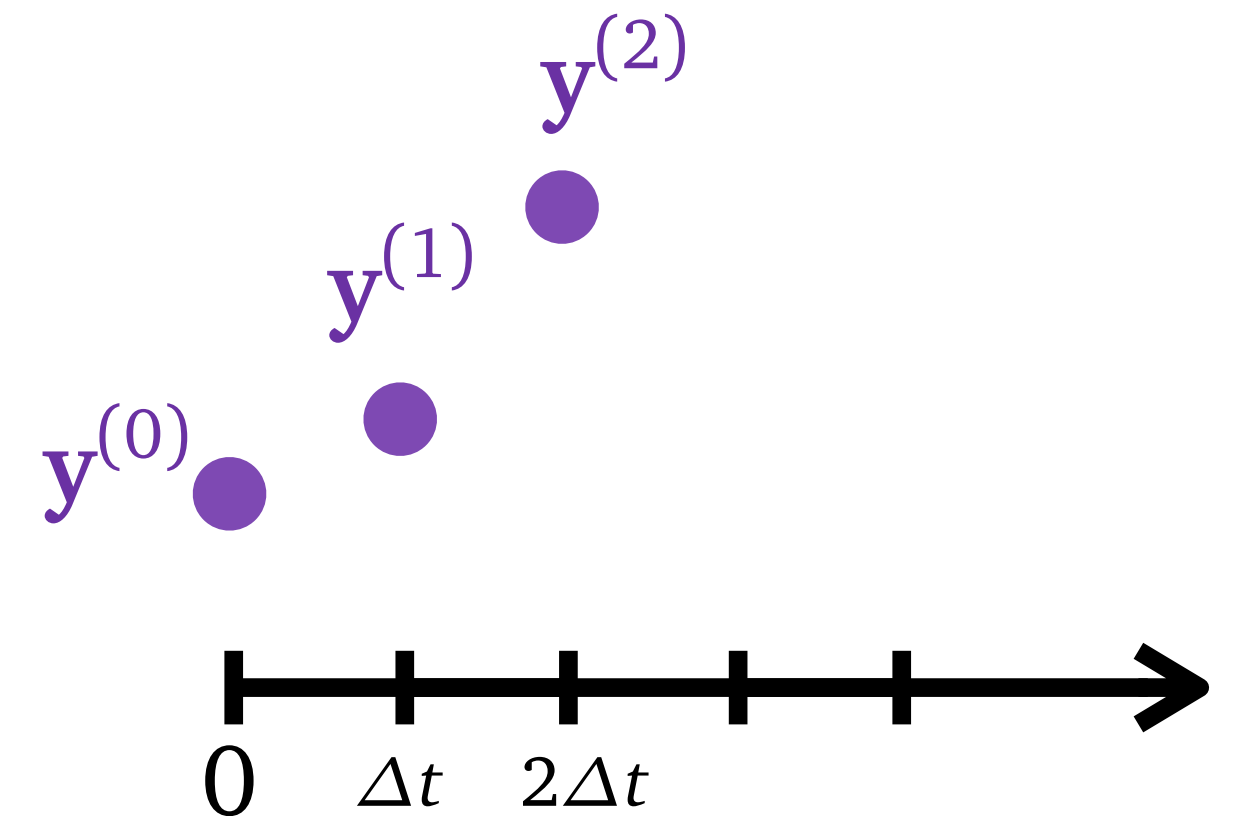
- ▶ Not very accurate unless Δt is tiny
- ▶ Can be energy increasing (unphysical)



Numerical ODE

- **(Forward) Euler method** $\frac{y^{(n+1)} - y^{(n)}}{\Delta t} \approx \mathbf{f}(y^{(n)})$

$$y^{(n+1)} \approx y^{(n)} + \Delta t \cdot \mathbf{f}(y^{(n)})$$



- **Backward Euler method**

$$\frac{y^{(n+1)} - y^{(n)}}{\Delta t} = \mathbf{f}(y^{(n+1)})$$

- **Limitations**

- ▶ Not very accurate unless Δt is tiny
- ▶ Have to solve for new state (**implicit**) instead of explicit update

Numerical ODE

- **Backward Euler method**

$$\frac{y^{(n+1)} - y^{(n)}}{\Delta t} = \mathbf{f}(y^{(n+1)})$$

- Limitations

- ▶ Not very accurate unless Δt is tiny
- ▶ Have to solve for new state (**implicit**) instead of explicit update

- Advantages

- ▶ Energy decreasing (dissipating), which looks physical
- ▶ Can take larger time steps Δt without instability
- ▶ Can incorporate collision (just add constraint to the implicit solves)

Numerical ODE

- **Euler method** $y^{(n+1)} \approx y^{(n)} + \Delta t \cdot \mathbf{f}(y^{(n)})$ $\dot{y} = \mathbf{f}(y)$

- **Runge–Kutta method (RK4)** (Accurate, stable, explicit)

$$\mathbf{k}_1 = \mathbf{f}(y^{(n)})$$

$$\mathbf{k}_2 = \mathbf{f}(y^{(n)} + \frac{\Delta t}{2} \mathbf{k}_1)$$

$$\mathbf{k}_3 = \mathbf{f}(y^{(n)} + \frac{\Delta t}{2} \mathbf{k}_2)$$

$$\mathbf{k}_4 = \mathbf{f}(y^{(n)} + \Delta t \mathbf{k}_3)$$

$$y^{(n+1)} = y^{(n)} + \frac{\Delta t}{6} (\mathbf{k}_1 + 2\mathbf{k}_2 + 2\mathbf{k}_3 + \mathbf{k}_4)$$

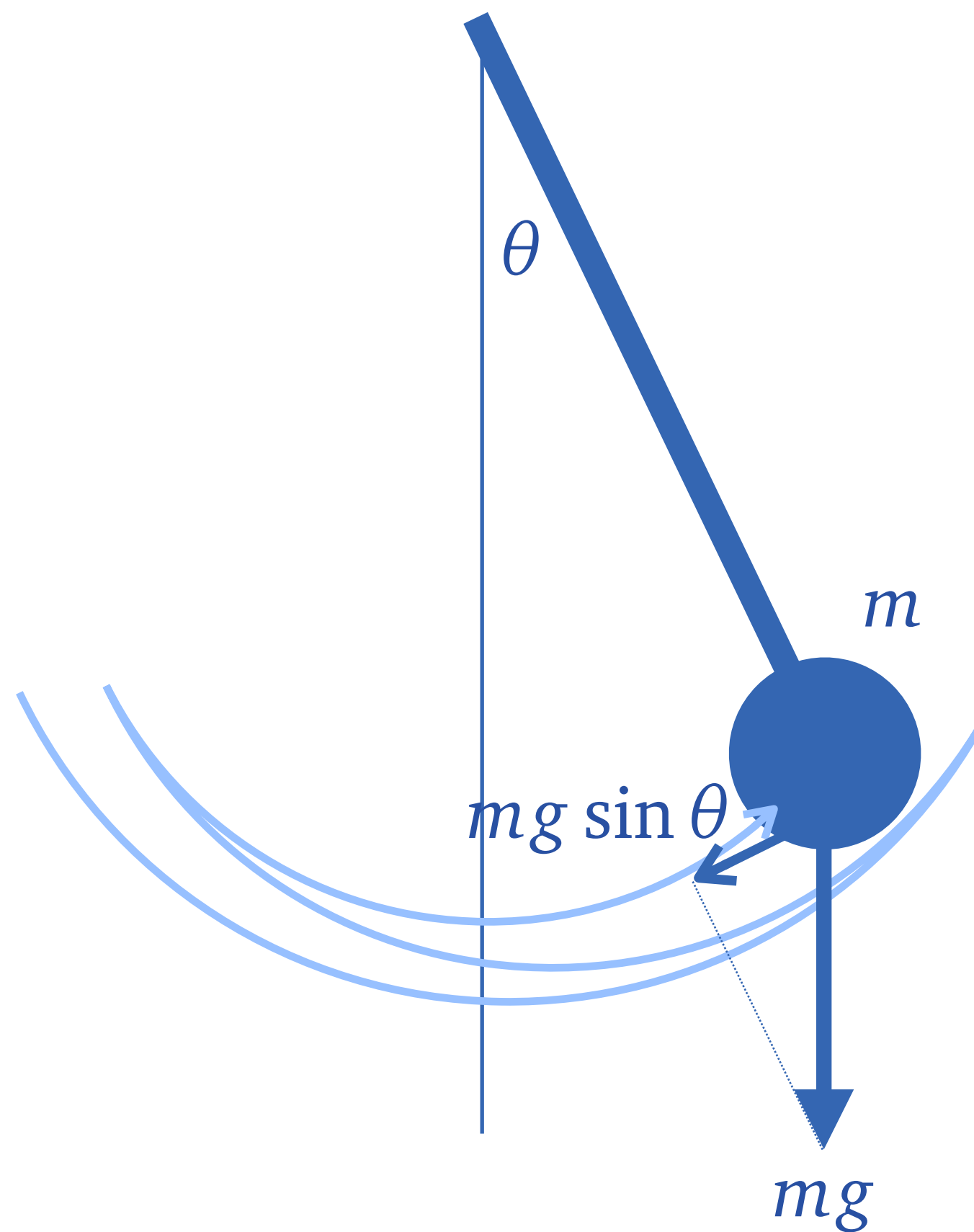
(collision handling is not as elegant as backward Euler)

Numerical ODE

- In most cases the RK4 method works very well
- Sometimes the underlying physical system has additional structures (energy conservation, momentum conservation)
- Special algorithm (non-RK4) aims at preserving energy or momentum
 - ▶ Variational integrator
 - ▶ Symplectic integrator
 - ▶ Lie group integrator

Pendulum equation

Example: pendulum equation.

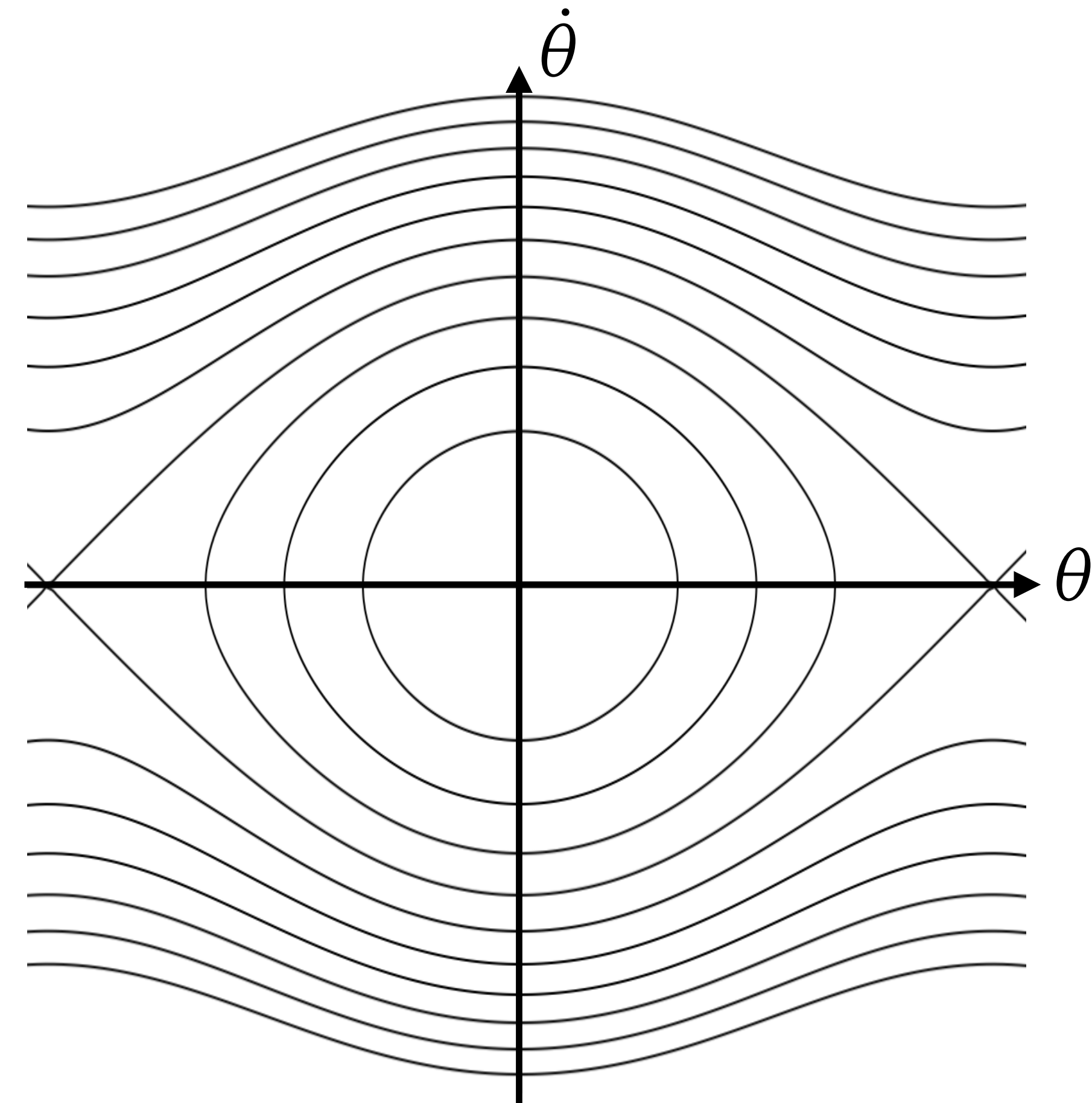


$$m\ddot{\theta} = -mg \sin \theta$$

✓ Energy conservation

$$\frac{1}{2}m\dot{\theta}^2 - mg \cos \theta = \text{const}$$

✓ Integrable system



Pendulum equation

Example: pendulum equation.

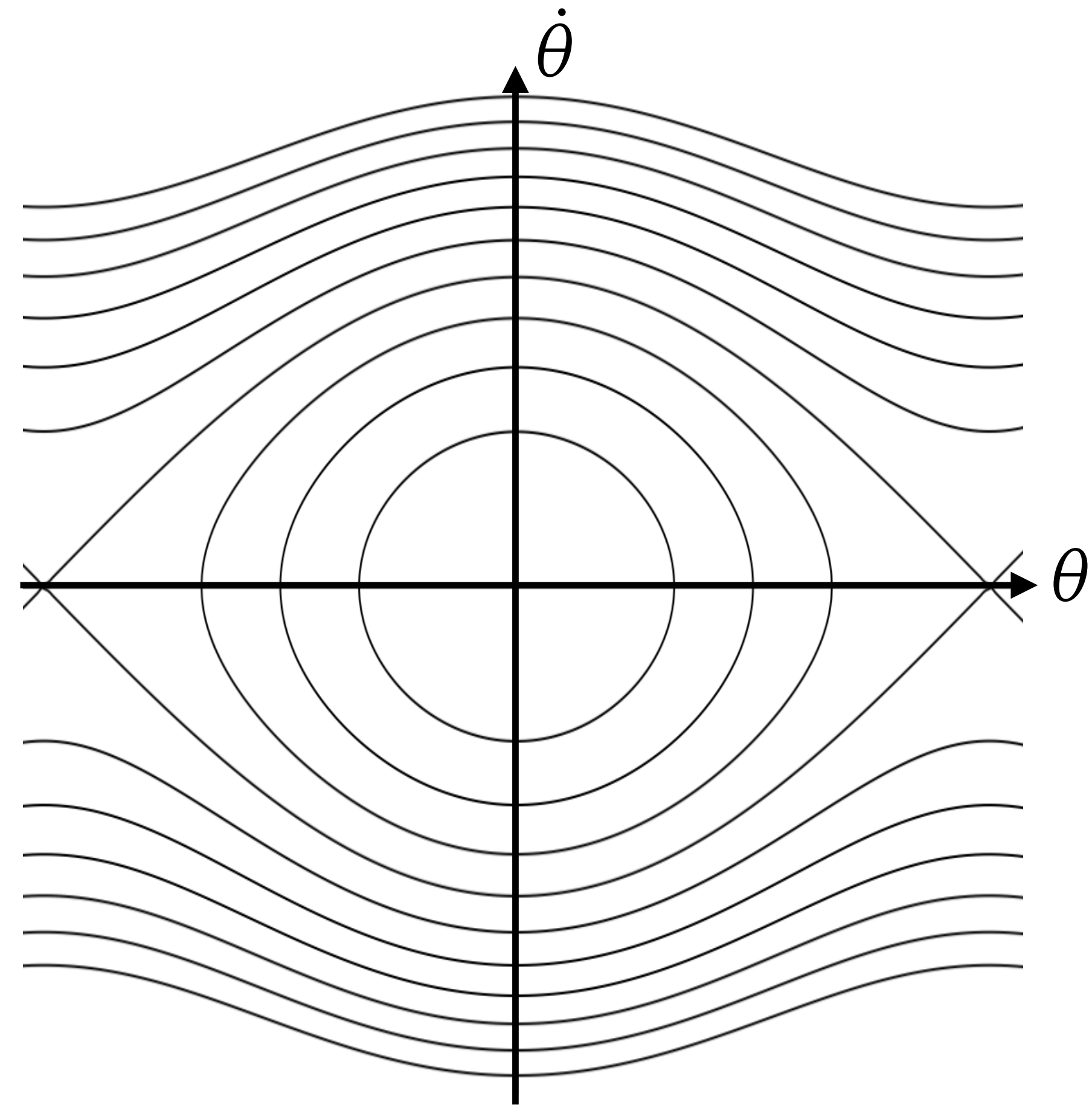
$$m\ddot{\theta} = -mg \sin \theta$$

- High order differential equation solver (4th order Runge–Kutta method)

Given $(\theta_i, v_i = \dot{\theta}_i)$

$$\begin{aligned} \triangleright \quad \theta_{i+1/2}^* &= \theta_i + \frac{\Delta t}{2} v_i & v_{i+1/2}^* &= v_i - \frac{\Delta t}{2} \sin \theta_i \\ \triangleright \quad \theta_{i+1/2}^{**} &= \theta_i + \frac{\Delta t}{2} v_{i+1/2}^* & v_{i+1/2}^{**} &= v_i - \frac{\Delta t}{2} \sin \theta_{i+1/2}^* \\ \triangleright \quad \theta_{i+1}^{***} &= \theta_i + \Delta t v_{i+1/2}^{**} & v_{i+1}^{***} &= v_i - \Delta t \sin \theta_{i+1/2}^{**} \end{aligned}$$

$$\begin{aligned} \text{Output} \quad \theta_{i+1} &= \theta_i + \frac{\Delta t}{6} \left(v_i + 2v_{i+1/2}^* + 2v_{i+1/2}^{**} + v_{i+1}^{***} \right) \\ v_{i+1} &= v_i - \frac{\Delta t}{6} \left(\sin \theta_i + 2 \sin \theta_{i+1/2}^* + 2 \sin \theta_{i+1/2}^{**} + \sin \theta_{i+1}^{***} \right) \end{aligned}$$



Pendulum equation

Example: pendulum equation.

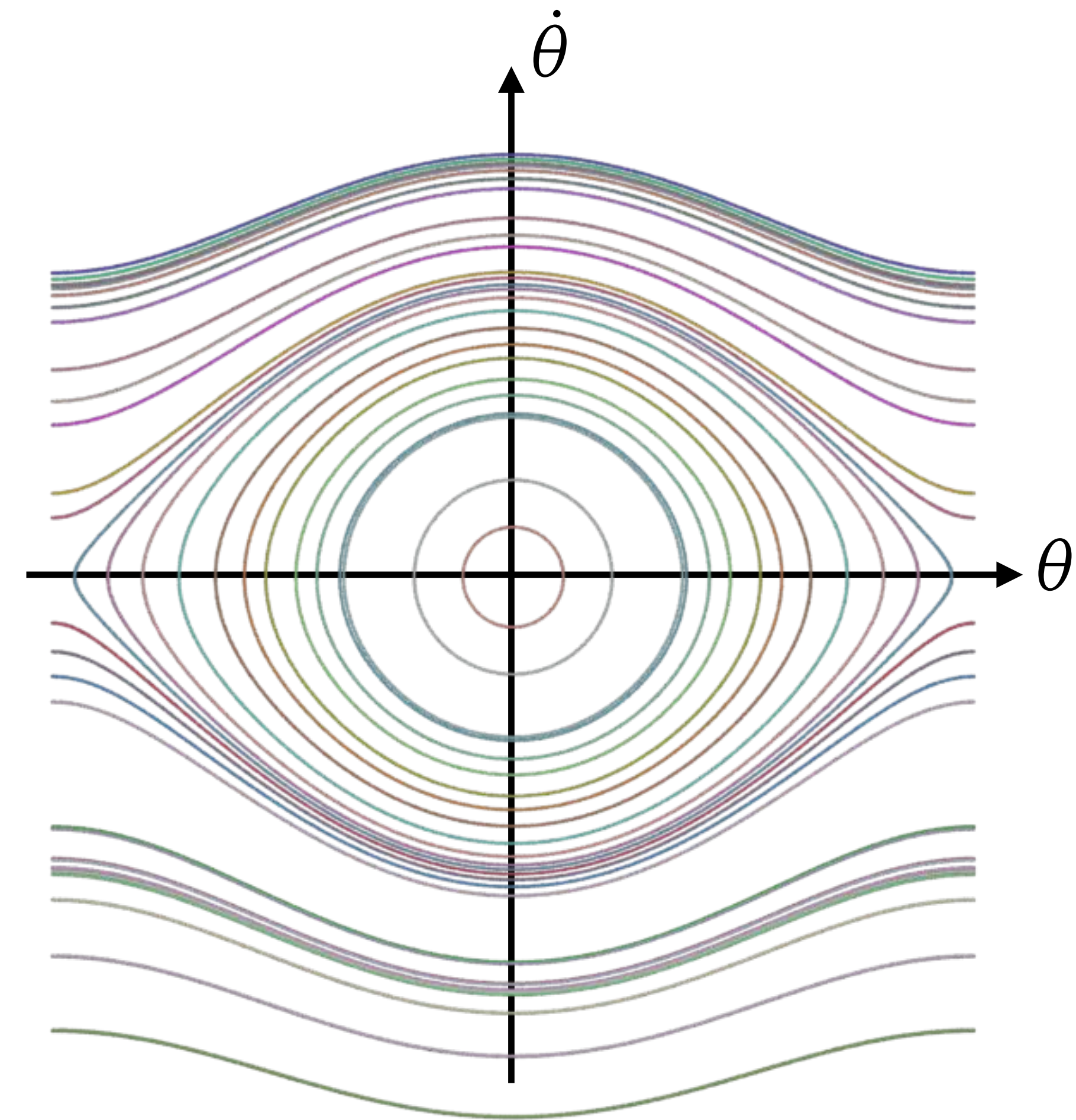
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$\Delta t = 0.1$

Pendulum equation

Example: pendulum equation.

$$m\ddot{\theta} = -mg \sin \theta$$

- High order differential equation solver (4th order Runge–Kutta method)

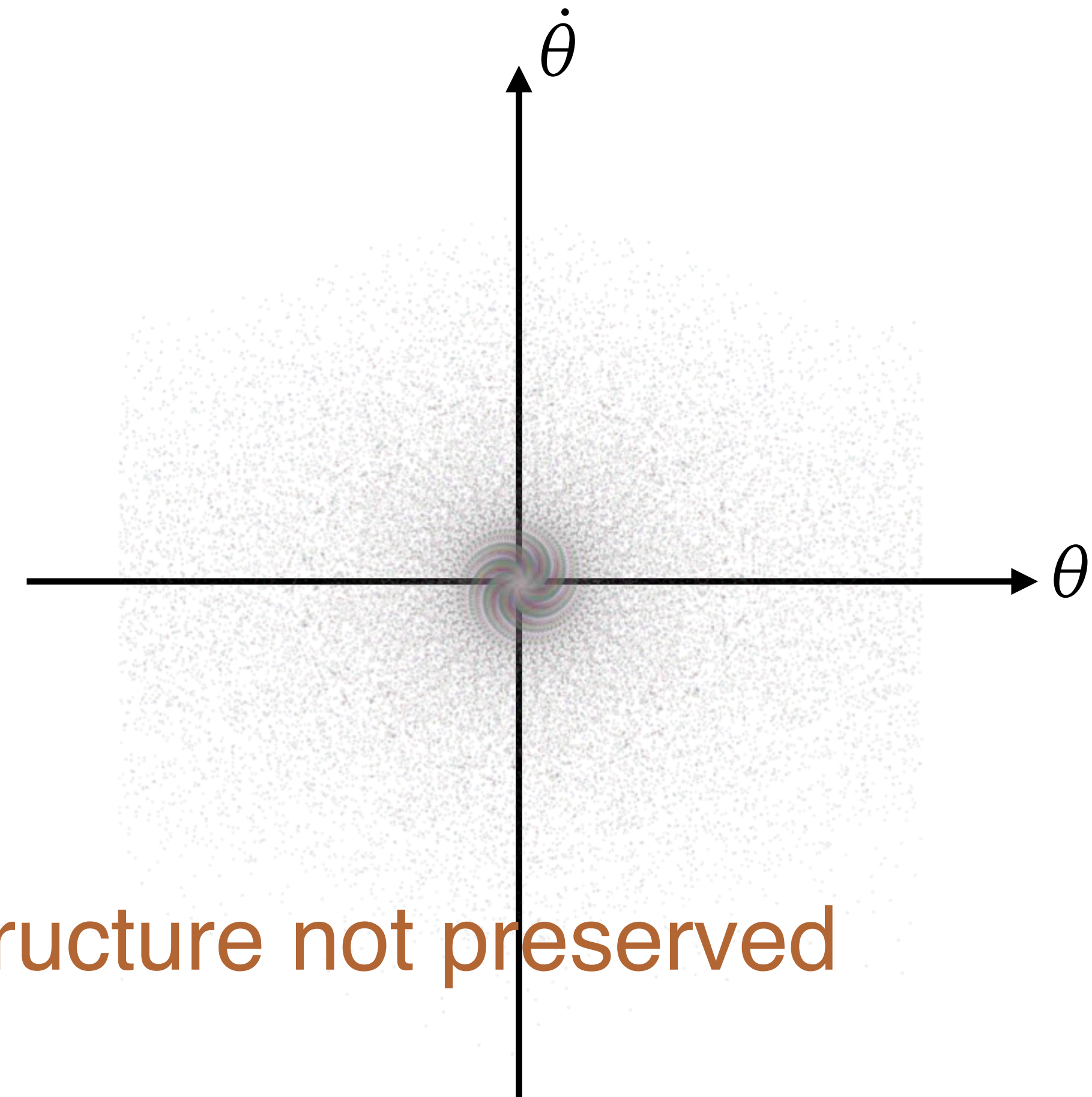
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$$\text{Output } \theta_{i+1} = \theta_i + \frac{\Delta t}{6} \left(v_i + 2v_{i+1/2}^* + 2v_{i+1/2}^{**} + v_{i+1}^{***} \right)$$

$$v_{i+1} = v_i - \frac{\Delta t}{6} \left(\sin \theta_i + 2 \sin \theta_{i+1/2}^* + 2 \sin \theta_{i+1/2}^{**} + \sin \theta_{i+1}^{***} \right)$$

$$\Delta t = 0.9$$



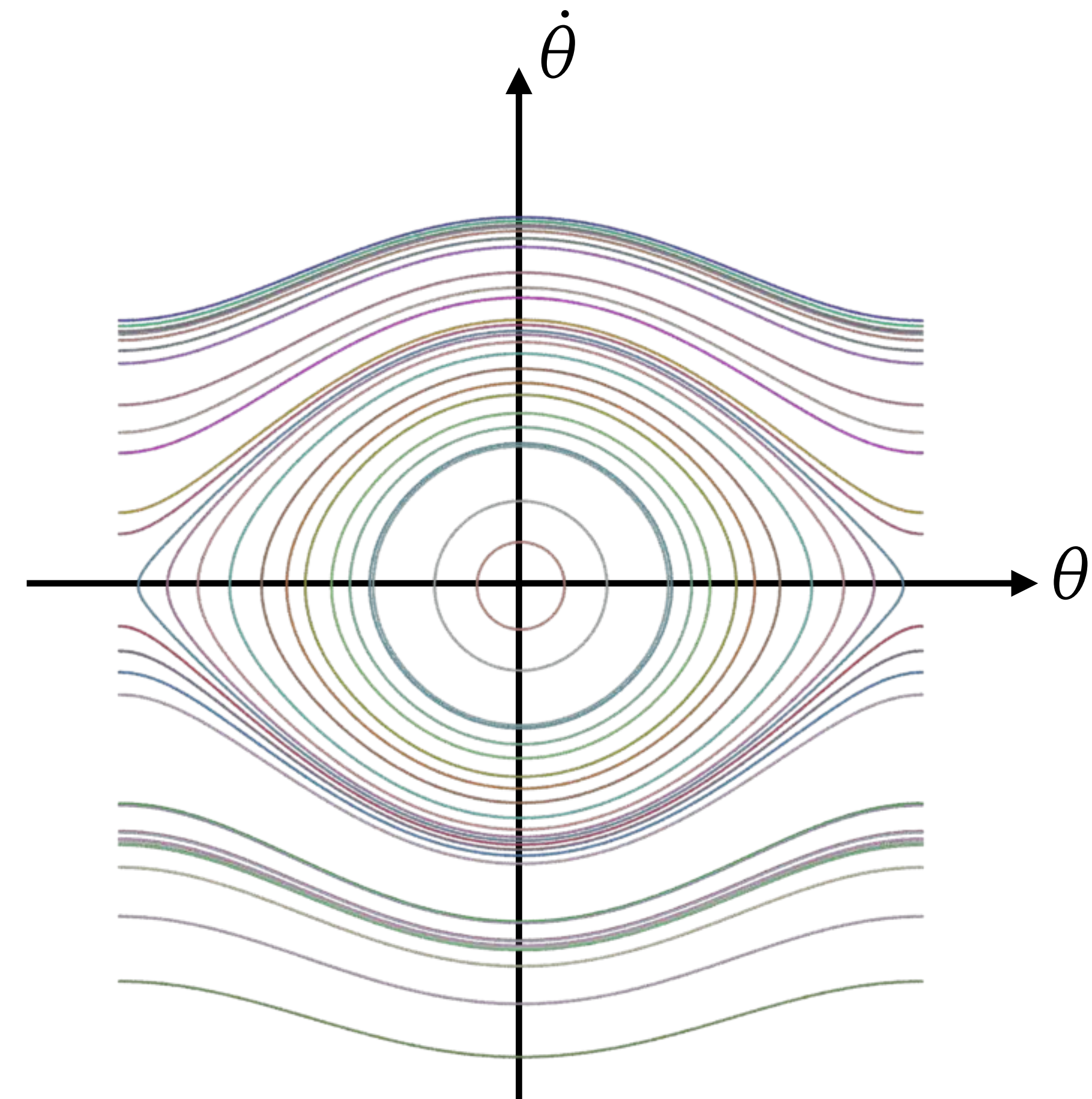
Pendulum equation

Example: pendulum equation.

$$m\ddot{\theta} = -mg \sin \theta$$

- A 2nd order discretization

$$\frac{\theta_{i-1} - 2\theta_i + \theta_{i+1}}{\Delta t^2} = -\sin \theta_i$$



$$\Delta t = 0.1$$

Pendulum equation

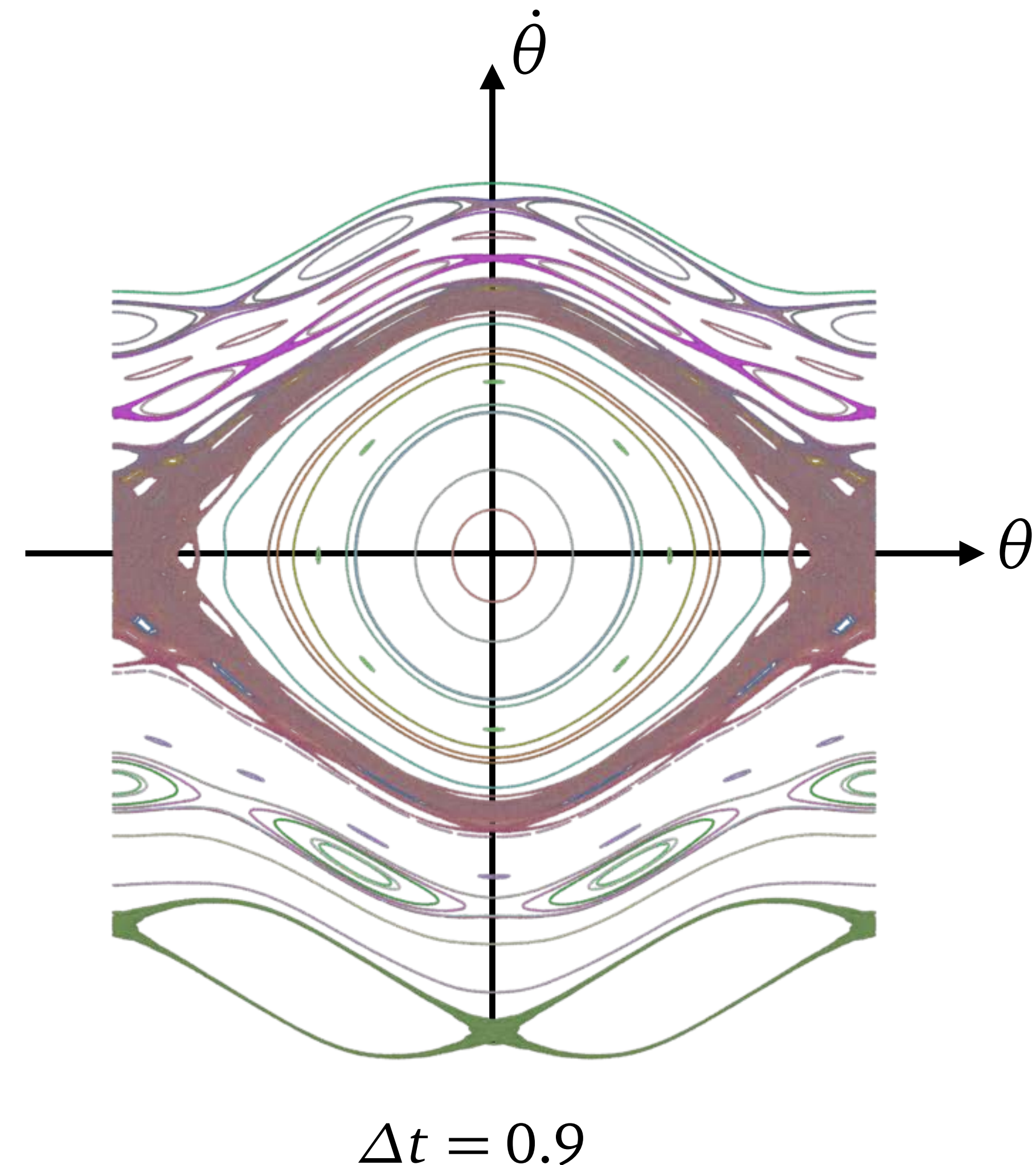
Example: pendulum equation.

$$m\ddot{\theta} = -mg \sin \theta$$

- A 2nd order discretization

$$\frac{\theta_{i-1} - 2\theta_i + \theta_{i+1}}{\Delta t^2} = -\sin \theta_i$$

✓ Energy conservation
in “asteroid belts”



Pendulum equation

Example: pendulum equation.

$$m\ddot{\theta} = -mg \sin \theta$$

Least action principle

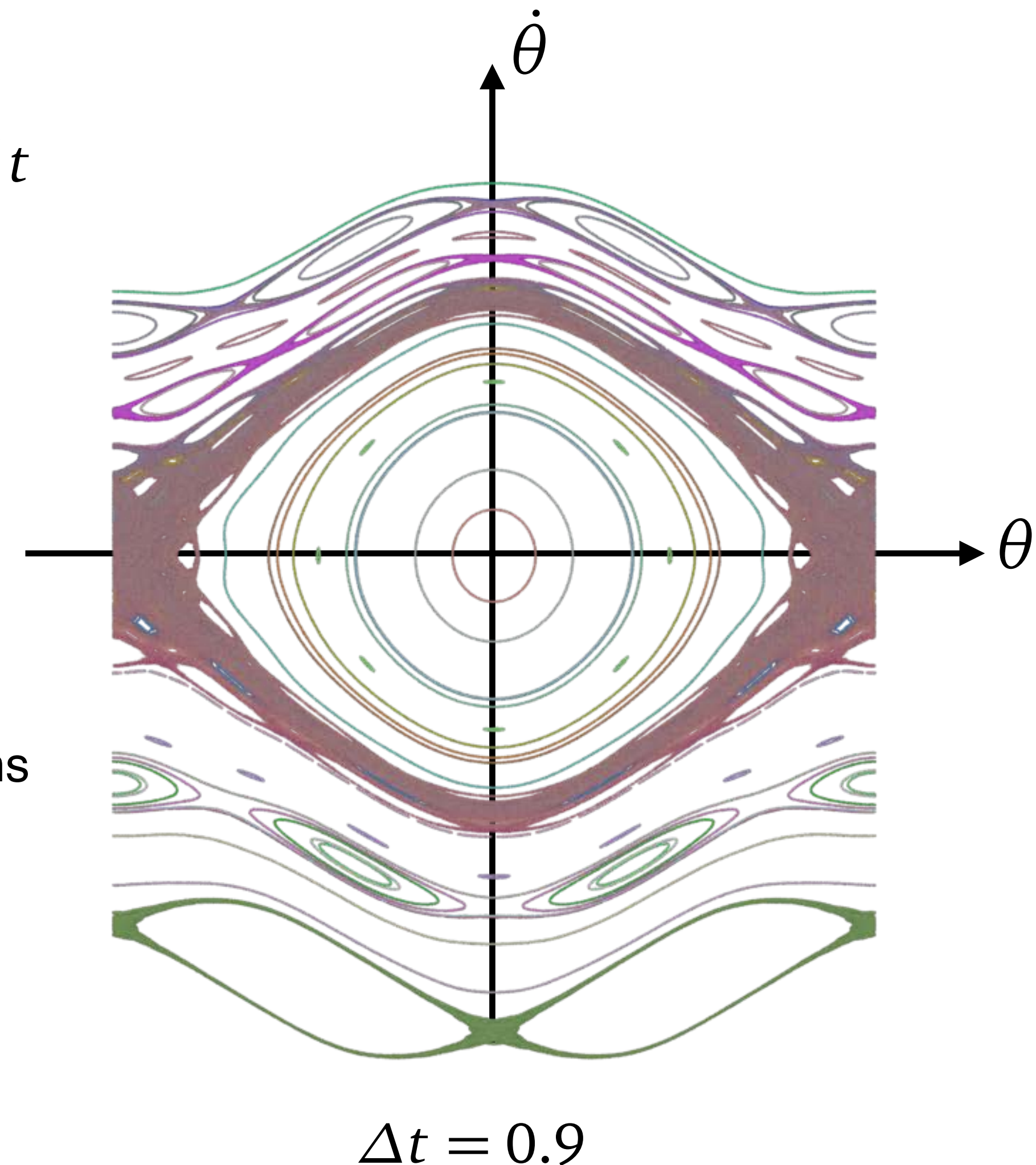
$$\int \left(\frac{m}{2} \dot{\theta}^2 + mg \cos \theta \right) dt$$

- A 2nd order discretization

$$\frac{\theta_{i-1} - 2\theta_i + \theta_{i+1}}{\Delta t^2} = -\sin \theta_i$$

First introduce discrete action,
then derive the least action paths

✓ Energy conservation
in “asteroid belts”



Pendulum equation

Example: pendulum equation.

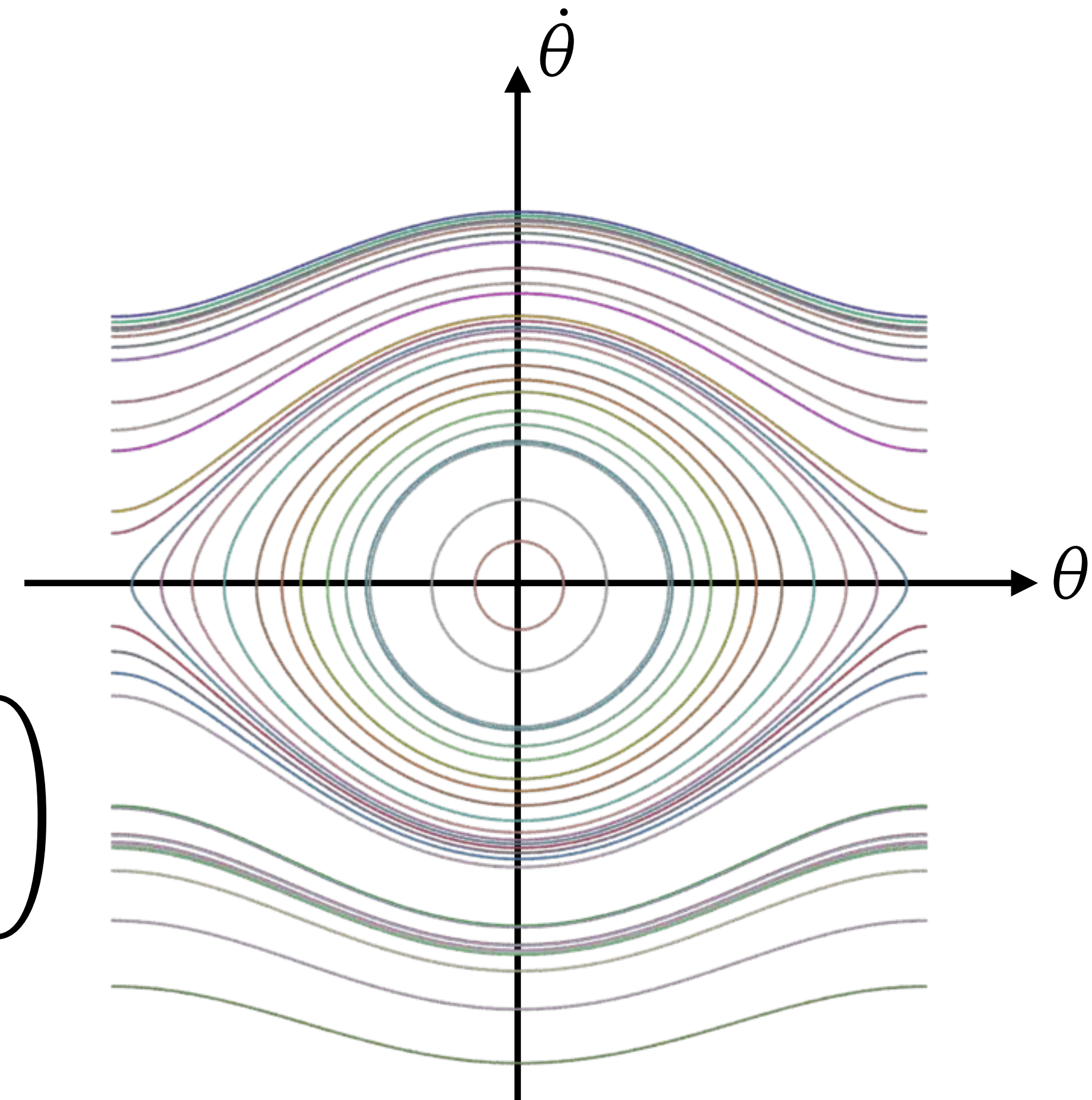
$$m\ddot{\theta} = -mg \sin \theta$$

- Another 2nd order discretization

$$\frac{\theta_{i-1} - 2\theta_i + \theta_{i+1}}{\Delta t^2} = -\sin \theta_i$$

$$\frac{\theta_{i-1} - 2\theta_i + \theta_{i+1}}{\Delta t^2} = 4 \arg \left(1 + \frac{\Delta t^2}{4} e^{-i\theta_i} \right)$$

✓ Integrable system



$\Delta t = 0.1$

Pendulum equation

Example: pendulum equation.

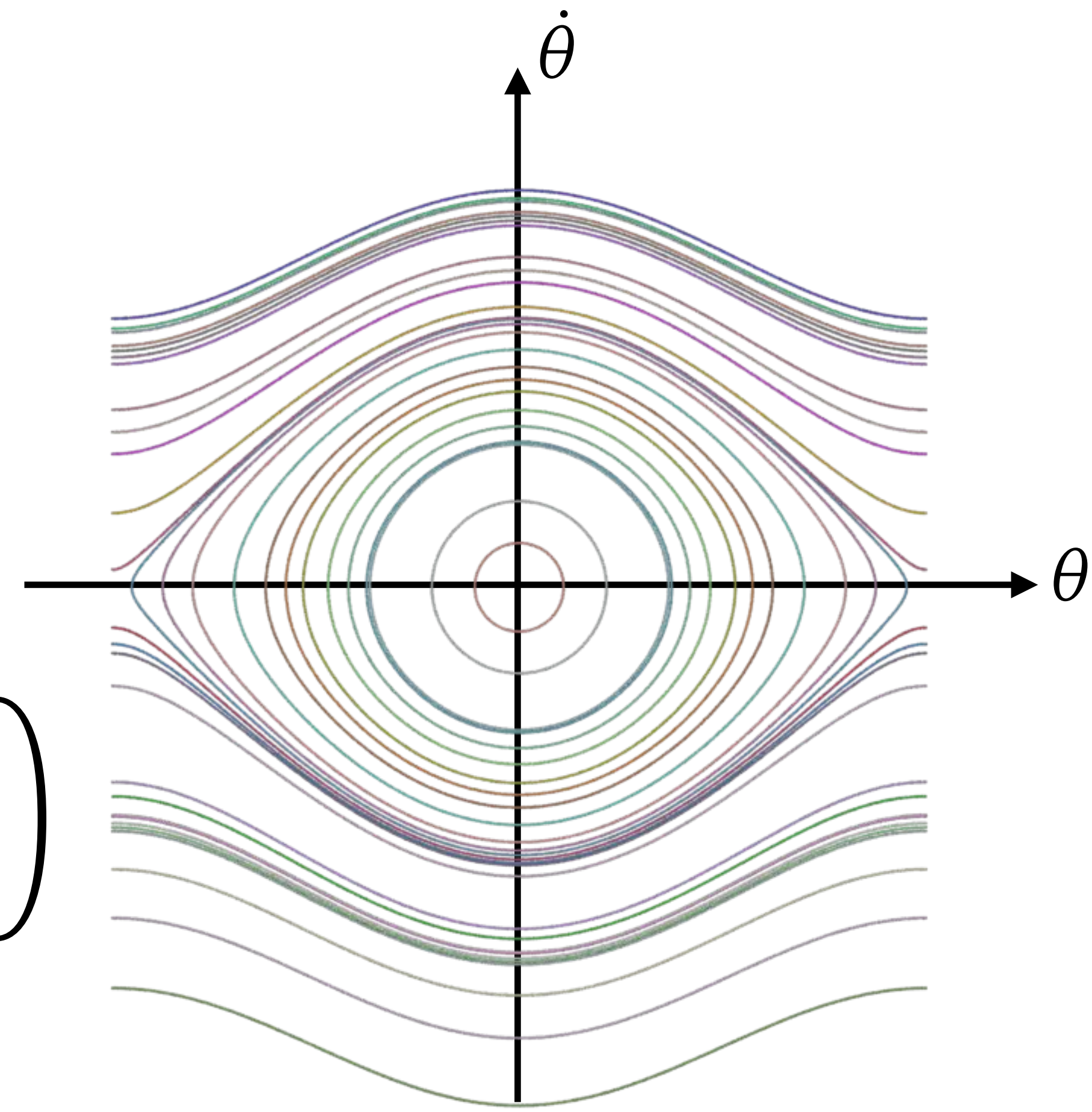
$$m\ddot{\theta} = -mg \sin \theta$$

- Another 2nd order discretization

$$\frac{\theta_{i-1} - 2\theta_i + \theta_{i+1}}{\Delta t^2} = -\sin \theta_i$$

$$\frac{\theta_{i-1} - 2\theta_i + \theta_{i+1}}{\Delta t^2} = 4 \arg \left(1 + \frac{\Delta t^2}{4} e^{-i\theta_i} \right)$$

✓ Integrable system



$\Delta t = 0.9$

Pendulum equation

general

4th order Runge–Kutta

Variational integrator

Discrete integrable system rare

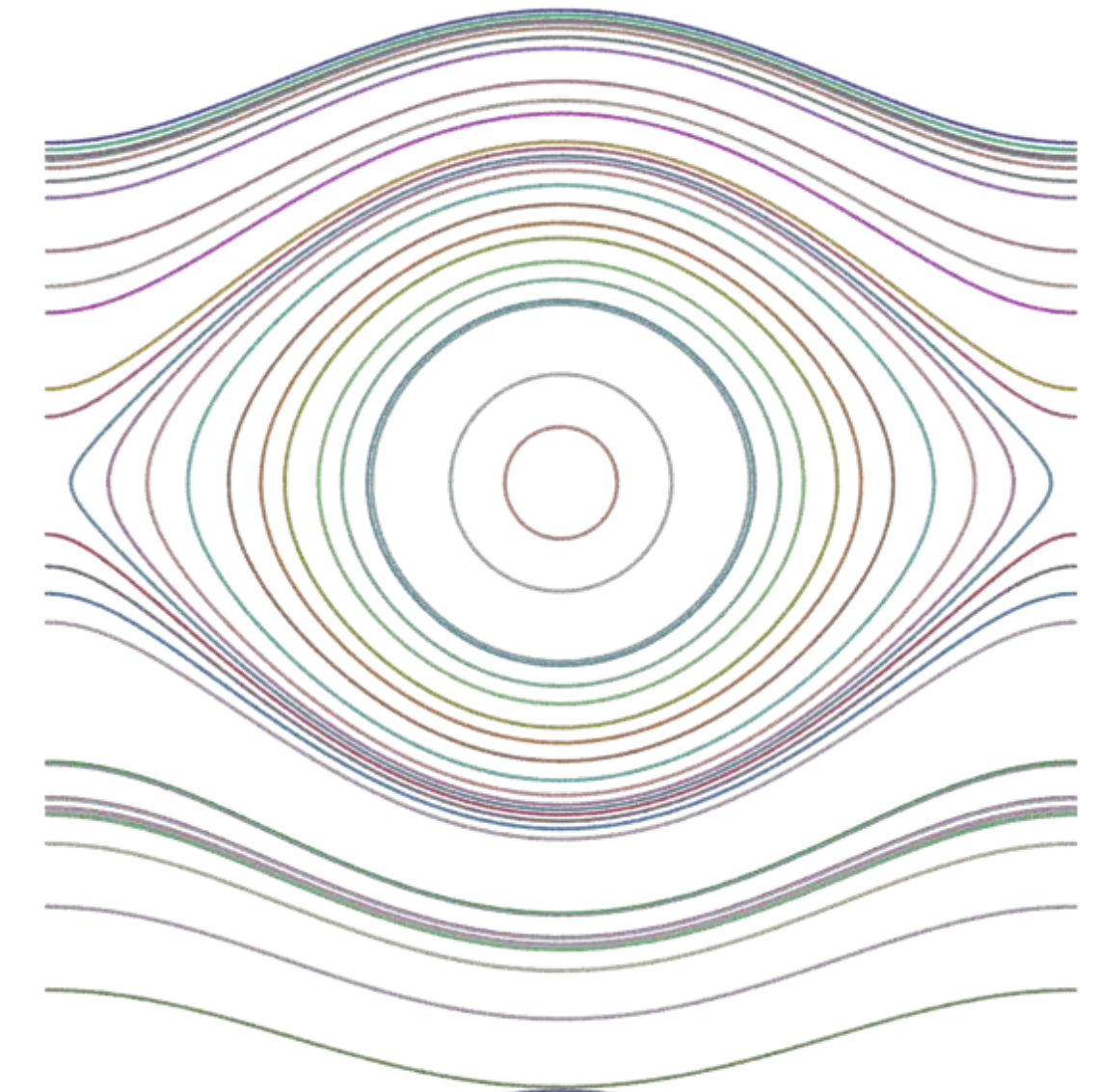
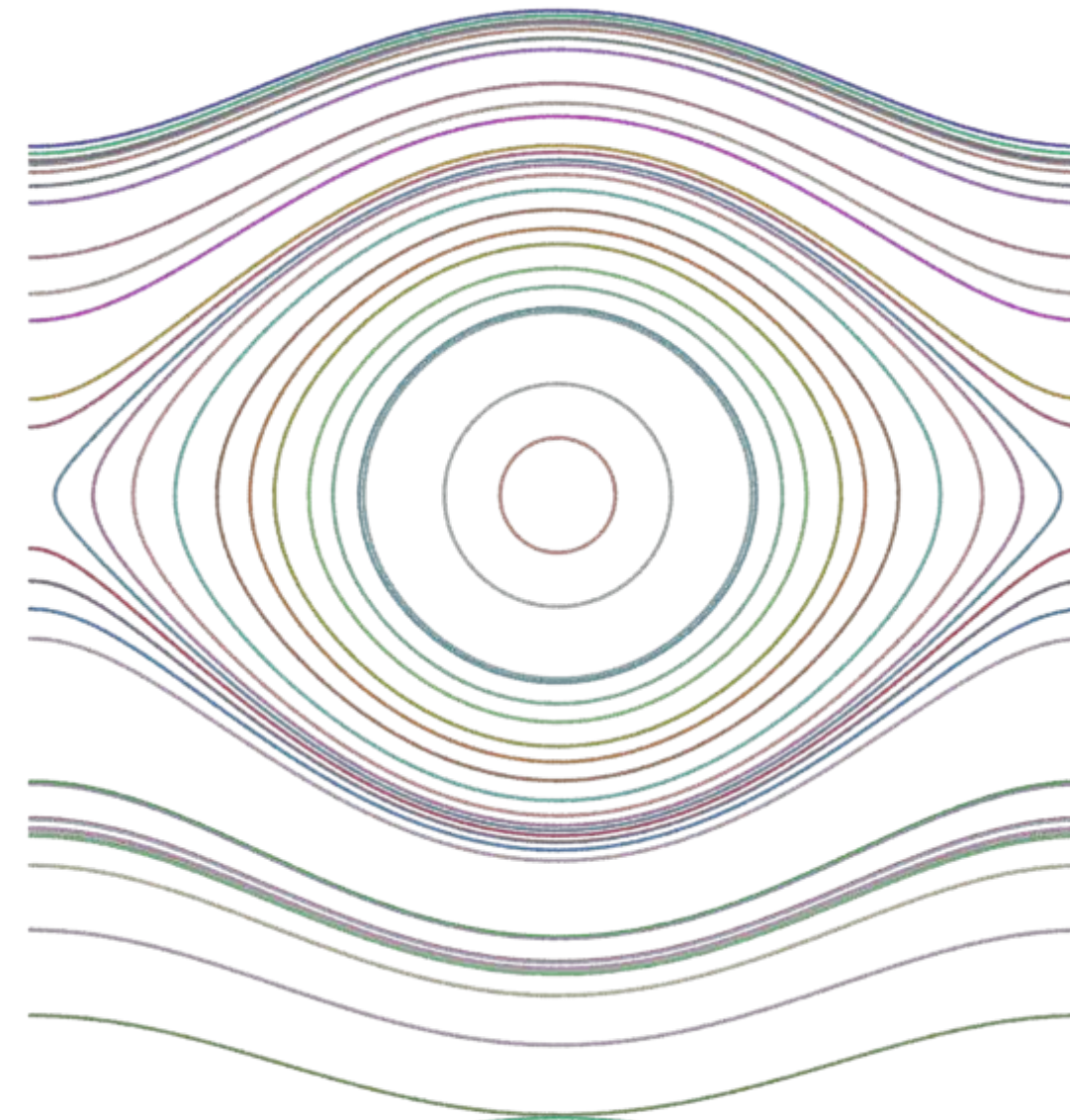
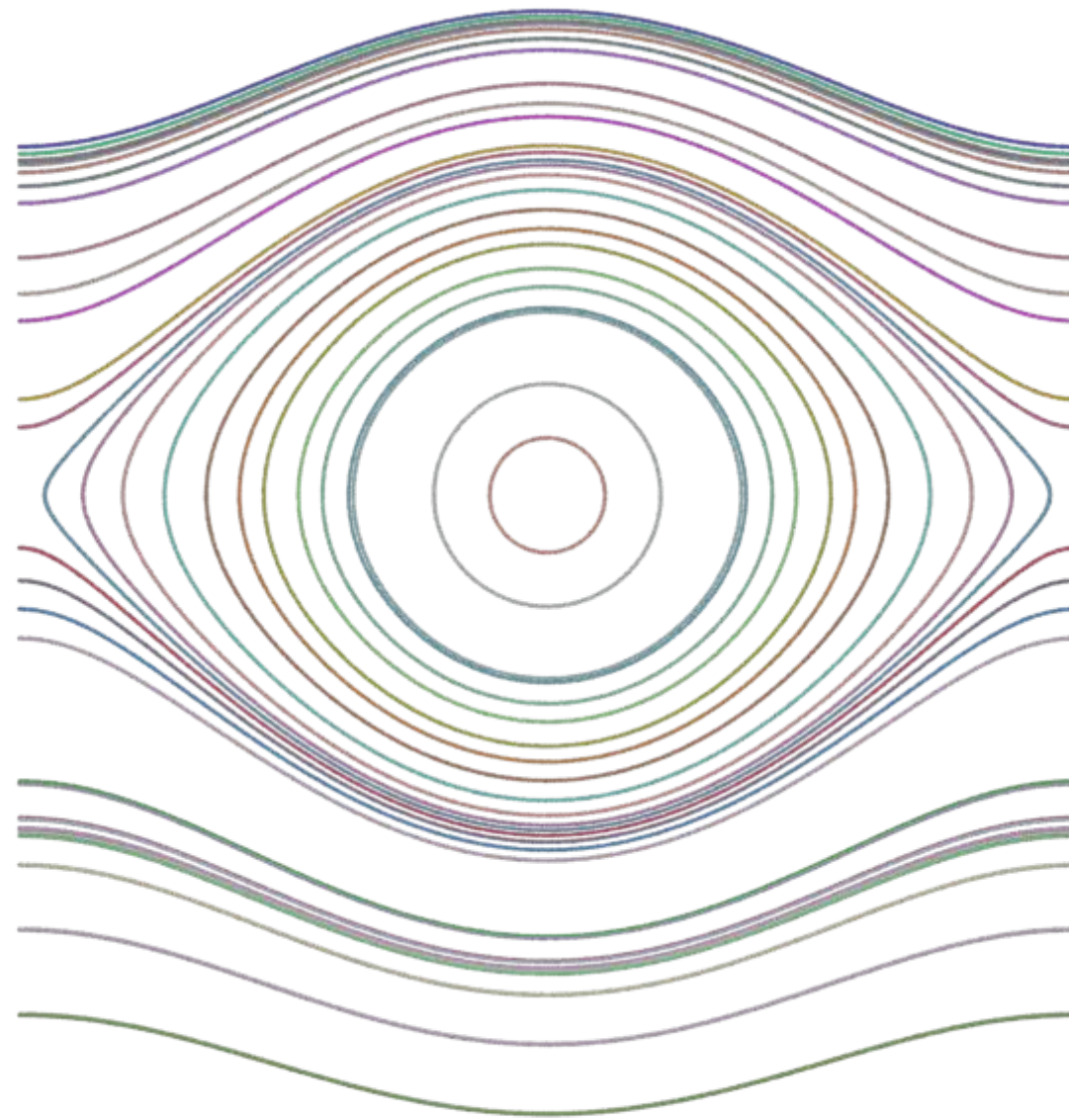
no-structure

quantitative
high-precision

structure
preserving

qualitative
exact

$\Delta t = 0.1$



$\Delta t = 0.9$

