Hairy Ball Theorem

- Hairy ball theorem
- Optimal sections of a bundle
- Curvature and singularities
- Gauge theory in physics
Hairy ball theorem
Theorem \textit{(Hairy ball theorem)}

Every continuous vector field on a topological sphere must have zeros (a.k.a. singularities).

(singularity in orientation) \& (continuity) forces the vector field dipping to zero.
**Theorem** (Hairy ball theorem)

Every continuous vector field on a topological sphere must have zeros (a.k.a. singularities).
Poincaré–Hopf theorem

**Theorem** (General hairy ball theorem)

Every continuous vector field on a closed surface must have total index equal to the Euler characteristic.

**Definition** The index of a singularity is the turning number of the vector field around the singularity. That is, it is the number of full (counterclockwise) turns the vector makes when we walk counterclockwise around a small circle around the singularity.
Recall:

$$(K \, dA)_i = 2\pi - \left( \text{turning angle of a loop around the point } i \right)$$
Poincaré index in angle defect

Recall:

\[(K \, dA)_i = 2\pi - \left(\text{turning angle of a loop around the point } i\right)\]

Total turning angle of \( v = 2\pi \) \( \sum \) index \( - \int \) region

Index observation from a distance
Proof of Poincaré–Hopf theorem

**Theorem** (General hairy ball theorem)
Every continuous vector field on a closed surface must have total index equal to the Euler characteristic.

Total turning angle of \( v = 2\pi \sum \text{index} - \int K \, dA \)

Let the enclosed region be the entire surface. Apply \( \int_M K \, dA = 2\pi \chi \)
Revising the notion of function

**Theorem** (General hairy ball theorem)
Every continuous vector field on a closed surface must have total index equal to the Euler characteristic.

- If vector field is like a function over a surface, what prevents us from assigning nonzero value everywhere?

- Unlike a traditional function $f : S_1 \to S_2$
  a vector field $\mathbf{u}$ takes value in different spaces $\mathbf{u}|_x \in T_x M$

- In fact a more unifying picture to talk about “function” is to look at the *function graph*. 
Revising the notion of function

Plotting $f : S_1 \rightarrow S_2$
Revising the notion of function

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Revising the notion of function

Plotting a tangent vector field $\mathbf{u}$ on $M$
Revising the notion of function

Plotting a tangent vector field $\mathbf{u}$ on $M$
Revising the notion of function

The “canvas” as a (higher-dimensional) manifold is called a **bundle**. Here we have the tangent bundle $TM$. $u \in \Gamma(TM)$ is called a section of the tangent bundle.
**Theorem** (Hairy ball theorem)

The tangent bundle of a closed surface is twisted in such a way that every pair of sections must have the total signed intersection equals to the Euler characteristic.
Optimal Sections of a Bundle

- Hairy ball theorem
- Optimal sections of a bundle
- Curvature and singularities
- Gauge theory in physics
**Goal** Find the optimal section of a bundle.

- Smoothest vector field
- Patterns best align with prescribed directions
- Automatically finds the optimal singularity locations.
Optimal section

Smoothest tangent vector fields

[Knöppel, Crane, Pinkall & Schröder 2013]
Globally optimal direction field
Optimal section

Impainting n-direction field

[Brandt, Scandolo, Eisemann & Hildebrandt 2018]
Modeling n-symmetric vector fields using higher-order energies
Optimal section

Non-orthogonal cross fields

[Diamanti, Vaxman, Panozzo & Sorkine-Hornung 2014]
Designing N-PolyVector fields with complex polynomials
Non-orthogonal cross fields with physical constraints

[Sageman-Furnas, Chern, Ben-Chen & Vaxman 2019]
Chebyshev nets from commuting polyvector field
N-direction field with physical constraints

[Vekhter, Zhuo, Fandino, Huang & Vouga 2019]
Weaving geodesic foliations
Optimal section

Complex phase field

[Knöppel, Crane, Pinkall & Schröder 2015]
Stripe patterns on surfaces
Optimal section

Complex phase field for vortex detection

[Weißmann, Pinkall & Schröder 2014]
Smoke rings from smoke

[Chern, Knöppel, Pinkall & Schröder 2017]
Inside fluid: Clebsch maps for visualization and processing
Optimal section

$C^2$ or $H$ phase field for quad meshing

[Fang, Bao, Tong, Desbrun & Huang 2018]
Quadrangulation through Morse-parametrization hybridization
Optimal section

Phase field for nearly conformal deformations

[Chern, Pinkall & Schröder 2015]
Close-to-conformal deformation of volumes
Optimal section

\( \mathbb{H} \) spinor field for nearly isometric reconstruction

[Chern, Knöppel, Pinkall & Schröder 2018]
Shape from metric
Optimal section

Frame field for hexahedral meshing

[Corman & Crane 2019]
Symmetric moving frame
Optimal section

Octahedral variety phase field for hexahedral meshing

[Palmer, Bommes & Solomon 2020]
Algebraic representations for volumetric frame fields
Common ingredient: gauge theory

- Find an algebraic representation of the value space $E_x$ at each $x \in M$.
- Describe the **connection** *(a.k.a. parallel transport)* between adjacent value spaces.
- The derivatives (finite differences) of the field respect the connections.
- Model some energy to minimize.
Discrete tangent bundle
Discrete tangent bundle

Each value space $E_i$

$\Psi_i \in E_i$
Discrete tangent bundle

Each value space $E_i$ is viewed as a complex plane via an arbitrary basis.

$\Psi_i \leftrightarrow \psi_i = r_i e^{i\theta_i} \in \mathbb{C}$

vector field $\leftrightarrow \mathbb{C}$-valued function
Discrete connection

For each adjacent value spaces

\[ 1_i \in E_i \]

\[ \Psi_i \leftrightarrow \psi_i = r_i e^{i\theta_i} \in \mathbb{C} \]

vector field \( \leftrightarrow \mathbb{C} \)-valued function
Discrete connection

For each adjacent value spaces

\[ \Psi_i \leftrightarrow \psi_i = r_i e^{i\theta_i} \in \mathbb{C} \]

\[ \Psi_j \leftrightarrow \psi_j = r_j e^{i\theta_j} \in \mathbb{C} \]
Discrete connection

For each adjacent value spaces $E_i$ and $E_j$:

$$\Psi_i \leftrightarrow \psi_i = r_i e^{i\theta_i} \in \mathbb{C}$$

$$\Psi_j \leftrightarrow \psi_j = r_j e^{i\theta_j} \in \mathbb{C}$$

parallel transported basis
Discrete connection

For each adjacent points $i, j$ encode the parallel transport by an angle $\alpha_{ij}$, with

$$\alpha_{ij} = -\alpha_{ji} \mod 2\pi$$

i.e. an angle-valued 1-form. It is called the connection as an additional structure for the bundle.
Covariant derivative

The derivative of the section $\Psi$ is denoted by $d^\nabla \Psi$.

In terms of the basis $1_i$

$$(d^\nabla \Psi)_{ij} \leftrightarrow e^{-i\alpha_{ij}} \psi_j - \psi_i$$

In terms of the basis $1_j$

$$(d^\nabla \Psi)_{ij} \leftrightarrow \psi_j - e^{i\alpha_{ij}} \psi_i$$

The magnitude $|{(d^\nabla \Psi)_{ij}}|$ is basis-choice independent.
Covariant derivative

The derivative of the section $\Psi$ is denoted by $d^\nabla \Psi$.

In terms of the basis $\mathbb{1}_i$

$$(d^\nabla \Psi)_{ij} \leftrightarrow e^{-i\alpha_{ij}} \psi_j - \psi_i$$

In terms of the basis $\mathbb{1}_j$

$$(d^\nabla \Psi)_{ij} \leftrightarrow \psi_j - e^{i\alpha_{ij}} \psi_i$$

The magnitude $|(d^\nabla \Psi)_{ij}|$ is basis-choice independent.
Dirichlet energy

The magnitude \(|(d\nabla \Psi)_{ij}|\) is basis-choice independent.

\[
\mathcal{E}[\Psi] = \sum_{ij \in \text{half edges}} w_{ij} |(d\nabla \Psi)_{ij}|^2
\]

connection derivative

\[
d\nabla = \begin{bmatrix}
i & j \\
\vdots & \vdots \\
e^{-\hat{i} \alpha_{ij}} & \ddots
\end{bmatrix}
\]
The magnitude $| (d^\nabla \Psi)_{ij} |$ is basis-choice independent.

$$E[\Psi] = \sum_{ij \in \text{half edges}} w_{ij} | (d^\nabla \Psi)_{ij} |^2$$

$$= \overline{\psi}^T \begin{bmatrix} d^\nabla \end{bmatrix}^T \begin{bmatrix} \star_1 \end{bmatrix} \begin{bmatrix} d^\nabla \end{bmatrix} \psi$$

$$L^\nabla \text{ connection Laplacian}$$
The Dirichlet energy $\mathcal{E}[^\psi] = \bar{\psi}^T L^\nabla \psi$ measures the smoothness of the section $\Psi$
(the smaller, the smoother with respect to the parallel transport)

**Ground state problem**

$$\min_{\psi} \bar{\psi}^T [L^\nabla] \psi$$

s.t. $\sum_{i} (\text{point}_{\text{area}})^i |\psi_i|^2 = 1$

$\Leftrightarrow$ 

$$\min_{\psi} \bar{\psi}^T [L^\nabla] \psi$$

$$\Leftrightarrow$$ 

$$L^\nabla \psi = \lambda \ast_0 \psi$$
Smoothest section

The Dirichlet energy $\mathcal{E}[\Psi] = \overrightarrow{\Psi}^T L\nabla \Psi$ measures the smoothness of the section $\Psi$ (the smaller, the smoother with respect to the parallel transport)

Ground state with pointwise norm=1 constraint

$$\min_{\Psi} \overrightarrow{\Psi}^T [L\nabla] \Psi$$

s.t. $|\psi_i|^2 = 1 \quad \forall i$

Ginzburg–Landau model

$$\min_{\Psi} \overrightarrow{\Psi}^T L\nabla \Psi + \frac{1}{\varepsilon^2} \sum_i \text{(point weight)}_i (|\psi_i|^2 - 1)^2$$
Complex line bundle setup summary

- The $\mathbb{C}$-valued function $\psi: \{\text{points}\} \to \mathbb{C}$ describes the geometric object called a section $\Psi \in \Gamma(E)$.

- We need to provide an angle-valued 1-form $\alpha$.

- The Laplacian for $\psi$ is

$$L^\nabla = d^\nabla^T \circ_1 d^\nabla$$

where $d^\nabla = \begin{bmatrix} i & j \r_i \theta_i \ci \Psi_i \in E_i \ci \Psi_j \in E_j \ci \Psi_j = r_j e^{i\theta_j} \in \mathbb{C} \ci \Psi_i = r_i e^{i\theta_i} \in \mathbb{C} \ci \Psi_i \leftrightarrow \Psi_j \ci \alpha_{ij} \ci \parallel\text{transported basis}\ci \parallel\text{parallel transported basis}\ci 1_i \in E_i \ci 1_j \in E_j \end{bmatrix}$
Effect of connection

If the complex line bundle is the tangent bundle,

then there is a canonical $\alpha$, known as the **Levi-Civita connection**, that describes the metric-induced parallel transport.
Effect of connection

\[ \alpha_{\text{LeviCivita}} \]

\[ \alpha_{\text{customized}} = \alpha_{\text{LeviCivita}} + \mathbf{v}^b \]
Levi-Civita connection

- Assign the polar angle $\phi_{ij}$ of each half-edge
  - In the picture $\phi_{ij} = \sum_{k=1}^{4} \beta'_k$ where $\beta'_k = \frac{\beta_k}{\sum_{\ell} \beta_\ell} 2\pi$
- The Levi-Civita connection is $\alpha_{ij} = \phi_{ji} - \phi_{ij} - \pi \mod 2\pi$
Gauge transformation

\[ \varphi_i \in \mathbb{R} \text{ (not angle-valued, but actual real-valued)} \]

\[ \widetilde{1}_i = e^{-i\varphi_i} 1_i \quad \widetilde{\psi}_i = e^{i\varphi_i} \psi_i \quad \widetilde{\alpha}_{ij} = \alpha_{ij} + (d\varphi)_{ij} \]

\[ = \alpha_{ij} + \varphi_j - \varphi_i \]
n-direction field
n-direction field
n-direction field

\[ \psi_i^{\text{vec}} = r_i e^{i \theta_i} \]

\[ \mathbf{\psi}_i^{\text{vec}} = \mathbf{r}_i e^{i \theta_i} \]

\[ \mathbf{z} \rightarrow \mathbf{z}^4 \]

\[ \psi_i^{\text{cross}} = (\psi_i^{\text{vec}})^4 \]
\[ \psi_i^{\text{cross}} = (\psi_i^{\text{vec}})^4 \]

Parallel transport for \( \psi_i^{\text{vec}} \):
\[ e^{i \alpha_{ij}^{\text{vec}}} \psi_i^{\text{vec}} \]

Parallel transport for \( \psi_i^{\text{cross}} \):
\[ e^{i \alpha_{ij}^{\text{cross}}} \psi_i^{\text{cross}} = e^{i 4 \alpha_{ij}^{\text{vec}}} (\psi_i^{\text{vec}})^4 \]

Therefore \( \alpha_{ij}^{\text{cross}} = 4 \alpha_{ij}^{\text{vec}} \)
\begin{align*}
\alpha_{ij}^{\text{cross}} &= 4\alpha_{ij}^{\text{vec}} \\
\text{Build} \quad L^\nabla &= d^\nabla \star_1 d^\nabla \\
\text{where} \quad d^\nabla &= \begin{bmatrix}
\delta_{ij} & -1 & e^{-i\alpha_{ij}^{\text{cross}}} \\
-1 & \ddots & \\
& \ddots & 
\end{bmatrix}
\end{align*}

Solve smallest eigenvalue problem

\[ L^\nabla \psi^{\text{cross}} = \lambda \star_0 \psi^{\text{cross}} \]
\[ \psi_i^{\text{vec}} = r_i e^{i \theta_i} \]

\[ -i \psi_i^{\text{vec}} \]

\[ z \rightarrow z^{1/4} \]

\[ \psi_i^{\text{cross}} = (\psi_i^{\text{vec}})^4 \]
Turning angle, curvature, singularities

- Hairy ball theorem
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Poincaré index in angle defect

Recall:

\[(K \, dA)_i = 2\pi - \left( \text{turning angle of a loop around the point } i \right)\]

Total turning angle of \( \mathbf{v} = 2\pi \sum \text{index} - \int \text{region enclosed} \]

Index observation from a distance
Poincaré index in angle defect

Total turning angle of $v = 2\pi$ \[ \sum_{\text{singularities enclosed}} \text{index} - \int_{\text{region enclosed}} K \, dA \]

For a general complex line bundle

- Turning angle from point $i$ to point $j$ \[ \text{arg} \left( \frac{\psi_j}{e^{i\alpha_{ij}} \psi_i} \right) \]

- Total turning angle after a round trip

\[ \text{arg} \left( \frac{\psi_{i_2}}{e^{i\alpha_{i_1,i_2}} \psi_{i_1}} \right) + \cdots + \text{arg} \left( \frac{\psi_{i_n}}{e^{i\alpha_{i_{n-1},i_n}} \psi_{i_1}} \right) \]
Poincaré index in angle defect

Total turning angle of $v = 2\pi \sum_{\text{singularities enclosed}} \text{index} - \int_{\text{region enclosed}} K dA$

For a general complex line bundle

- Turning angle from point $i$ to point $j$ $\arg\left( \frac{\psi_j}{e^{i\alpha_{ij}} \psi_i} \right)$

- Total turning angle after a round trip

\[
\arg\left( \frac{\psi_{i_2}}{e^{i\alpha_{i_1,i_2}} \psi_{i_1}} \right) + \cdots + \arg\left( \frac{\psi_{i_1}}{e^{i\alpha_{i_n,i_1}} \psi_{i_n}} \right) \mod 2\pi = -\text{index}(\alpha_{i_1,i_2} + \cdots + \alpha_{i_n,i_1})
\]
Poincaré index in angle defect

Total turning angle of \( \mathbf{v} = 2\pi \) \( \sum \) \( \text{index} \) \( - \int \) \( K \, dA \)

\[
\arg\left( \frac{\psi_i}{e^{i\alpha_{i_1,i_2}} \psi_i} \right) + \cdots + \arg\left( \frac{\psi_i}{e^{i\alpha_{i_n,i_1}} \psi_i} \right) \mod 2\pi = -i\left( \alpha_{i_1,i_2} + \cdots + \alpha_{i_n,i_1} \right)
\]

The curvature of the bundle enclosed by the loop is

\[
\int K \, dA = \beta = \sum_{\text{around the loop}} \alpha_{i_k,i_{k+1}} \mod 2\pi
\]

Knowing which 2 pi branch of curvature for every face can give us Poincaré index.
The curvature of the bundle enclosed by the loop is

$$\int K \, dA = \beta = \sum_{\text{around the loop}} \alpha_{i_k, i_{k+1}} \mod 2\pi$$

Knowing which 2 pi branch of curvature for every face can give us Poincaré index.

Curvature is gauge invariant
A complex line bundle over a triangle mesh is equipped with

- angle-valued 1-form $\alpha$
- real-valued 2-form $\beta$ so that $d\alpha = \beta \mod 2\pi$

A complex-valued 0-form $\psi$ and the connection $\alpha$
are relative to an arbitrary reference basis. Under basis transformation,

$$\psi \mapsto e^{i\varphi} \psi \quad \alpha \mapsto \alpha + d\varphi$$

The turning angle of $\psi$ along an edge is $\arg\left(\frac{\psi_j}{e^{i\alpha_{ij}}\psi_i}\right)$

The index formula: $(\text{Total turning angle}) = 2\pi \sum \text{index} - \int \beta$
Complex line bundle

The index formula: \( (\text{Total turning angle}) = 2\pi \sum \text{index} - \int \beta \)

Hairball theorem for complex line bundle

Over the entire closed surface

\[
2\pi \sum \text{index} = \int_M \beta
\]

Definition. The degree of a complex line bundle is the integer

\[
\frac{1}{2\pi} \int_M \beta
\]
Hairball theorem for complex line bundle

Over the entire closed surface

\[ 2\pi \sum \text{index} = \int_M \beta \]

Paint some 2-form total integral being $2\pi$ integer, can you build the rest of the ingredients? (i.e 1-form $\alpha$)
Singularities in optimal $n$-direction field

$n=1$

$\alpha = \alpha^{LC}$

$\beta = K \, dA$

$\text{degree} = 2$
Singularities in optimal n-direction field

\[ n = 2 \]
\[ \alpha = 2\alpha^{LC} \]
\[ \beta = 2K\,dA \]
degree = 4
Singularities in optimal n-direction field

$n=3$

$\alpha = 3\alpha^{\text{LC}}$

$\beta = 3K \, dA$

degree $= 6$
Singularities in optimal n-direction field

\[ n = 10 \]
\[ \alpha = 10\alpha^{LC} \]
\[ \beta = 10K \, dA \]

degree = 20
Singularities in optimal n-direction field

n=20
\[ \alpha = 20\alpha^{LC} \]
\[ \beta = 20K \text{ } dA \]
degree = 40
Singularity of optimal field

Independent random samples
Gauge theory in physics

- Hairy ball theorem
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Continuous theory

- In the discrete setting
  \[(d^\nabla \psi)_{ij} = e^{-i\alpha_{ij}} \psi_j - \psi_i\]

- Towards continuous
  \[
  \psi_j \approx \psi_i + d\psi(i\tilde{j}) + \cdots \\
  e^{-i\alpha_{ij}} \approx 1 - i\alpha(i\tilde{j}) + \cdots \\
  (d^\nabla \psi)(i\tilde{j}) \approx (1 - i\alpha(i\tilde{j}))(\psi_i + d\psi(i\tilde{j})) - \psi_i \\
  = d\psi(i\tilde{j}) - i\alpha(i\tilde{j})\psi_i + O(|i\tilde{j}|^2)
  \]

\[d^\nabla \psi = d\psi - i\alpha\psi\]
Continuous theory

- Continuous covariant derivative

\[ d^\nabla \psi = d\psi - \imath \alpha \psi \]

\[ d^\nabla = d - \imath \alpha \wedge \]

: \( \Omega^k(M; \mathbb{C}) \to \Omega^{k+1}(M; \mathbb{C}) \)

- Double covariant exterior derivative

\[ d^\nabla d^\nabla \psi = (d - \imath \alpha \wedge)(d - \imath \alpha)\psi \]

\[ = dd\psi - \imath d(\alpha \psi) - \imath \alpha \wedge d\psi - \alpha \wedge \alpha \psi \]

\[ = -\imath \beta \psi \quad \text{curvature of the connection} \]
Electrodynamics

charged particle
Electrodynamics

Coulomb electric force

\[ F = q_2 E \]

Electric field

\[ E \in \Omega^1(M; \mathbb{R}) \]

Charge density

\[ d \star E = \rho \]

n-form
Electrodynamics

\[ F = q_2 E \]

**Magnetic field**

\[ \beta \in \Omega^2(M; \mathbb{R}) \]

\[ d\beta = 0 \]

**Electric current flux form**

\[ d \star \beta = J \]
Electrodynamics

Lorentz magnetic force

\[ F = q_2 v_2 \times B \]

\[ F = q_2 E \]

\[ \beta \in \Omega^2(M; \mathbb{R}) \]

\[ d \beta = 0 \]

\[ d \star \beta = J \]

electric current flux form
This figure seems to violate the conservation of momentum.
Vector potential

- The magnetic field $\beta \in \Omega^2(M)$ satisfies $d\beta = 0$

- Kelvin (1851) and Helmholtz (1858): by Hodge decomposition
  \[ \beta = d\alpha \]

- The potential $\alpha \in \Omega^1(M)$ is not unique
  \[ \alpha, \quad \alpha + d\varphi \quad \text{are equality valid representation.} \]

- A change of choice $\alpha \mapsto \tilde{\alpha} = \alpha + d\varphi$ is called a gauge transformation

- All physically observable quantities should be gauge-invariant.
Electrodynamics

This figure seems to violate the conservation of momentum.

Problem solved if we revise the momentum as $p^b = m v^b + q \alpha$

(but now the momentum is not gauge invariant)
Quantum mechanics
Quantum mechanics

\[ \psi_1 = r_1 e^{i \theta_1} \]

\[ \psi_2 = r_2 e^{i \theta_2} \]

\[ \mathbf{p}^b = r^2 d\theta \]
Quantum mechanics

\[ \psi = re^{i\theta} \]
\[ p^b = r^2 d\theta \]
\[ = \overline{\psi}(-i\partial)\psi \]

Total kinetic energy is our Dirichlet energy

\[ \int \frac{1}{2m} |p^b|^2 = \frac{1}{2m} \int \overline{\psi} d* d\psi \]
Quantum electrodynamics

$$\Psi = r e^{i\theta}$$

$$p^b = r^2 d\theta + r^2 q\alpha$$

$$= \bar{\Psi}(-i d)\Psi + q\alpha \bar{\Psi}\Psi = \bar{\Psi}(-i(d + iq\alpha))\Psi = \bar{\Psi}(-id^\nabla)\Psi$$

Total kinetic energy is our Dirichlet energy

$$\int \frac{1}{2m} |p^b|^2 = \frac{1}{2m} \int \bar{\Psi} d^* d\Psi$$
Quantum electrodynamics

\[ \psi = re^{i\theta} \]
\[ \mathbf{p}^b = r^2 d\theta + r^2 q\alpha \]
\[ = \overline{\psi}(-i\mathbf{d})\psi + q\alpha \overline{\psi}\psi = \overline{\psi}(-i(\mathbf{d} + iq\alpha))\psi = \overline{\psi}(-i\mathbf{d}^\nabla)\psi \]

Total kinetic energy is our Dirichlet energy

\[ \int \frac{1}{2m} |\mathbf{p}^b|^2 = \frac{1}{2m} \int \overline{\psi}(\mathbf{d}^\nabla^T \star \mathbf{d}^\nabla)\psi \]
Gauge transform in quantum electrodynamics

\[ \psi \rightarrow e^{i\varphi}\psi \]
\[ \alpha \rightarrow \alpha + d\varphi \]

Curvature of the bundle

\[ \beta = d\alpha \text{ is gauge invariant} \]

Interference pattern is gauge invariant

Singularity location is gauge invariant
What determines the complex line bundle? By Hodge decomposition

\[ \beta = d\alpha \quad \text{and} \quad \int_{\text{loop}} \alpha \quad \text{allows us to reconstruct} \quad \alpha \quad \text{up to a factor of} \quad d\varphi \]
Optimal singularity placement

- Prescribe magnetic field $\beta$ (e.g. as Gaussian curvature) and global period $\int_{\text{global}} \alpha$

- Construct $\alpha$ and thus

$$L^\nabla = d^\nabla^T \star_1 d^\nabla$$

- Find the singularities of the ground state.
Optimal singularity placement

[Knöppel, Crane, Pinkall & Schröder 2015]
Stripe patterns on surfaces
Optimal singularity placement

Complex phase field for vortex detection

[Weiβmann, Pinkall & Schröder 2014]
Smoke rings from smoke

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Inside fluid: Clebsch maps for visualization and processing