Exercise 3.1 — 15 pts. A spherical polygon is a region on the sphere bordered by great circles. In this exercise you will show that the sum $\sum_i \theta_i$ of the exterior angles $\theta_i$ of a spherical polygon is $2\pi - A$, where $A$ is the area enclosed by the spherical polygon.

(a) 5 pts. Show that the area $A$ of a spherical triangle on the unit sphere with interior angles $\alpha_1, \alpha_2, \alpha_3$ is

$$A = \alpha_1 + \alpha_2 + \alpha_3 - \pi. \quad (1)$$

Hint Consider the areas $A_1, A_2, A_3$ of the three spherical lunes.

(b) 5 pts. Show that the area of a spherical $n$-gon with interior angles $\beta_1, \ldots, \beta_n$ is

$$A = (2 - n)\pi + \sum_{i=1}^{n} \beta_i. \quad (2)$$

Hint Partition the polygon into triangles.
(c) 5 pts. Conclude that the area of a spherical $n$-gon with exterior angles $\theta_1, \ldots, \theta_n$ is

$$A = 2\pi - \sum_{i=1}^{n} \theta_i.$$  

(3)

Exercise 3.2 — 10 pts. Consider a discrete space curve $\gamma_0, \gamma_1, \ldots, \gamma_{n-1} \in \mathbb{R}^3$. Each edge $(i, i + 1)$ is associated with a unit tangent vector $T_{i,i+1} = \frac{\gamma_{i+1} - \gamma_i}{|\gamma_{i+1} - \gamma_i|}$. A frame at edge $(i - 1, i)$ is a unit normal vector $U_{i-1,i} \perp T_{i-1,i}$. For consecutive edges, we say $U_{i,i+1}$ is the \textbf{parallel transport} of $U_{i-1,i}$ if $U_{i,i+1}$ is obtained by rotating $U_{i-1,i}$ about the axis $T_{i-1,i} \times T_{i,i+1}$. (If $T_{i-1,i} = T_{i,i+1}$, then $U_{i,i+1} = U_{i-1,i}$.)

Equivalently, we may also view this notion using the trajectory of $T$ on the unit sphere $S^2$: The parallel transport $U_{i,i+1} \in T^+_{i,i+1}$ from the frame $U_{i-1,i} \in T^+_{i-1,i}$ is obtained by “translating” the tangent plane of $S^2$ along a great circle $C_i$ connecting $T_{i-1,i}$ and $T_{i,i+1}$. During this transportation along a great circle, the angle between $U$ and the great circle is fixed.

Given an initial frame $U_{0,1} \in T^+_{0,1}$ we obtain a \textbf{parallel frame} by parallel transporting the frame iteratively over the entire curve. Over a closed curve, in general, the parallel frame does not return to the same frame after a round trip. Show that this angular gap is given by the \textbf{total torsion}, defined by $2\pi$ minus the area of the spherical polygon $T_{0,1}, \ldots, T_{n-2,n-1}$.

**Hint** What are the angles between $U$ and the paths connecting the $T$’s on the unit sphere?
Exercise 3.3 — 10 pts. For a discrete triangle surface we define the **Gaussian curvature form** \((K \, dA)\), at a vertex \(v\) as the area \(A\) on the unit sphere bounded by a spherical polygon whose vertices are the unit normals \(n_i\) of the \(m\) faces around the vertex.

**(a) 5 pts.** Show that this area is equal to the **angle defect**

\[
(K \, dA)_v = A = 2\pi - \sum_{i=1}^{m} \theta_i
\]

where \(\theta_i\) is the interior angle of the face \(i\) at the vertex.

**Hint** Consider planes that contain two consecutive normals and their intersection with the unit sphere.

**(b) 5 pts.** Show that the total angle defect on a closed triangle surface with \(|V|\) vertices, \(|E|\) edges and \(|F|\) faces is

\[
\sum_{v \in V} (K \, dA)_v = 2\pi \chi
\]

where \(\chi = |V| - |E| + |F|\) is the **Euler characteristic** of the surface.

**Stability Analysis**

When it comes to discretizing a time evolution process into an iteration (such as gradient descent method for numerical optimization) it is helpful to have a general sense of the stability of the resulting discrete time sequence.

Take the heat equation \(\frac{\partial u}{\partial t} = \Delta u\) for example. If we discretize space we obtain

\[
\frac{\partial u}{\partial t} = - \star_0^{-1} d_0^T \star_1 d_0 \, u.
\]

Now discretize the time into \(\{\ldots, t_{n-1}, t_n, t_{n+1}, \ldots\}\) with timestep \(t_n - t_{n-1} = \Delta t\). Let \(u^{(n)}\) denote the solution at time \(t_n\). The **forward Euler scheme** is

\[
\frac{u^{(n+1)} - u^{(n)}}{\Delta t} = - \star_0^{-1} Lu^{(n)} \quad \Rightarrow \quad u^{(n+1)} = u^{(n)} - \Delta t \star_0^{-1} Lu^{(n)}
\]

while the **backward Euler scheme** is

\[
\frac{u^{(n+1)} - u^{(n)}}{\Delta t} = - \star_0^{-1} Lu^{(n+1)} \quad \Rightarrow \quad (\star_0 + \Delta tL)u^{(n+1)} = \star_0 u^{(n)}.
\]
The von Neumann stability analysis is to perform the following steps. First of all consider the eigenvalues $\lambda_j$ and eigenvectors $\psi_j$ of the discrete Laplace matrix $\star_0^{-1}L$ (yes here with the $\star_0^{-1}$ in front)

\[(\star_0^{-1}L)\psi_j = \lambda_j \psi_j \quad \text{(that is } L\psi_j = \lambda_j \star_0 \psi_j \text{ in the form of HW1 implementation)}, \quad (9)\]

and let us take the known fact that

\[\lambda_j \text{ ranges from 0 to } 1/\ell^2; \quad (10)\]

where $\ell$ represents an estimate of the edge length of the mesh. Now, plug in to the time iteration (7) and (8) the Ansatz

\[u^{(n)} = r^n \psi_j \quad (11)\]

where $r \in \mathbb{C}$ is to be determined and it represents the rate of growth (if $|r| > 1$) or rate of decay $|r| < 1$ for the particular mode $\psi_j$. For the forward Euler method (7)

\[\frac{r^{n+1} \psi_j - r^n \psi_j}{\Delta t} = -(\star_0^{-1}L)r^n \psi_j = -r^n \lambda_j \psi_j \implies \frac{r - 1}{\Delta t} = -\lambda_j \implies r = 1 - \Delta t \lambda_j. \quad (12)\]

For the backward Euler method (8)

\[\frac{r^{n+1} \psi_j - r^n \psi_j}{\Delta t} = -(\star_0^{-1}L)r^{n+1} \psi_j \implies \frac{r - 1}{\Delta t} = -r \lambda_j \implies r = \frac{1}{1 + \Delta t \lambda_j}. \quad (13)\]

The method is stable if all possible values of $r$ satisfy $|r| \leq 1$. Using the estimation that $\lambda_j$ ranges from 0 to $1/\ell^2$ we have that the $r$ in (12) falls in the range $[1 - \frac{\ell^2}{\ell^2}, 1]$. Therefore for forward Euler method is stable only when $\Delta t \leq 2\ell^2$. For the backward Euler method (13) $r$ lies in the range $[\frac{1}{1 + \Delta t/\ell^2}, 1]$ which always yield $|r| \leq 1$ no matter the choice of $\Delta t$.

**Exercise 3.4 — 10 pts.** Here is another time discretization for the heat equation $\frac{\partial u}{\partial t} = -\star_0^{-1}L u$. The time derivative $\frac{\partial u}{\partial t}$ at time $t_n$ is in fact better approximated by $\frac{u^{(n+1)} - u^{(n-1)}}{2\Delta t}$, which also looks more symmetric. So, why not consider the following time discretization?

\[\frac{u^{(n+1)} - u^{(n-1)}}{2\Delta t} = -(\star_0^{-1}L)u^{(n)}. \quad (14)\]

Using the fact that the eigenvalues of $(\star_0^{-1}L)$ range from 0 to $1/\ell^2$, determine for what range (if any) of timestep $\Delta t > 0$ is (14) stable?  

\[
\text{Conformal Coordinate}
\]

Let $M$ be a 2-dimensional manifold with metric. Define $J = R^{90^\circ}: T_xM \to T_xM$ denote the counterclockwise $90^\circ$ rotation in each tangent plane. Useful facts include $J^2 = -I, J^{-1} = -J$.  

\[4\]
Exercise 3.5 — 5 pts. Suppose $\alpha$ is a 1-form on this 2-dimensional manifold with metric. Check that
\[
(- \star \alpha)(v) = \alpha(Jv)
\]
for all $v \in T_xM$. In other words, $-\star$ is the adjoint of the 90-degree rotation operator with respect to the covector-vector dual pairing: $\langle -\star (\cdot)|(\cdot) \rangle = \langle (\cdot)|J(\cdot) \rangle$.

**Hint** Take an orthonormal basis $e_1, e_2$ for $T_xM$ and express $\alpha = \alpha_1 e_1^\flat + \alpha_2 e_2^\flat$ and $v = v_1 e_1 + v_2 e_2$.

---

**Figure 1** A texture map is a complex-valued function $z: M \to \mathbb{C}$. In constrast to a general map (top), a conformal map (bottom) is locally a scaling and rotation which preserves angles and all the little shapes (in this case squares) in the texture. The right-hand side figures are flattened; the shading is just for hinting the correspondence to the face, instead of representing any 3D bumps.

A texture coordinate is an $\mathbb{R}^2$-valued map $(u, v): M \to \mathbb{R}^2$. Let us combine $u(x), v(x)$ into a complex number $z(x) = u(x) + iv(x)$. In other words, a texture coordinate is a complex-valued function, i.e. a complex-valued 0-form
\[
z: M \to \mathbb{C}, \quad z \in \Omega^0(M; \mathbb{C}).
\]
Its exterior derivative is a complex-valued 1-form
\[
dz|_x: T_xM \xrightarrow{\text{linear}} \mathbb{C}, \quad dz \in \Omega^1(M; \mathbb{C})
\]
describing the linear transformation the texture map is doing infinitesimally. The texture maps that have minimal shearing distortion are the ones that only infinitesimally scale and rotate. Such texture maps are called conformal texture coordinates. They are characterized by “the maps which preserve the notion of 90° rotation” formulated below. The complex-valued function $z$ is said to be holomorphic or conformal if

$$dz(fv) = lv dz(v)$$

for all tangent vector $v$.

That is (by Eq. (15))

$$- \star dz = lv dz, \quad \text{or} \quad dz - lv dz = 0.$$  

Eq. (19) is also known as the Cauchy–Riemann equation. On the contrary, a map is said to be antiholomorphic if

$$\star dz = iv dz, \quad \text{or} \quad dz + iv dz = 0.$$  

We call $\bar{\partial}$ (del-bar operator) and $\partial$ (del operator)

$$\bar{\partial}z = \frac{1}{2}(dz - iv dz), \quad \partial z = \frac{1}{2}(dz + iv dz).$$

Note that $dz = \bar{\partial}z + \partial z$. In particular $\bar{\partial}$ measures the deviation from being conformal. The conformality condition is encapsulated as $\partial z = 0$. We mention $\partial$ just to emphasize that every map has a conformal and an anticonformal part; they constitute $dz = \bar{\partial}z + \partial z$.

To seek a texture map $z: M \to \mathbb{C}$ as conformal as possible, we minimize $\|\partial z\|^2$.

Here for a $\mathbb{C}$-valued 1-forms $\alpha, \beta \in \Omega^1(M; \mathbb{C})$,

$$\|\alpha\|^2 = \langle \alpha, \alpha \rangle, \quad \text{and} \quad \langle \alpha, \beta \rangle = \int_M \text{Re} (\overline{\alpha} \wedge \star \beta).$$

(23)

Here $(\cdot)$ is the complex conjugate.

**Exercise 3.6 — 10 pts.** Here we will show that (23) is a natural extension of the $L^2$ product $\langle \cdot, \cdot \rangle = \int_M \langle \cdot \rangle \wedge \star \langle \cdot \rangle$ originally only for real-valued forms.

**(a)** Show that when writing the $\mathbb{C}$-valued form in $\alpha = \alpha_{\text{Re}} + i\alpha_{\text{Im}}$ where $\alpha_{\text{Re}}, \alpha_{\text{Im}}$ are $\mathbb{R}$-valued forms, and similarly for $\beta$, we have

$$\langle \alpha, \beta \rangle = \langle \alpha_{\text{Re}}, \beta_{\text{Re}} \rangle + \langle \alpha_{\text{Im}}, \beta_{\text{Im}} \rangle.$$

(24)

**(b)** Check that $\langle \alpha, \beta \rangle = \langle \beta, \alpha \rangle$ and $\langle i\alpha, \beta \rangle = \langle \alpha, -i\beta \rangle$.

**Exercise 3.7 — 5 pts.** Let $z = u + iv: M \to \mathbb{C}$ be a map from a 2-dimensional manifold $M$ to the complex plane. Show that the total area of the image of $z$ is given by

$$\text{Area}[z] = -\frac{1}{2} \int_M \text{Re} (\bar{\partial}z \wedge \bar{\partial}z).$$

(25)

All you need to check is that for each two tangent vectors $w_1, w_2 \in T_xM$ at a point $x \in M$,
the 2-form \( \omega = -\frac{1}{2} \text{Re}(\overline{dz} \wedge \overline{dz}) \) is evaluated into

\[
\omega(w_1, w_2) = \det \begin{bmatrix} du(w_1) & du(w_2) \\ dv(w_1) & dv(w_2) \end{bmatrix}.
\]  

(26)

Exercise 3.8 — 10 pts. Show that the to-be-minimized conformal energy \( \|\overline{\partial z}\|^2 \) for a texture map \( z = u + iv: M \to \mathbb{C} \) is the Dirichlet energy minus the image area

\[
\|\overline{\partial z}\|^2 = \frac{1}{2}\|du\|^2 + \frac{1}{2}\|dv\|^2 - \text{Area}[z].
\]  

(27)

Throughout the entire theory for conformal texture, the only metric-related structure that we have used is \( \star \) (equivalently the \( J \) operator on the tangent spaces). None of notion of conformality, Dirichlet energy or area functional depends on \( \star \). So the whole theory can be weaken to just for manifold with \( J \) (namely, without the full notion of metric). A 2-dimensional manifold with just a \( J \) operator (with the property \( J^2 = -I \) ) defined for each tangent space is called a Riemann surface (not to be confused with Riemannian surface).

You cannot measure the norm of a tangent vector on a Riemann surface. You cannot measure the inner product between vectors, nor can you measure the area spanned by two vectors. But you can measure the angle between two vectors: The angle between \( u \) and \( v \) is whatever \( \theta \) that makes the rotated \( u, e^{i\theta}u = \cos(\theta)u + \sin(\theta)v \), parallel to \( v \).

The complex plane is a Riemann surface. It is \( \mathbb{R}^2 \) equipped with \( J = i \). A conformal map is a map that preserves this \( J \) structure (\( dz(f(\cdot)) = i dz(\cdot) \)). In particular, it will be an angle-preserving map (since the notion of angle is a derived concept from the \( J \) structure). Angle preservation is a good quality to have for a texture map.

Implementation: Spectral Conformal Flattening (50 pts)

In this implementation, modify your code from HW2. Take a triangle mesh with boundary. Compute a conformal texture coordinate \( u, v \) as point attributes.

- Set a vector point attribute \( v@uv = \text{set}(f@u, f@v, 0) \); Houdini recognizes the vector attribute named “uv” for texture coordinate for rendering.
- You can set \( v@P = v@uv \) to speculate how the flattened mesh looks.
- You can use the node “uv quick shade” to preview your texture coordinate on your surface.
In material setting, for a principled shader you can assign texture image. For example in the “Textures tab” you can choose an image for the base color. You can also assign image for “Bump/Normal” and “Displacement.”

Discretizing conformal energy
To obtain a conformal coordinate we find $z = u + iv: \mathcal{P} \rightarrow \mathbb{C}$ (here $\mathcal{P}$ is the set of points) that minimizes a discrete version of the conformal energy

$$C[z] = \frac{1}{2} \|d\bar{z}\|^2 - \text{Area}[z].$$

(28)

The Dirichlet energy is discretized as

$$\frac{1}{2} \|d\bar{z}\|^2 = \sum_{e \in E} \frac{1}{2} (\star 1)_{e \in \mathcal{E}} |z_{\text{dst}(e)} - z_{\text{src}(e)}|^2 = \sum_{e \in E} \frac{1}{2} (\star 1)_{e \in \mathcal{E}} (\bar{z}_{\text{dst}(e)} - \bar{z}_{\text{src}(e)})(z_{\text{dst}(e)} - z_{\text{src}(e)})$$

(29)

$$= \frac{1}{2} \begin{bmatrix} \bar{z}_0 & \bar{z}_1 & \cdots & \bar{z}_{|\mathcal{P}|-1} \end{bmatrix} \begin{bmatrix} d_0 & \star_1 & \cdots & d_0 \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \\ \vdots \\ z_{|\mathcal{P}|-1} \end{bmatrix}$$

(30)

$$= z^H \left( \frac{1}{2} L \right) z$$

(31)

where $L$ is the Laplace matrix that you have already implemented. Here

$$X^H = X^\top$$

is called the Hermitian transpose (conjugate transpose).

(32)

What about Area[z]? It is the sum of all triangle (signed) area on the complex plane when we realize the mesh with point positions $z \in \mathbb{C}$. Let’s consider triangle $ijk$. 

\[\]
We illustrate $ijk$ in clockwise order since if you call the Houdini function `primpoints` it will give you the list of points of a given triangle in the clockwise order. The area of this triangle is given by

$$\text{Area}(z_i, z_j, z_k) = \frac{1}{2} \text{Re} \left( \frac{(z_j - z_i) \overline{(z_k - z_i)}}{(z_k - z_i)} \right)$$

Therefore, we may write the area functional also as a quadratic form in $z$

$$\text{Area}[z] = \begin{bmatrix} z_0 & \bar{z}_1 & \cdots & \bar{z}_{|P|-1} \end{bmatrix} \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \\ \vdots \\ z_{|P|-1} \end{bmatrix}$$

where the sparse matrix $A$ has for each triangle $ijk$

$$A_{jk} = +\frac{1}{4}, \quad A_{ji} = -\frac{1}{4}, \quad A_{ik} = -\frac{1}{4}, \quad A_{kj} = +\frac{1}{4}, \quad A_{ji} = +\frac{1}{4}, \quad A_{ki} = +\frac{1}{4}. \quad (36)$$

(You should build the sparse matrix by setting arrays of row, column and entry values and calling `scipy.sparse.csr_matrix`. Repeated entries will automatically get summed when the matrix is built.)

Finally, the conformal energy is

$$C[z] = z^H C z, \quad \text{where the Hermitian matrix } C = \frac{1}{2} L - A.$$

Solve for a nontrivial minimizer of $C$

We look for $z: \mathcal{P} \to \mathbb{C}$ that minimizes $z^H C z$. Well, this will just give us $z = \text{constant}$. That is, $z$ maps to a single point, and $z^H C z = 0$. To get a nontrivial solution, we add an artificial constraint that $z^H \star_0 z = \langle z, z \rangle = \text{a nonzero constant}$. (Yes, $\star_0$ is not part of the original conformal theory.) Then the resulting optimization problem becomes the minimization of the Rayleigh quotient

$$\min_z \frac{z^H C z}{z^H \star_0 z}$$

whose solutions are given by the eigenvalue problem

$$C z = \lambda \star_0 z.$$
In the smooth theory, there exists many perfect conformal maps to the plane, so there are in principle many linearly independent $z$ such that $Cz = 0$; that is, $C$ has many zero eigenvalues. In the discrete setting, there are many eigenvalues clustered around zero. Only the constant $z$ will truly be the zero eigenvector.

The desired $z$ is the eigenvector to (39) corresponding to the smallest nonzero eigenvalue.

Assign the real and imaginary part of the solution to your $u,v$ point attributes.

**A more efficient way to call scipy.sparse.linalg.eigsh**

If you want to find the $n$ smallest-magnitude eigenvalues/eigenvectors of (39) then you could call

```python
la.eigsh(C, k=n, M=star0, which="SM")
```

However it will be extremely slow since all the eigenvalues are clustered around zero. We should call the much more efficient

```python
la.eigsh(C, k=n, M=star0, which="LM", sigma=0.0001)
```

which finds the largest-magnitude eigenvalues of $(C - \sigma \star 0)^{-1}$, i.e. it finds the eigenvalues near $-\sigma$. It took only 1 second on a 30k mesh.